Hopf algebra gauge theory and Kitaev lattice models

Workshop on

Noncommutative Field Theory and Gravity

Corfu, September 26 2015

Catherine Meusburger
Department Mathematik, Universität Erlangen-Nürnberg

partly joint work with Derek Wise

EUROPEAN COOPERATION
IN SCIENCE AND TECHNOLOGY

Motivation

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⇨ **understand Kitaev-lattice models and relation to gauge theory:** [Kitaev, ‘03]

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2. Construction of a local Hopf algebra lattice gauge theory

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5. Kitaev models as local Hopf algebra gauge theories

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lattice gauge theory for a group

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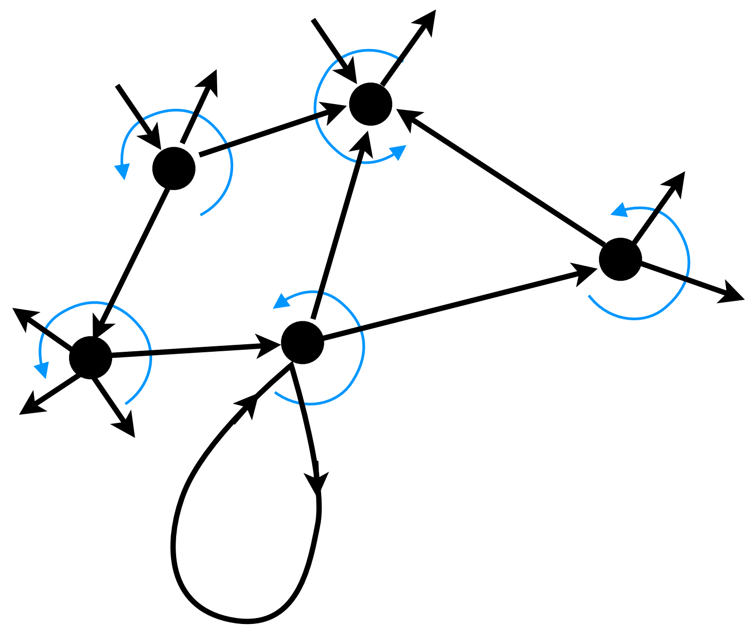

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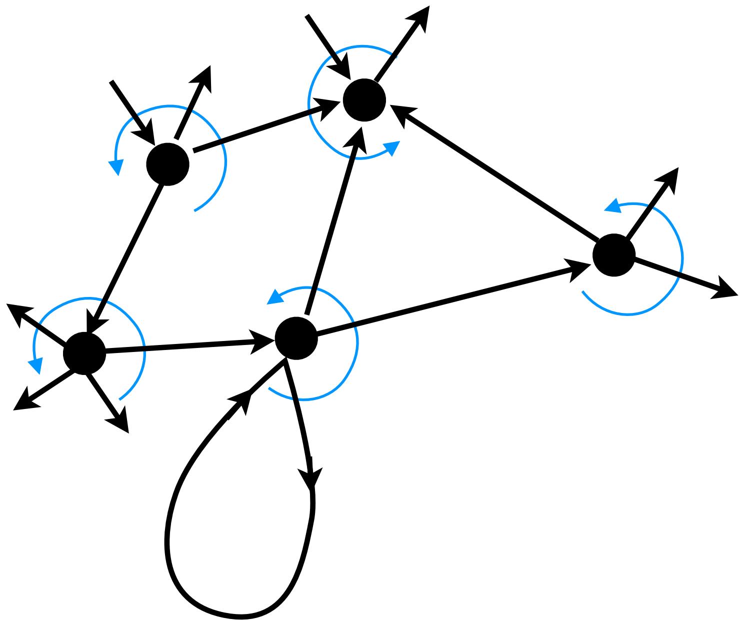
gauge fields set $G^{\times E}$



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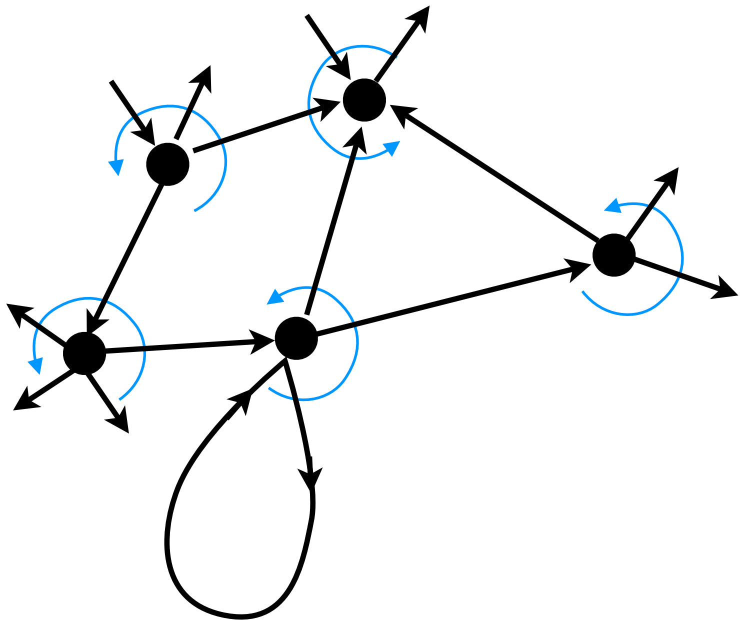

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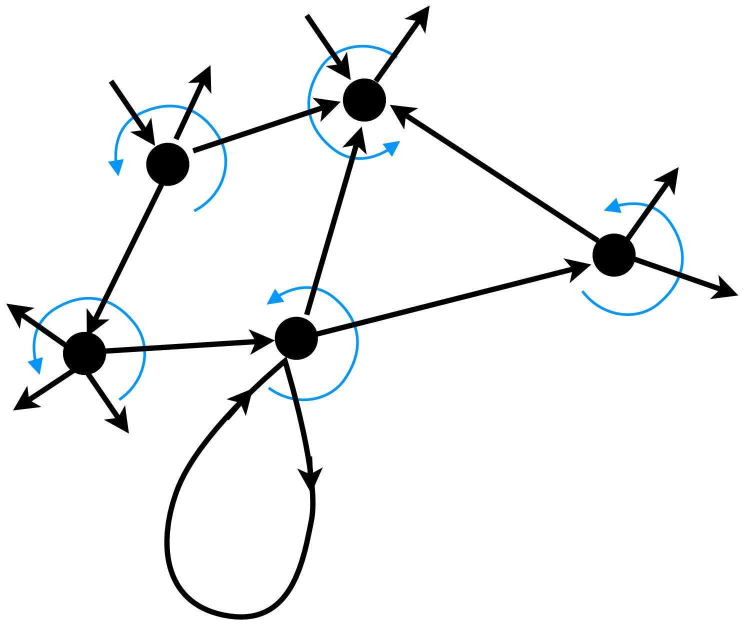

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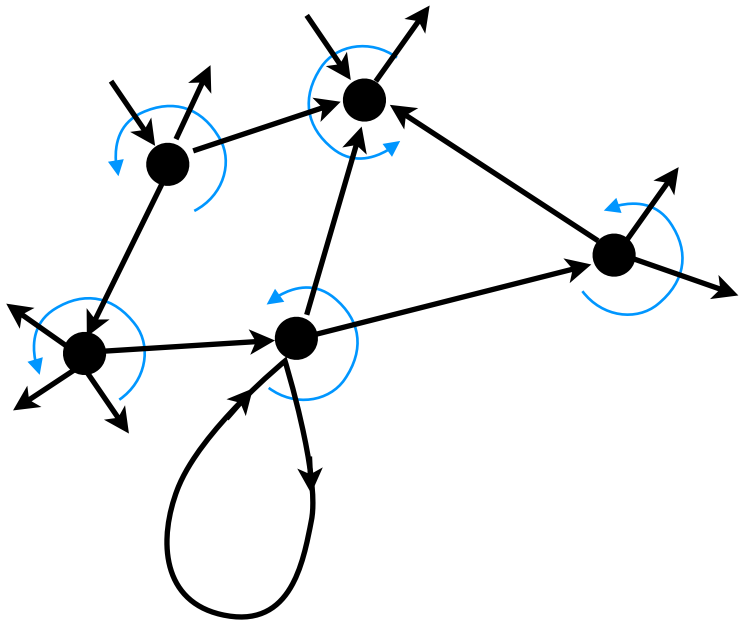

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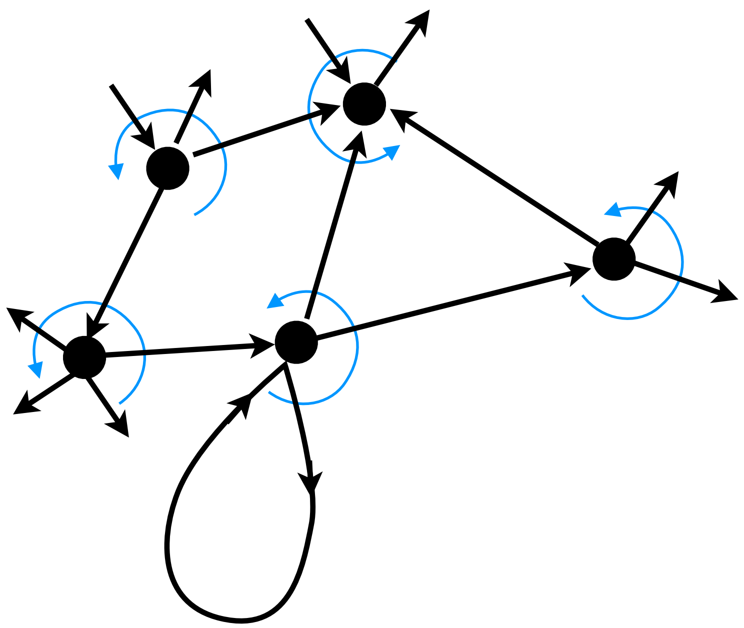

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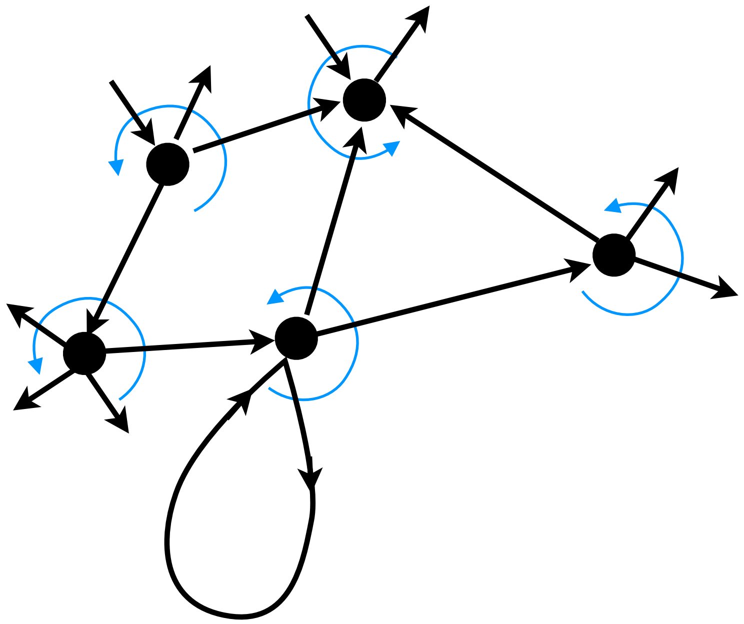

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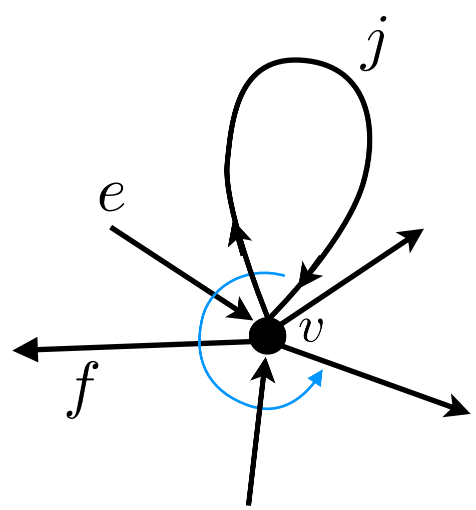

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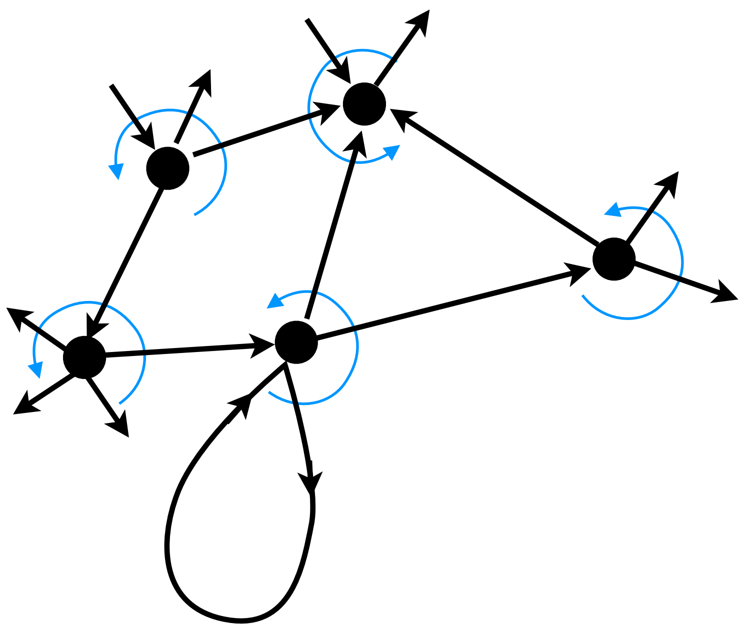

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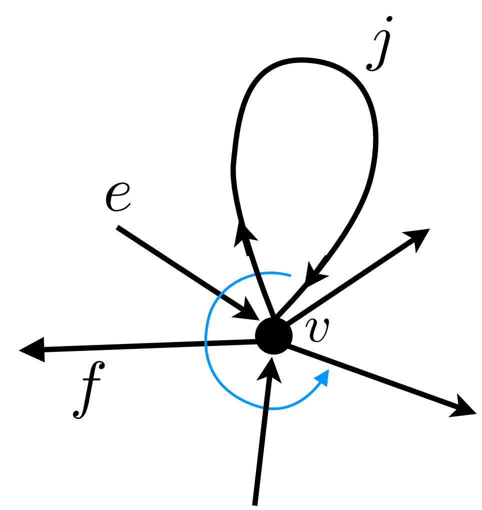

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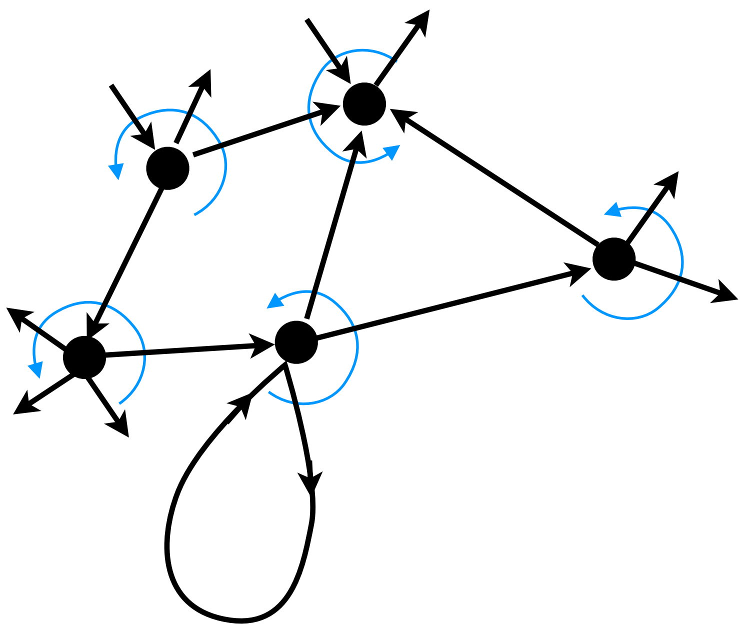

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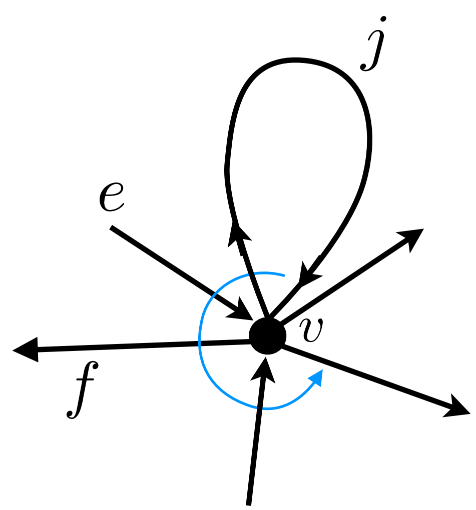

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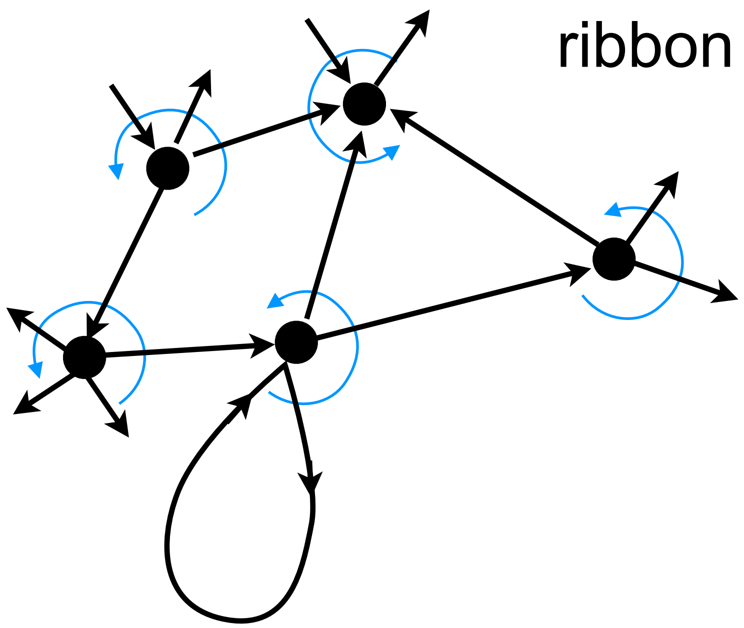

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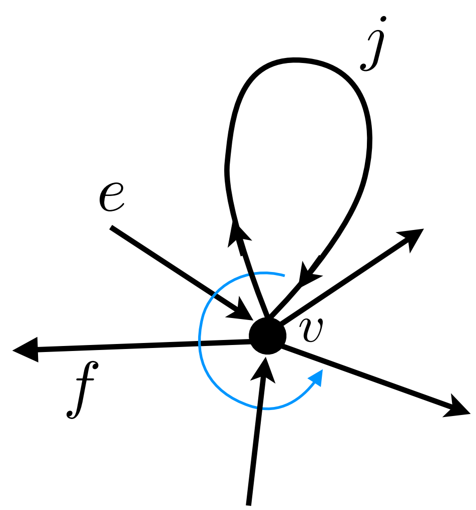

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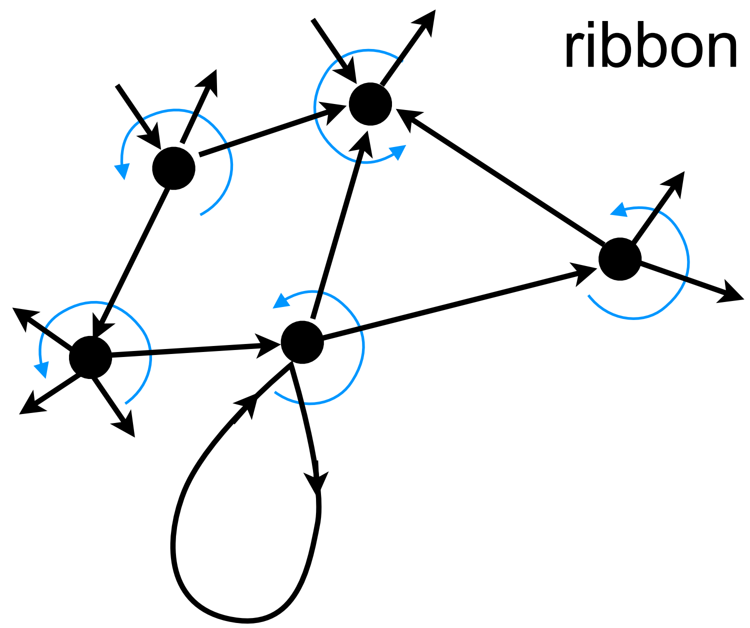

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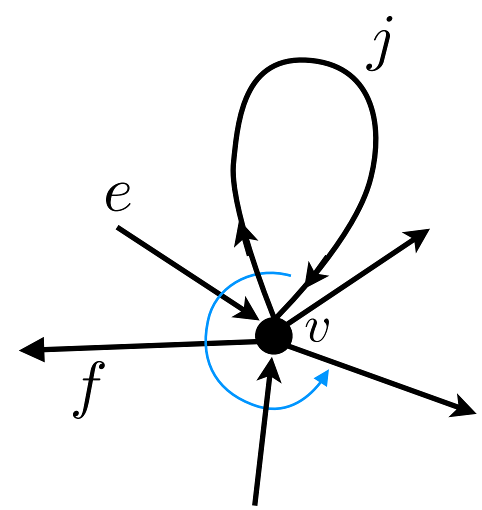

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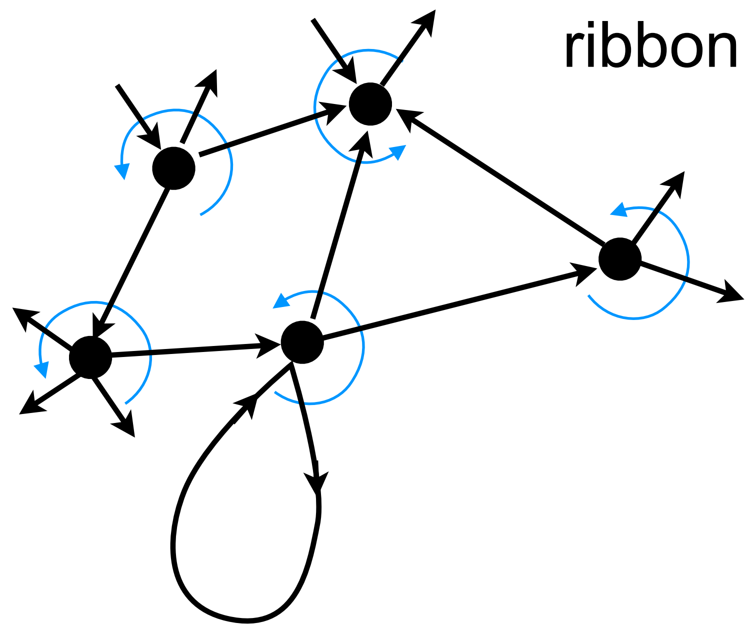

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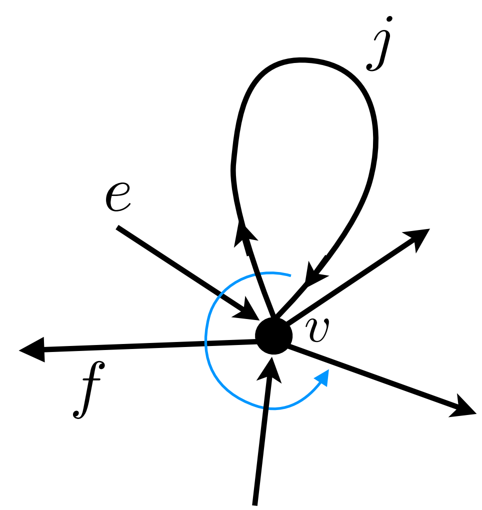

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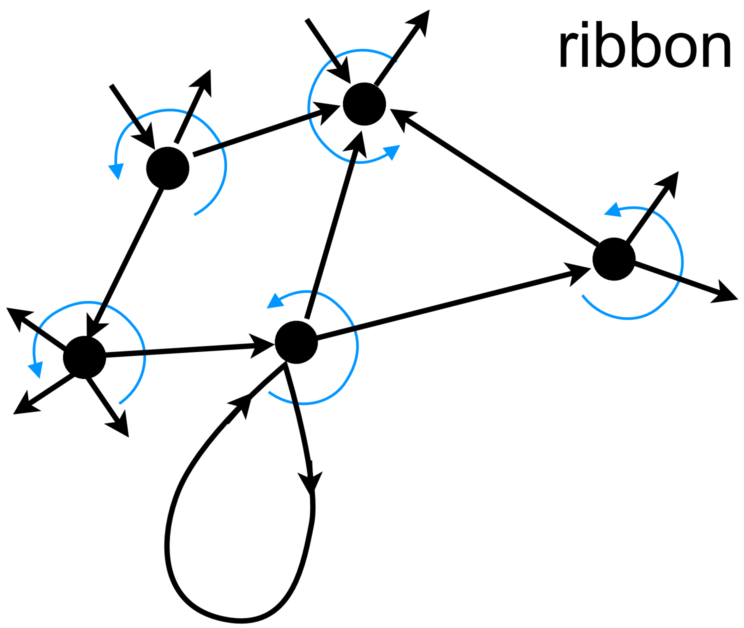

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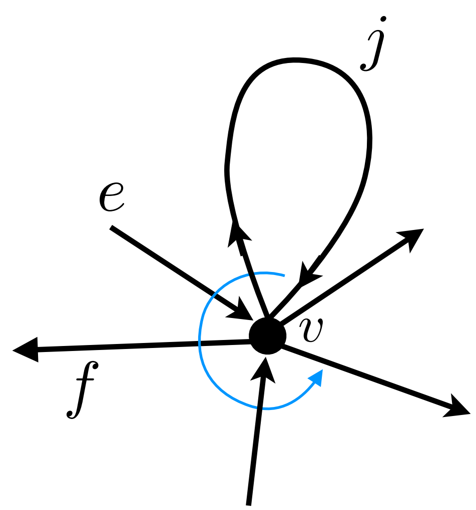

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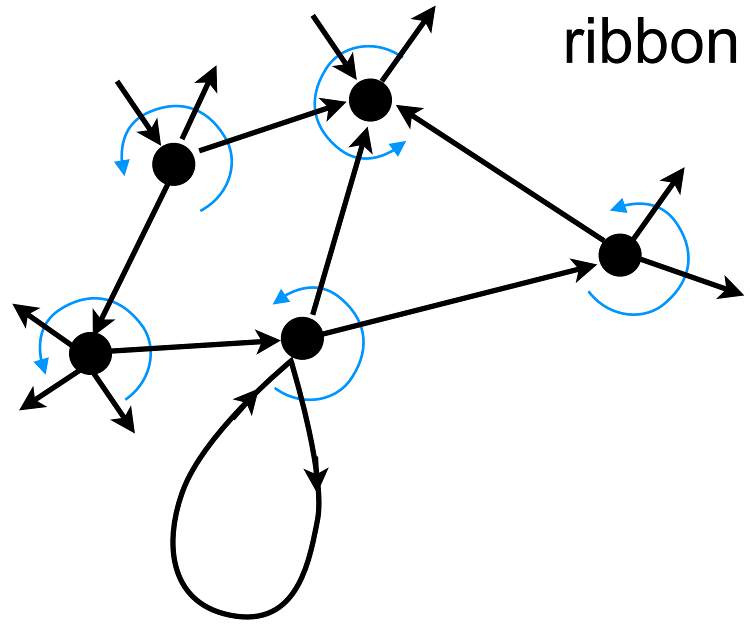

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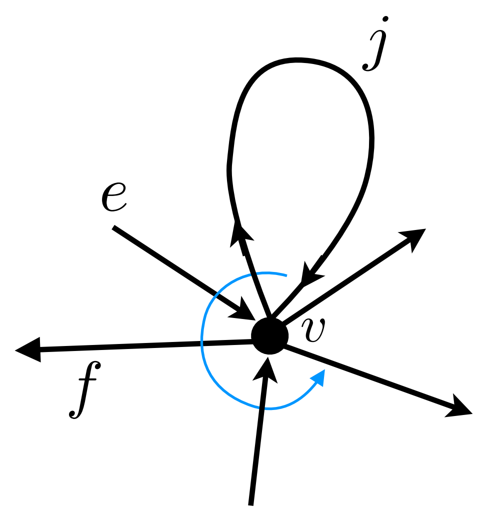

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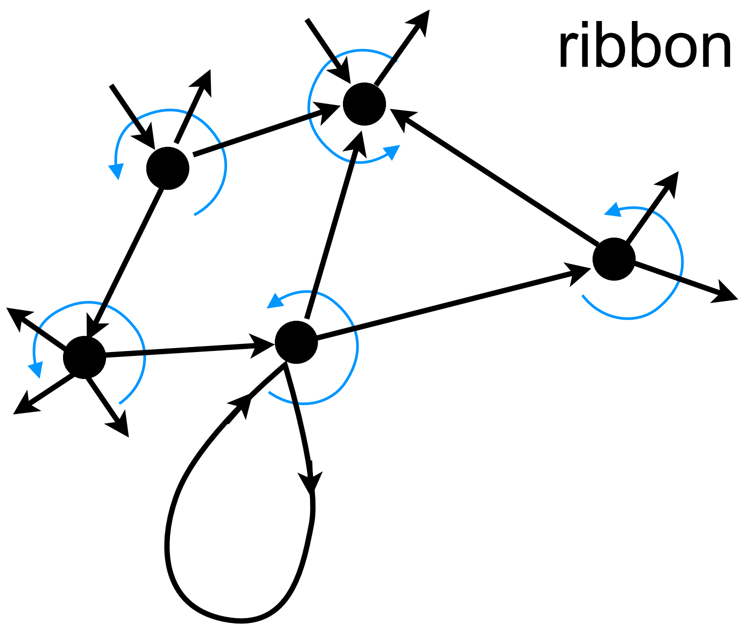

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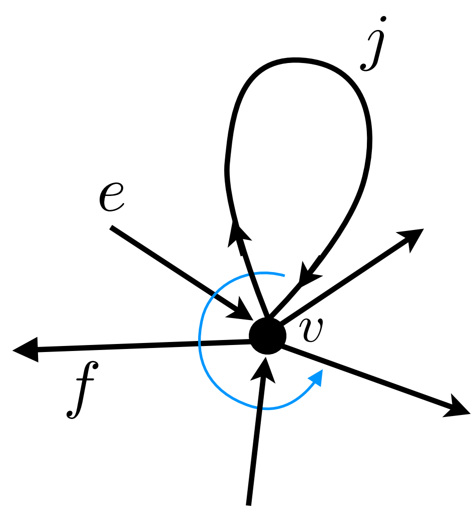

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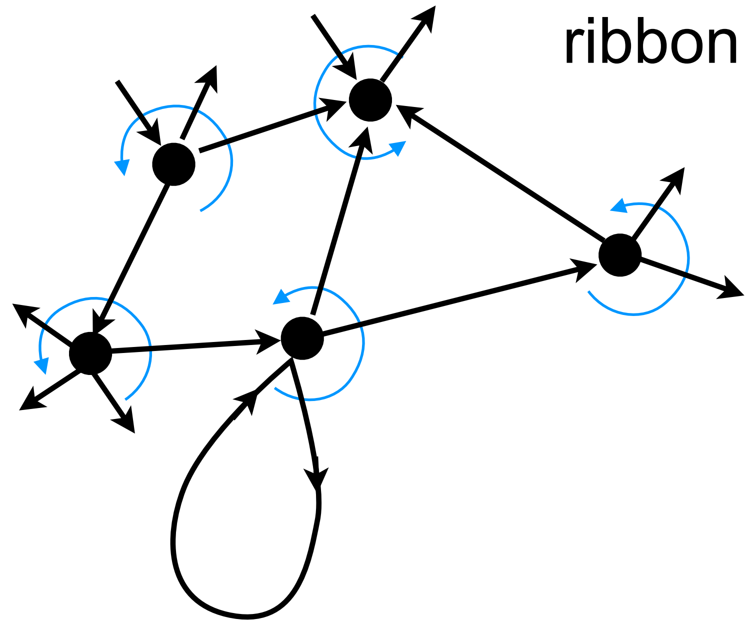

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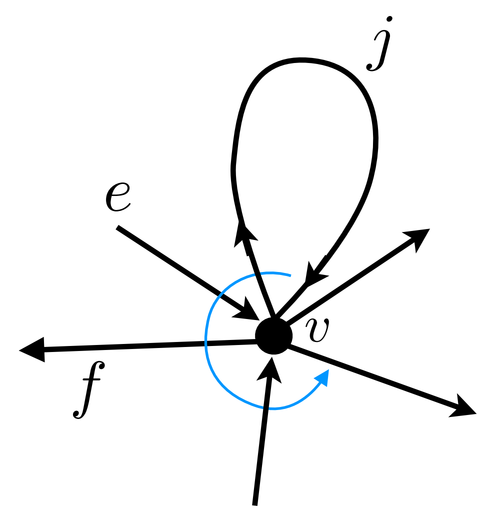

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invariants $\alpha \in K^{*\otimes E}$

$\alpha \lhd h = \epsilon(h)\, \alpha \quad \forall h \in K^{\otimes V}$

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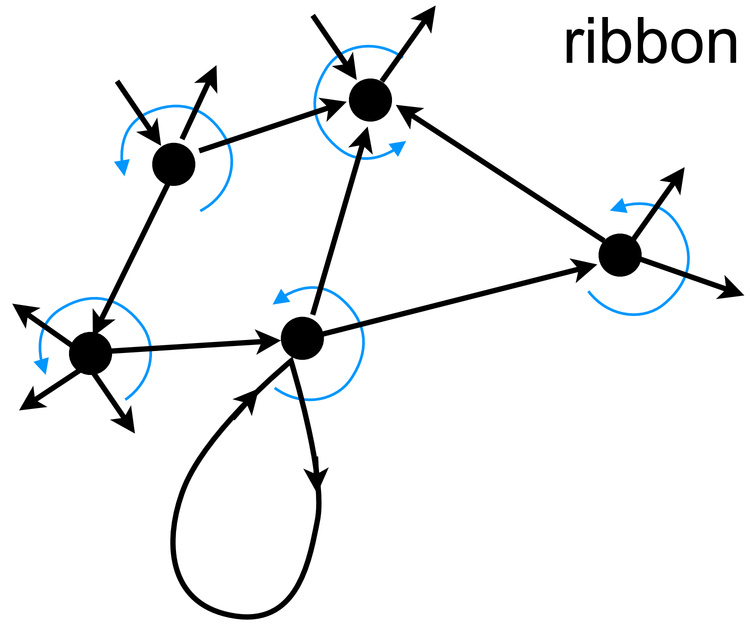

gauge fields **set** $G^{\times E}$

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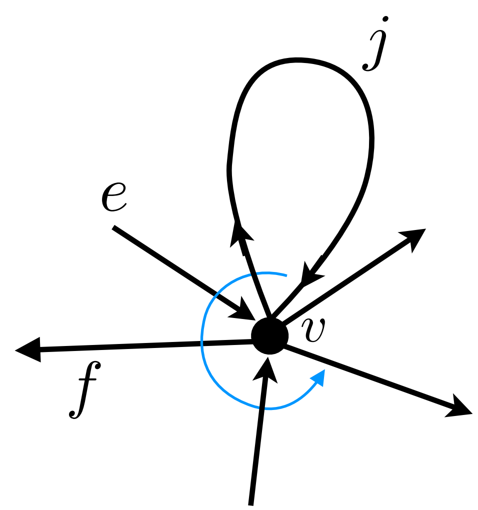

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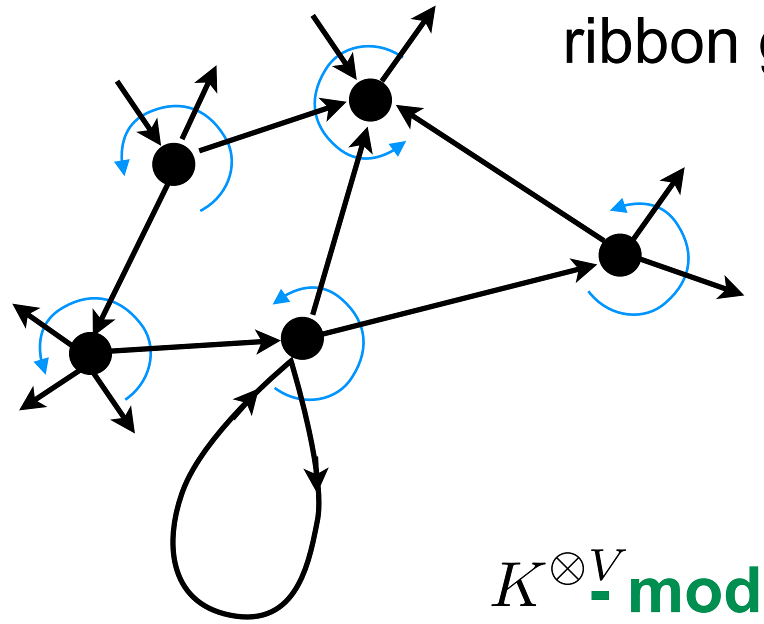
1. Hopf algebra gauge theory

lattice gauge theory for a group

ribbon graph Γ + group G



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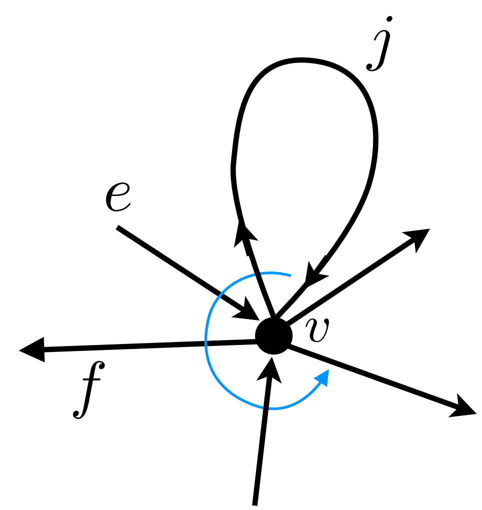

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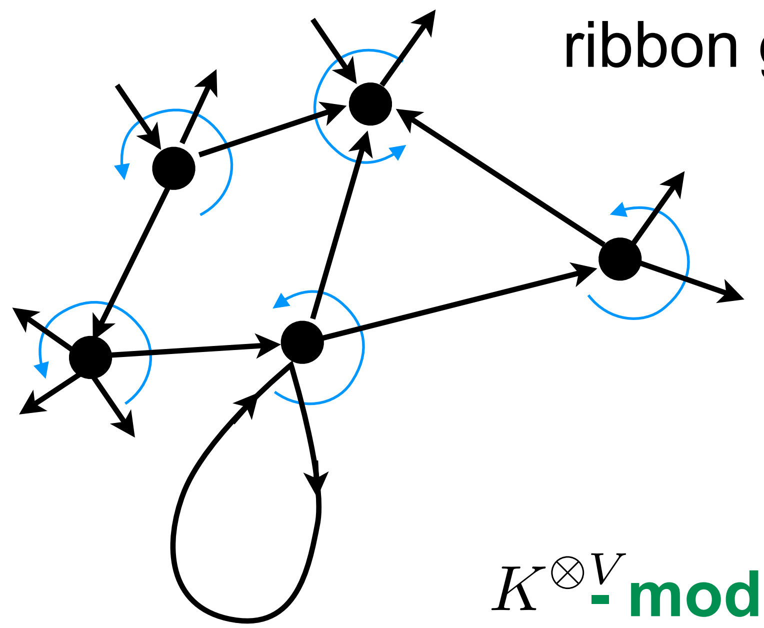

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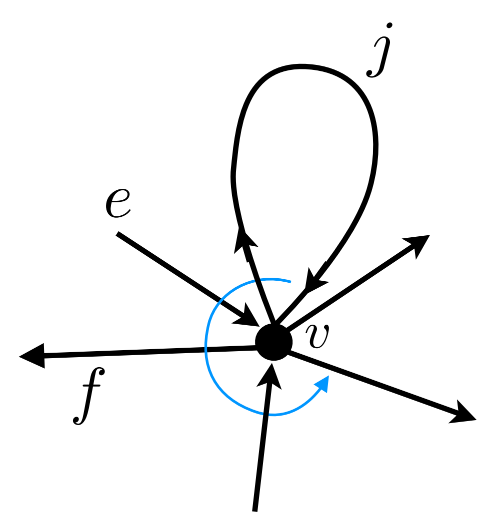

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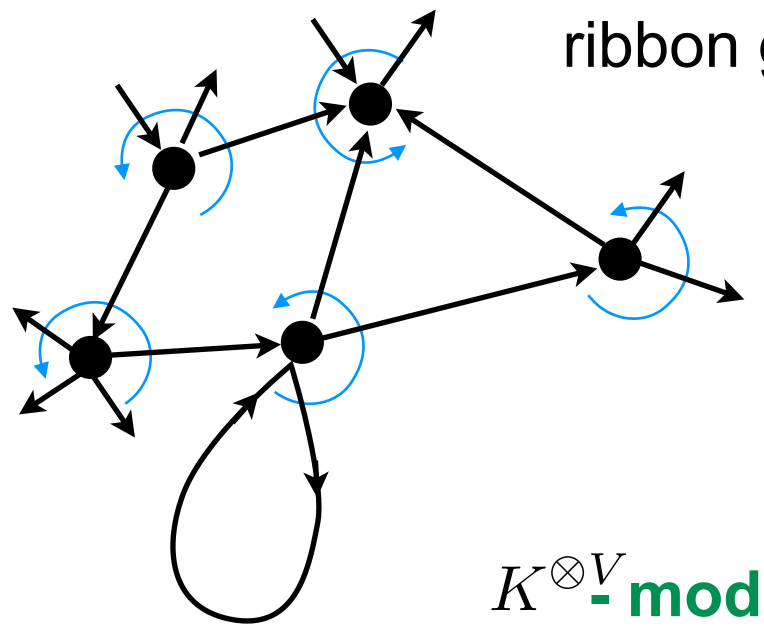

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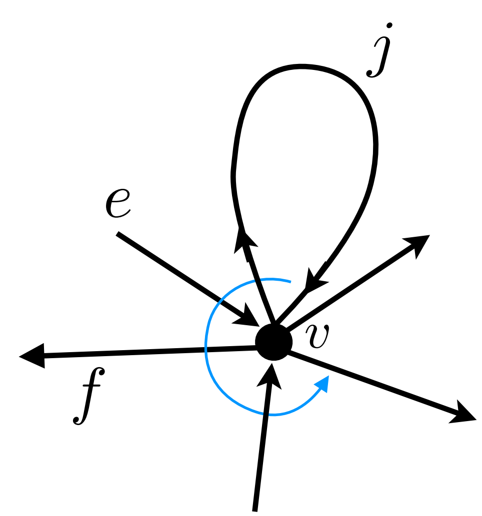

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= module algebra over a Hopf algebra

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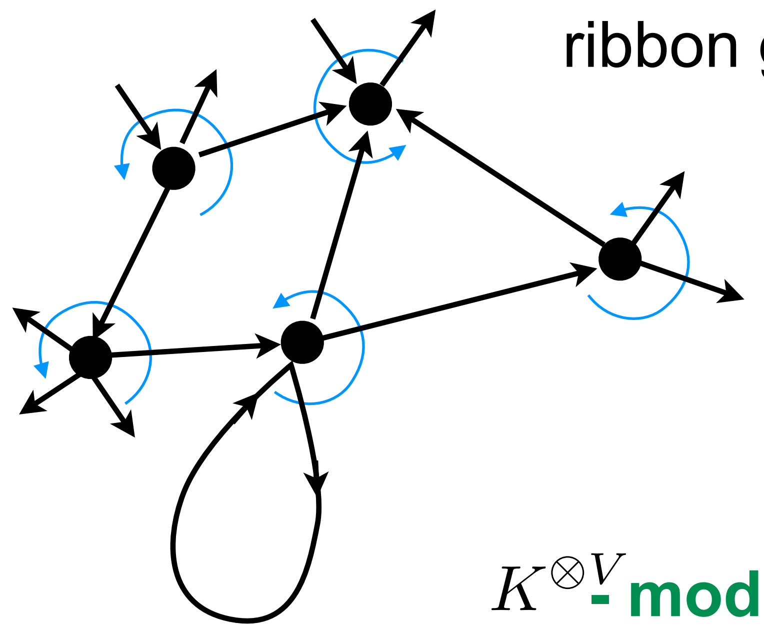

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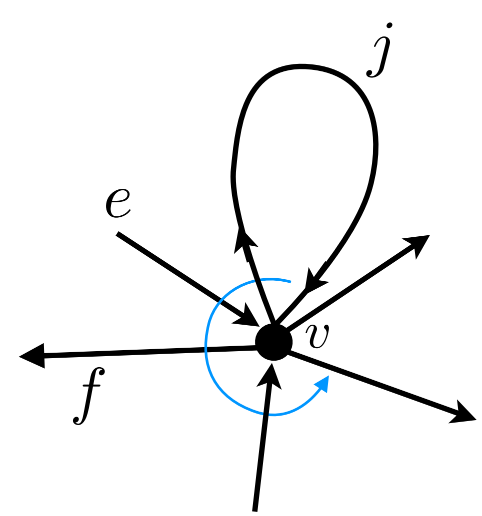

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Locality condition

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- cut ribbon graph in vertex discs

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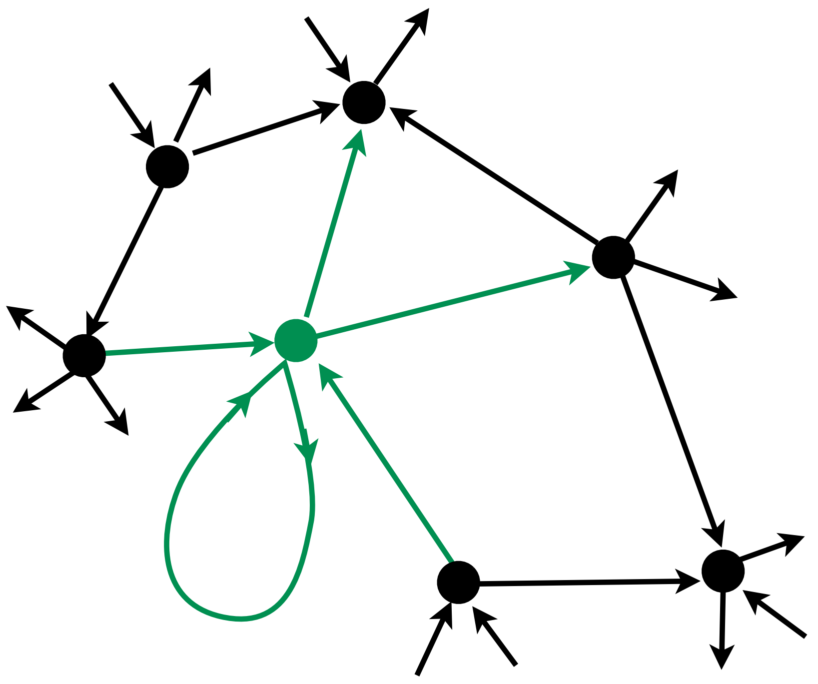

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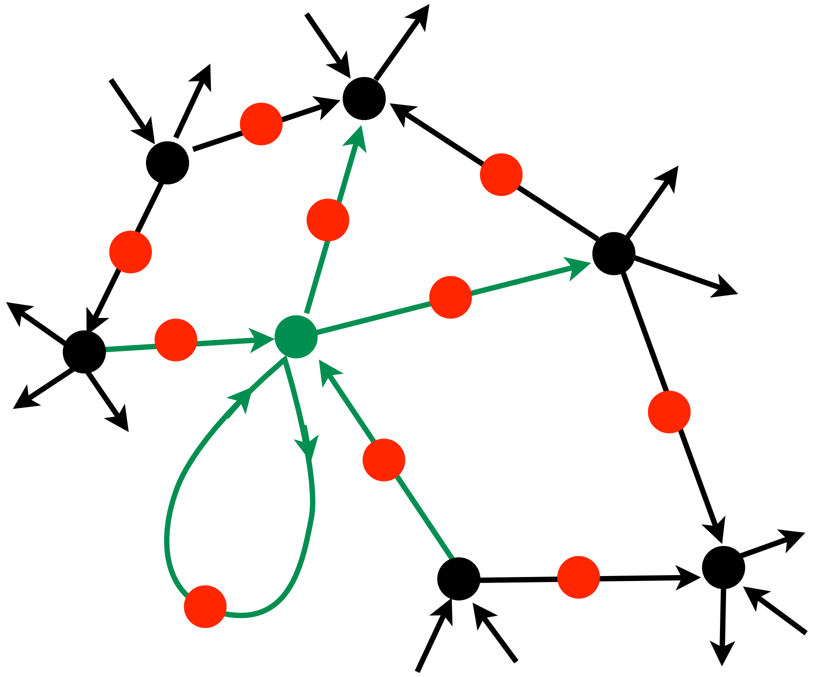

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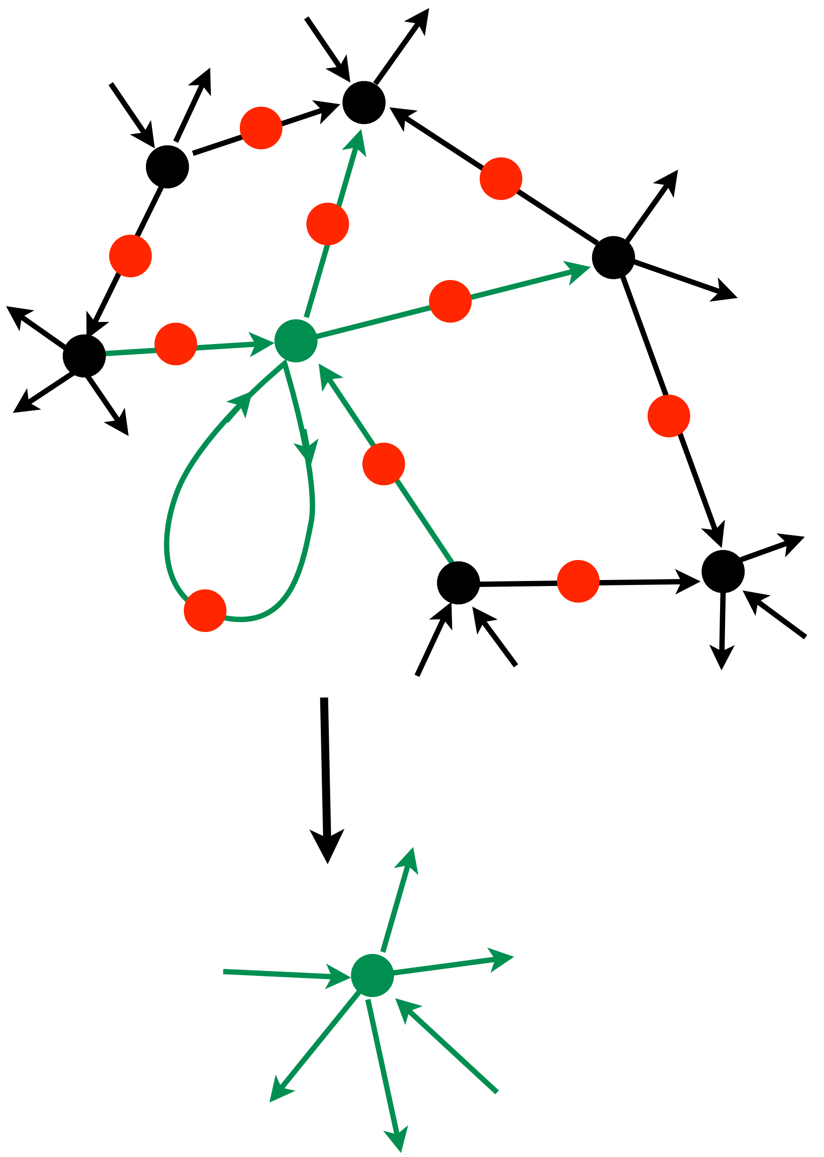

Locality condition

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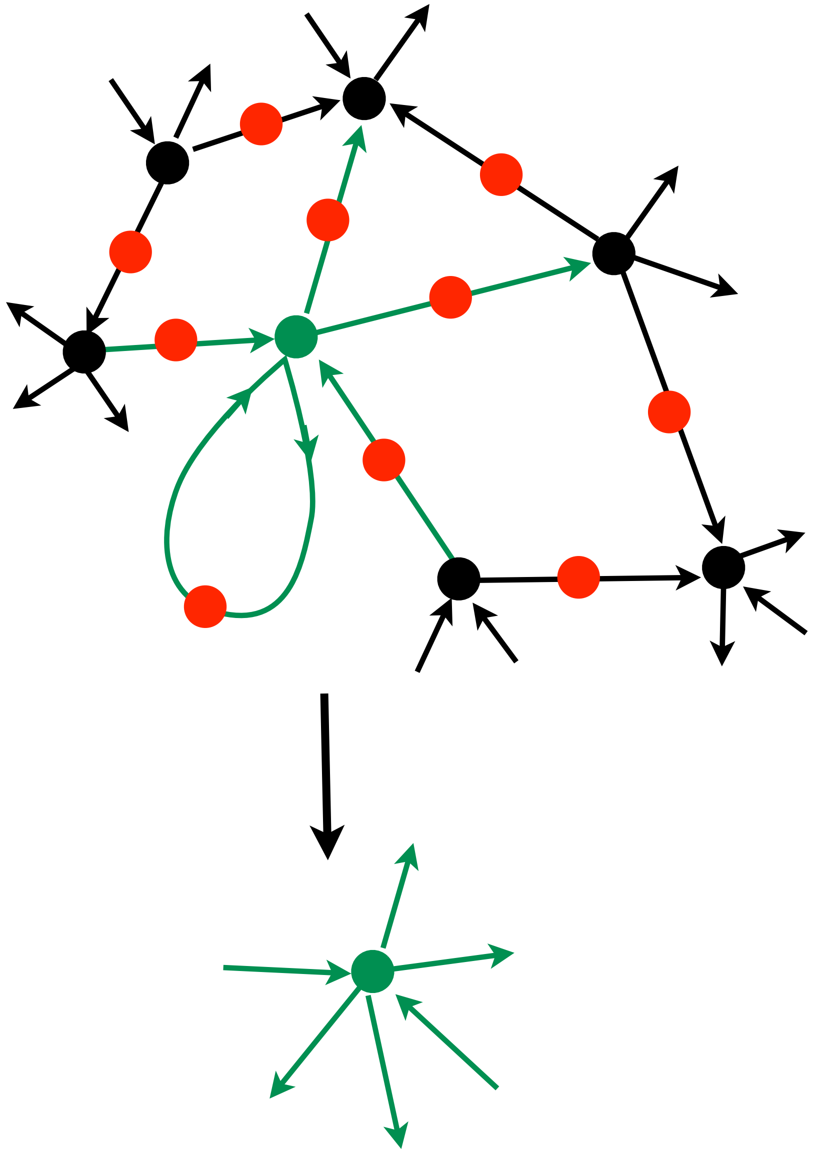
- condition: $K^{\otimes V}$- module algebra structure on $K^{*\otimes E}$ induced by K -module algebra stuctures on vertex discs



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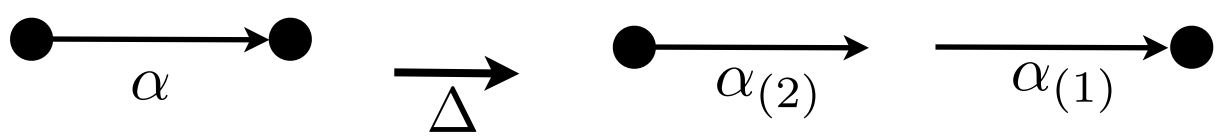

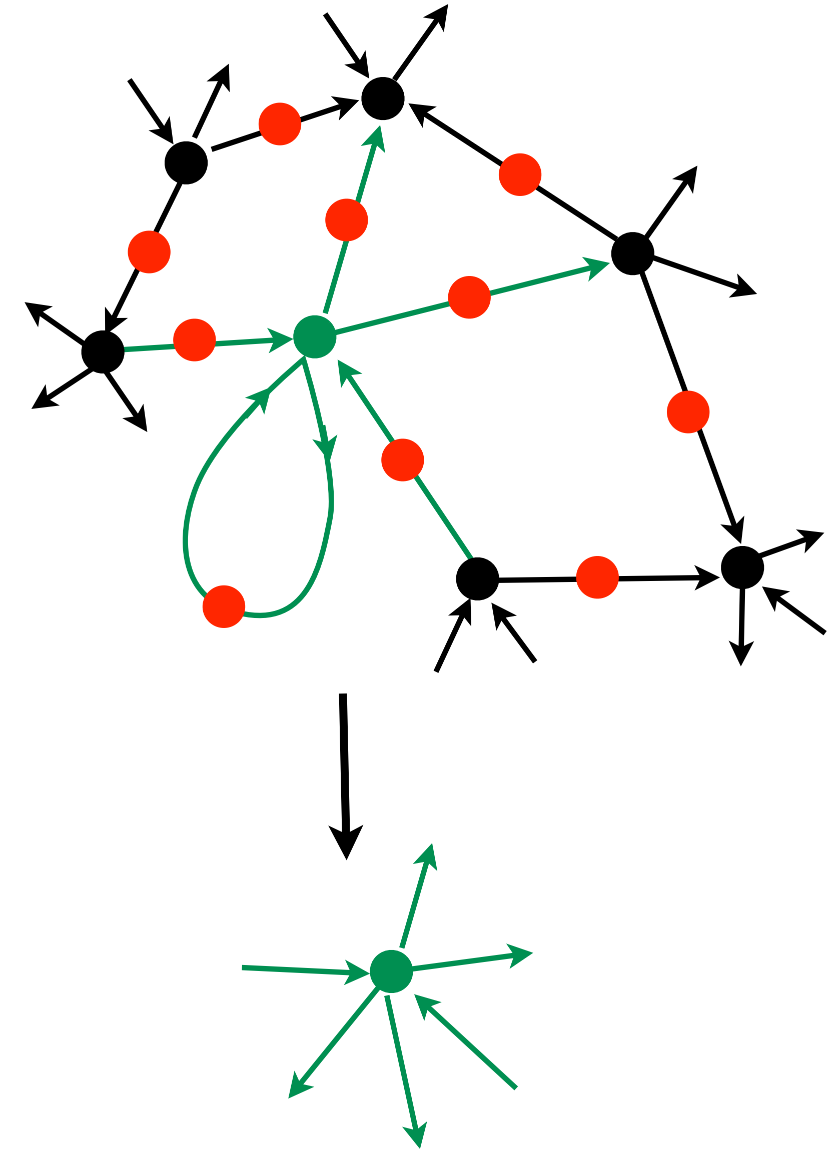

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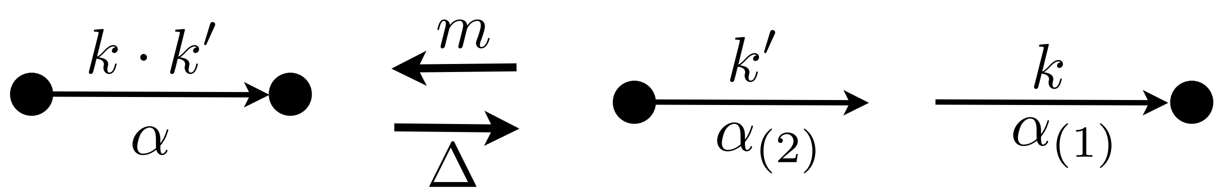

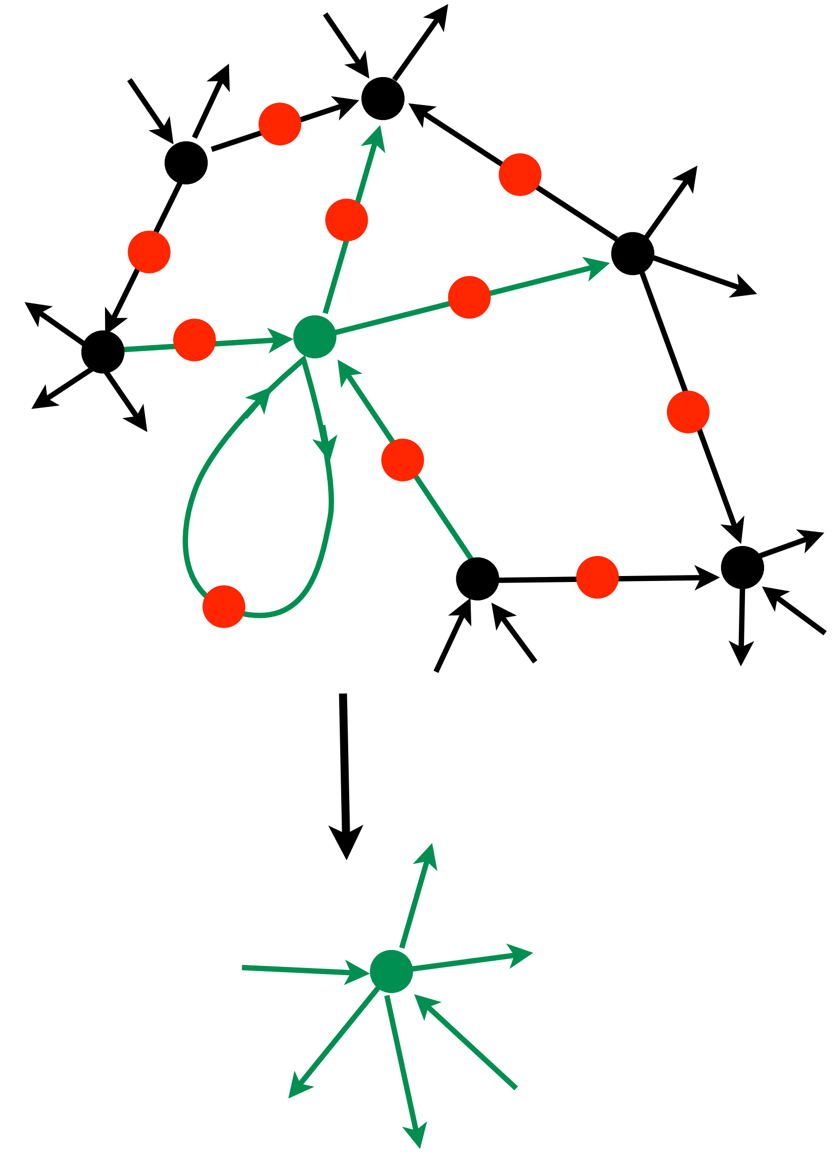

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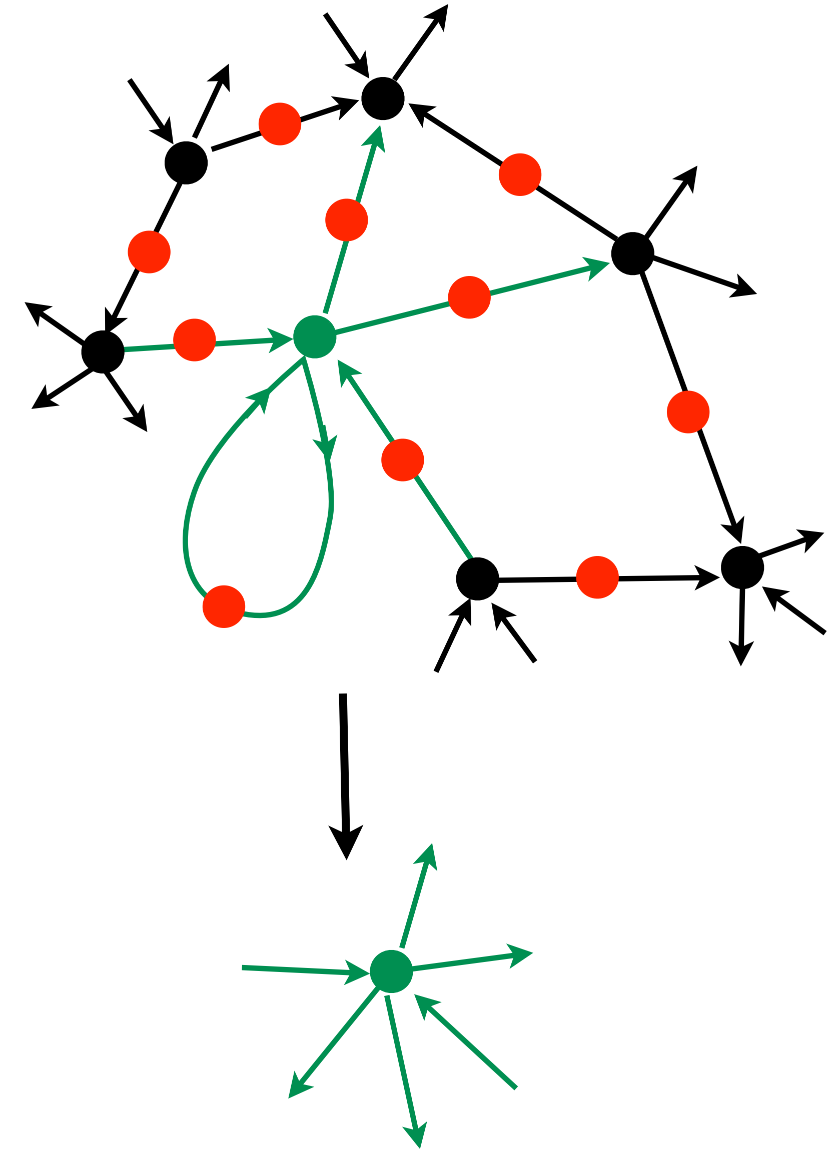

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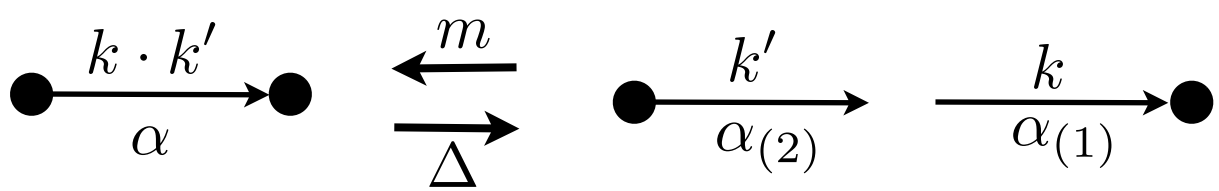
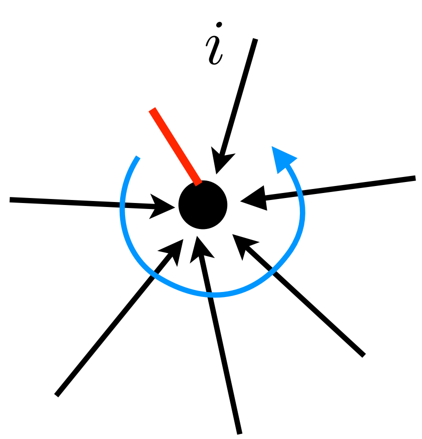


Hopf algebra gauge theory on a vertex disc



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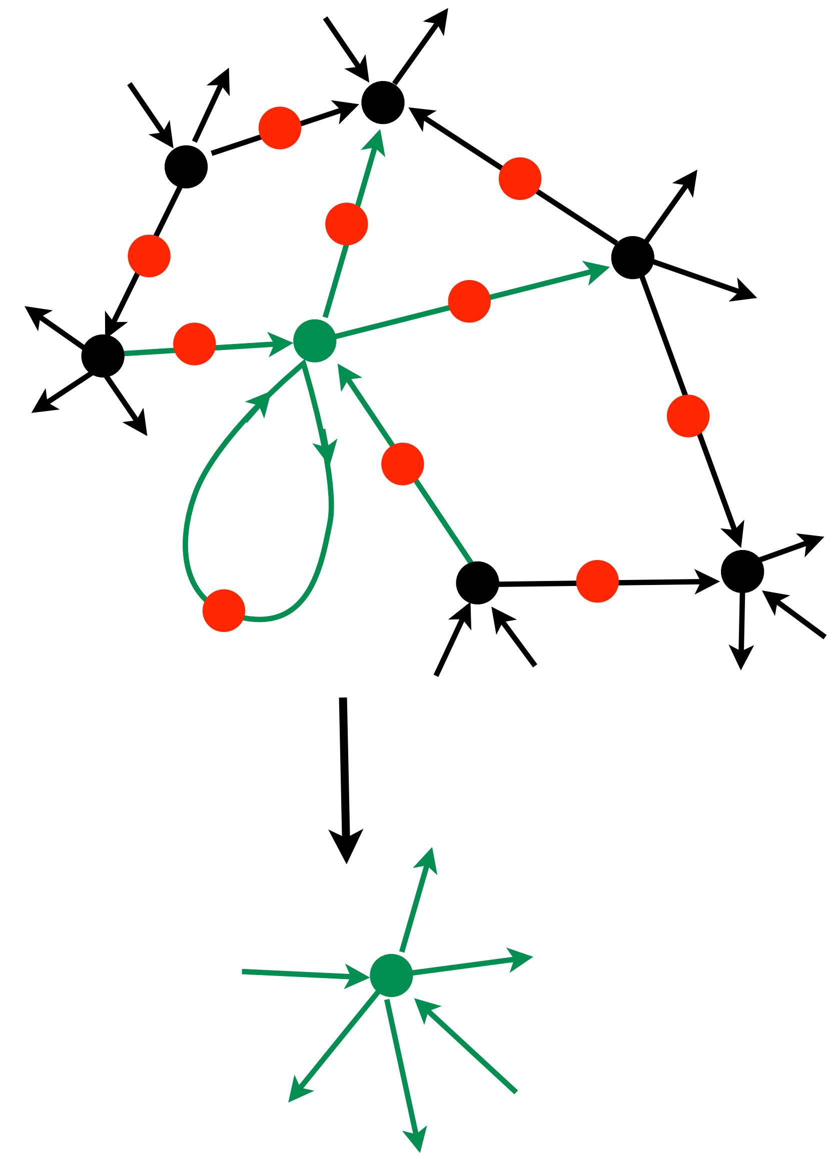

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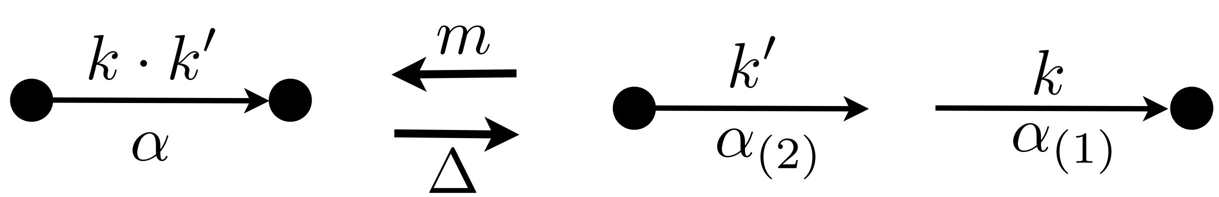

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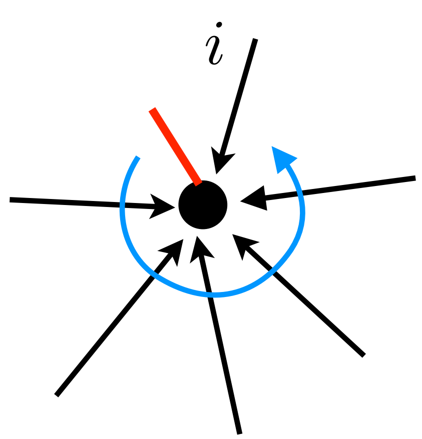

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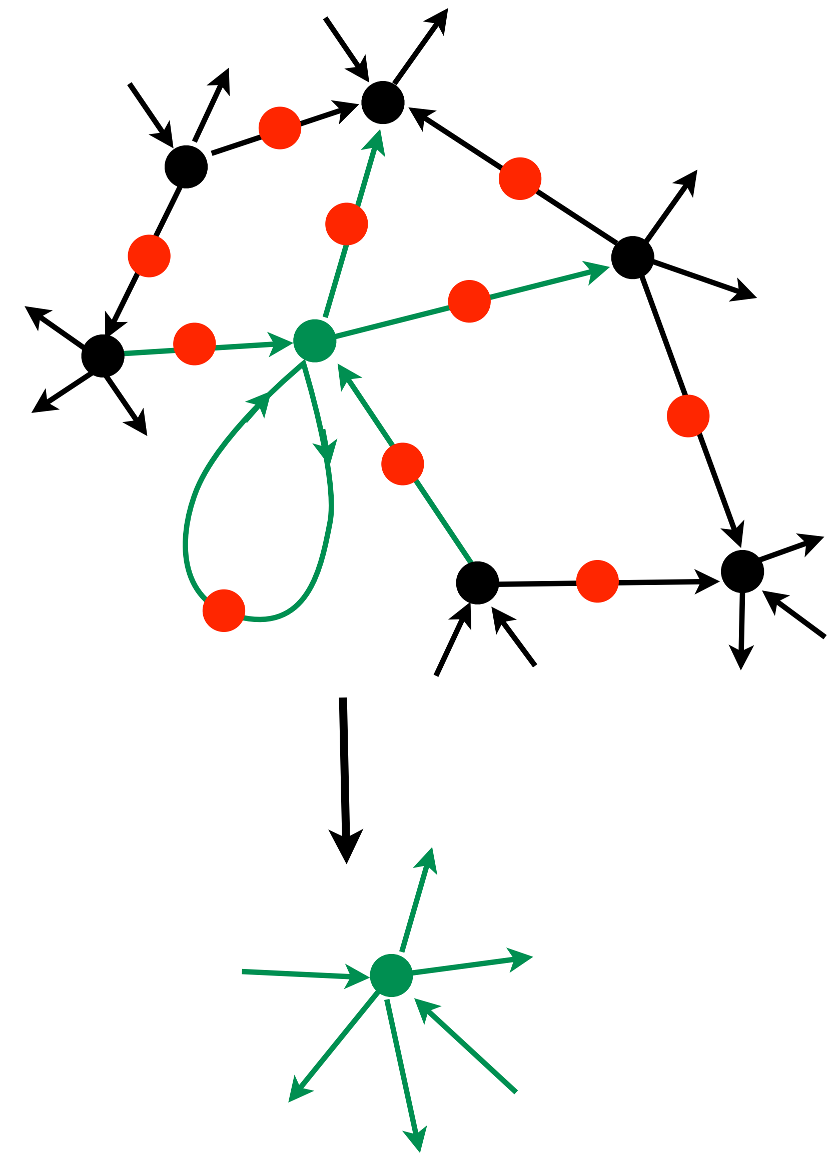

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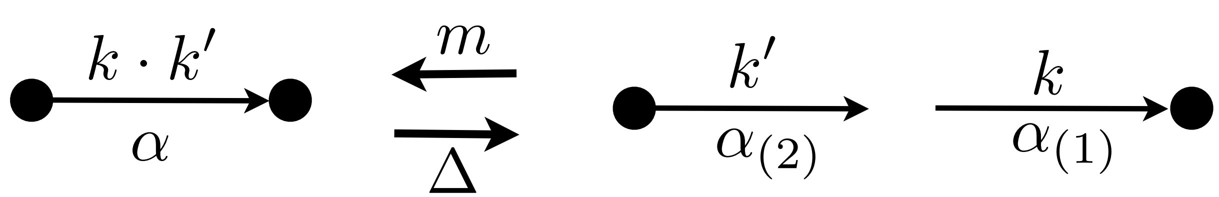

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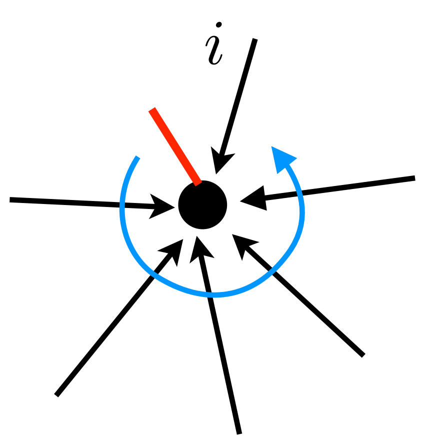

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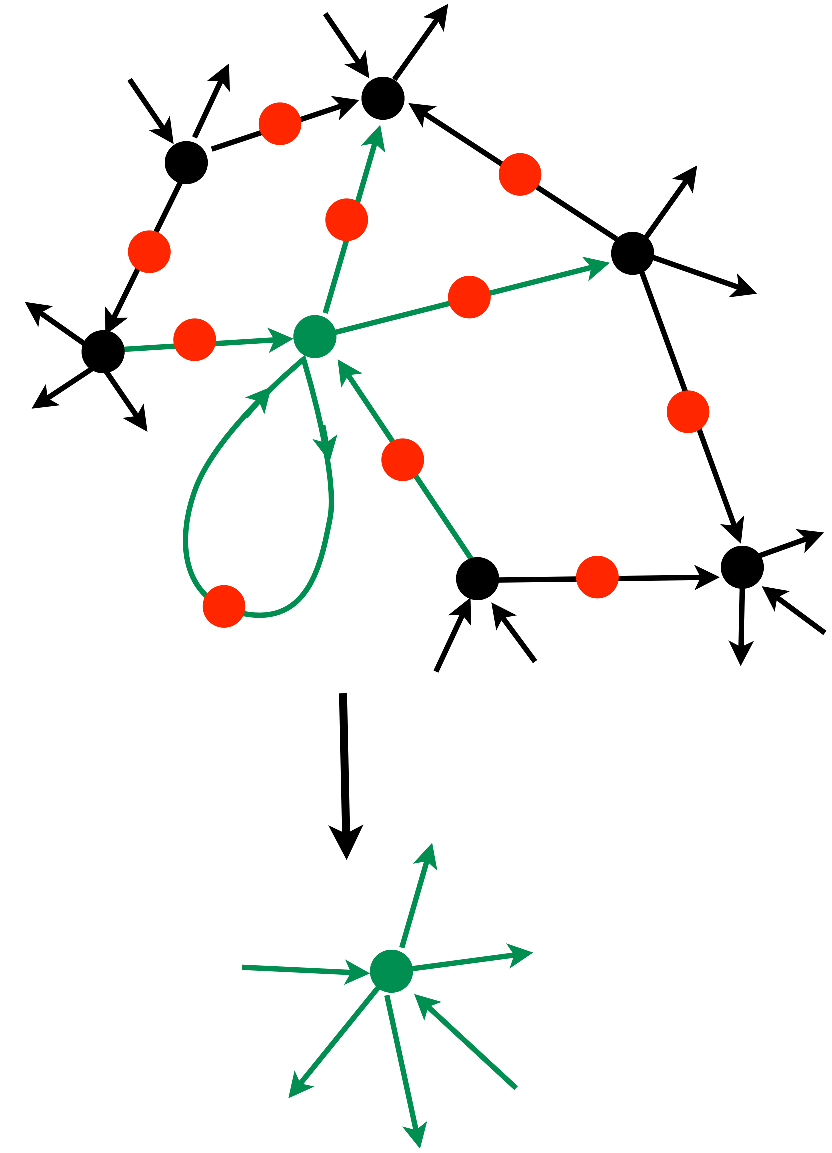

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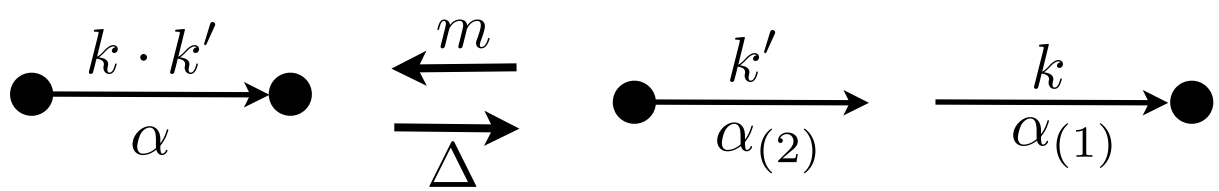

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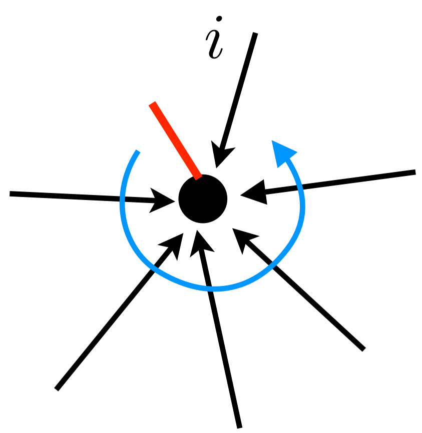

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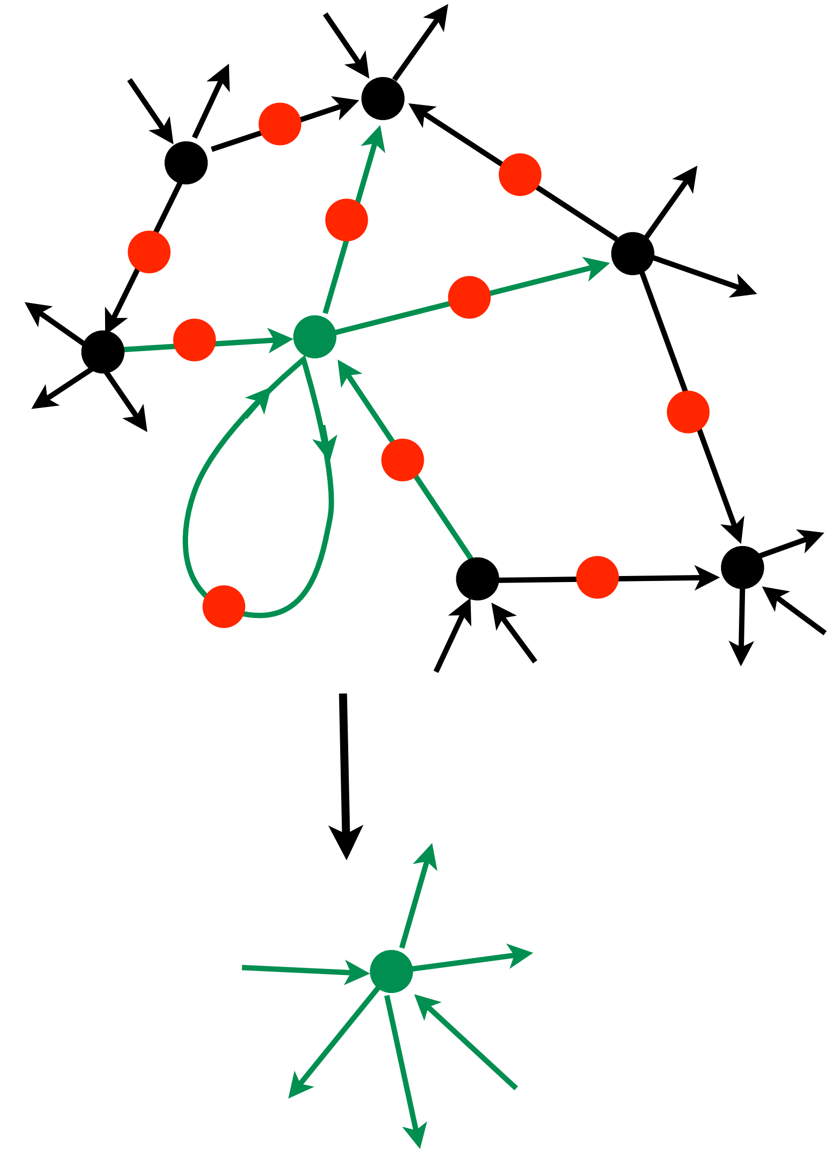

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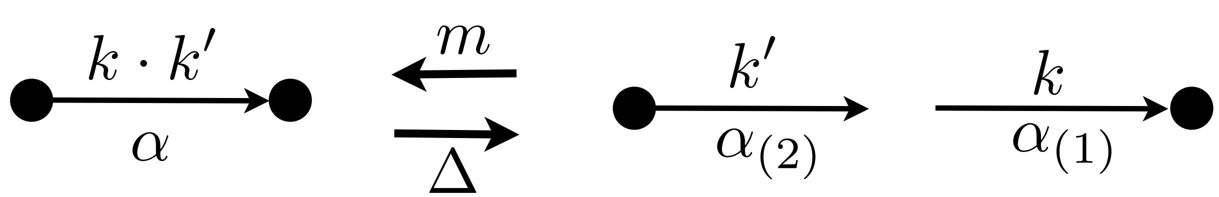

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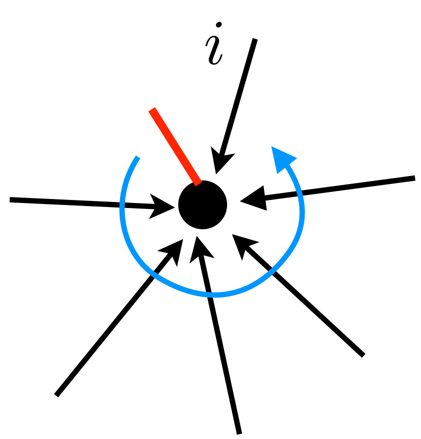

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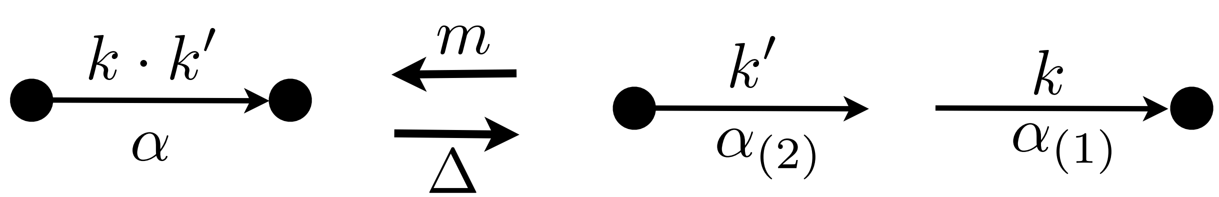

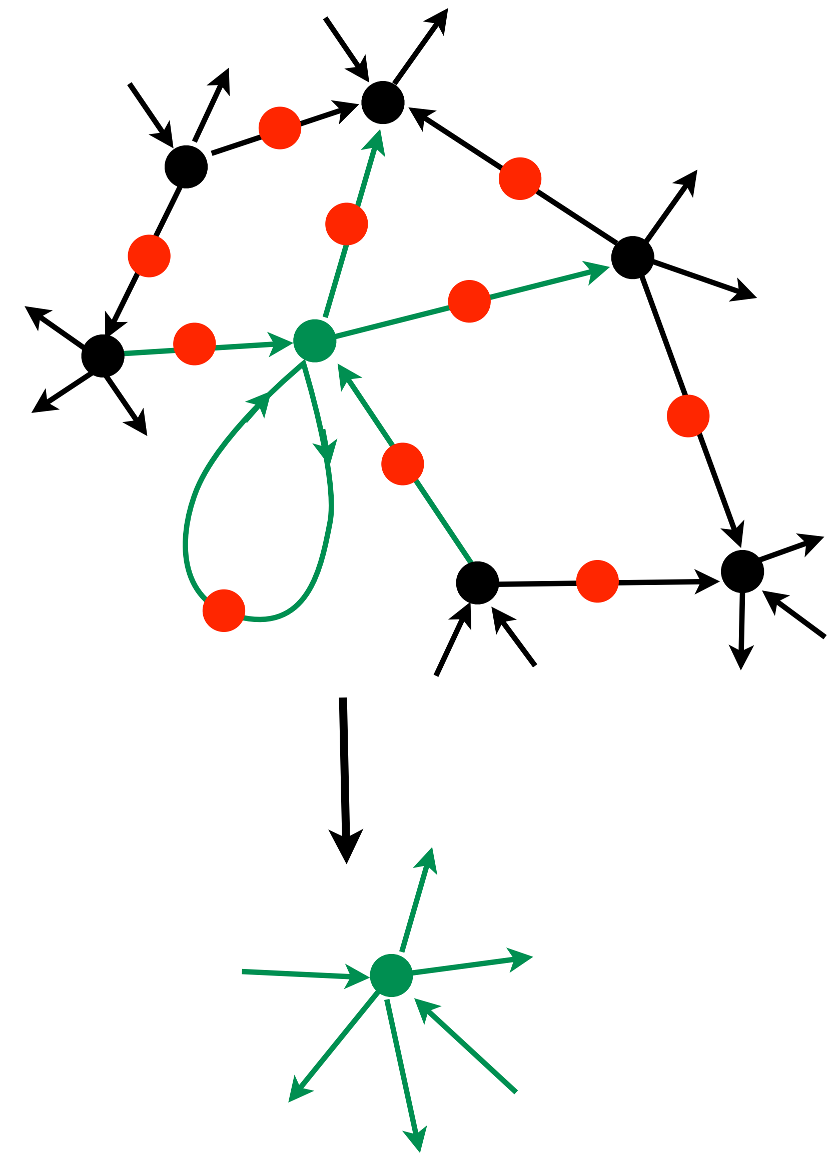

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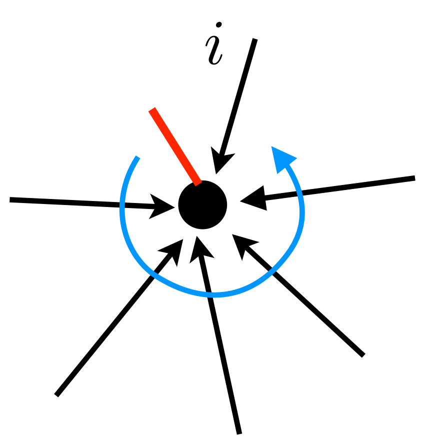

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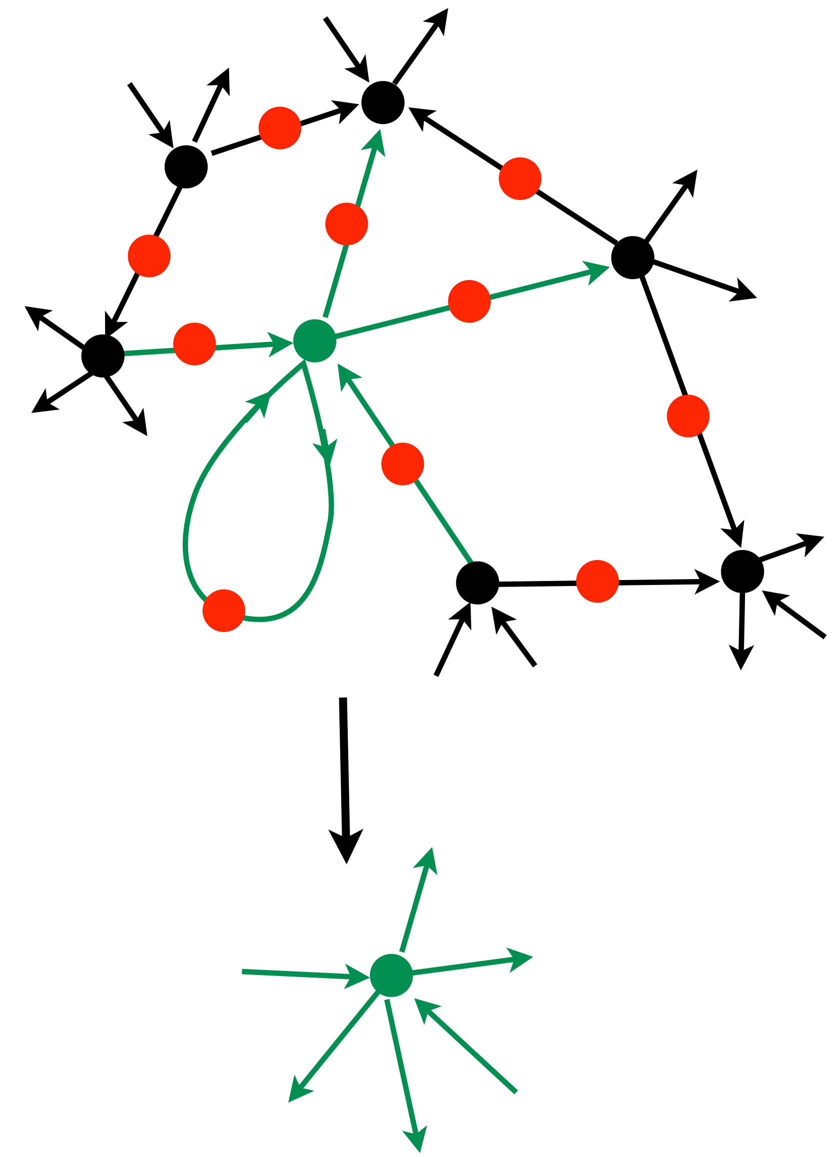

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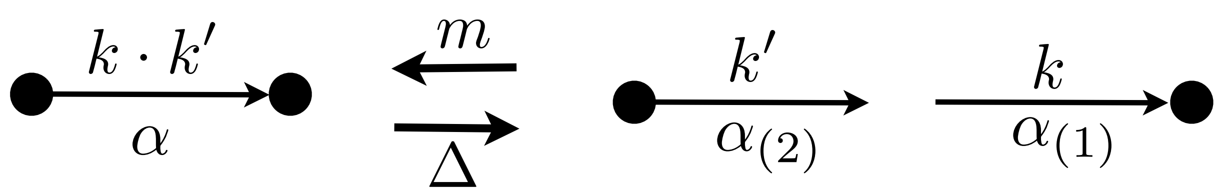

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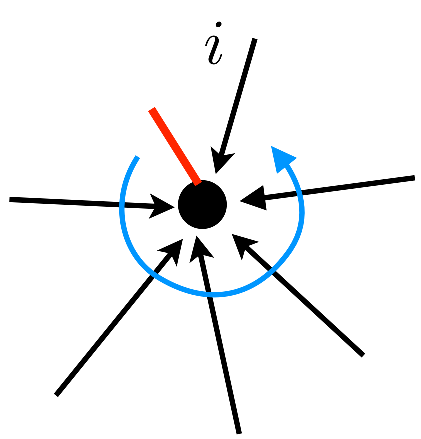

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⇨ require struture to relate $\Delta(h) = h_{(1)}\otimes h_{(2)}$ and $\Delta^{op}(h) = h_{(2)}\otimes h_{(1)}$

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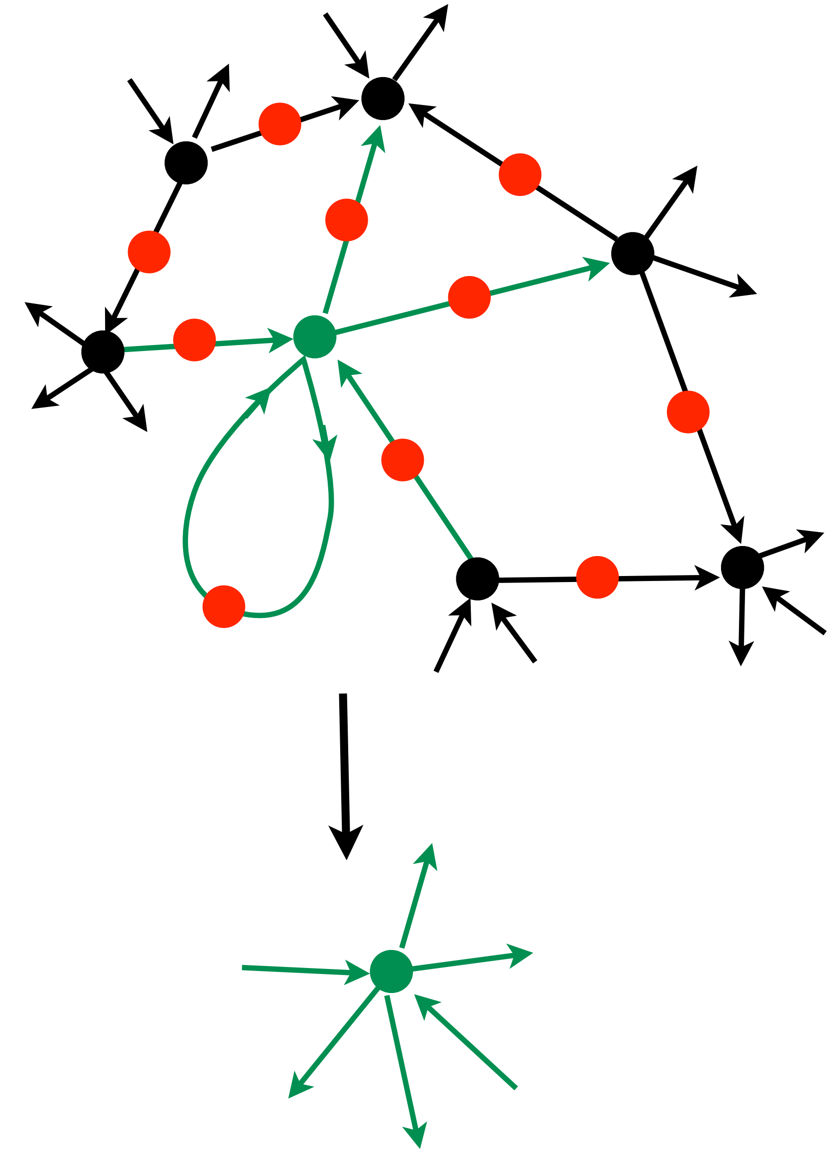

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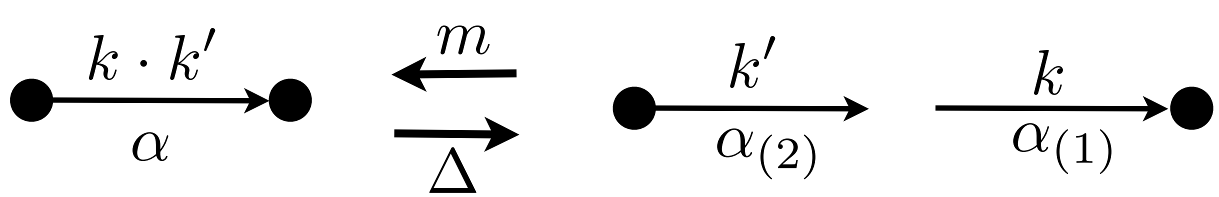
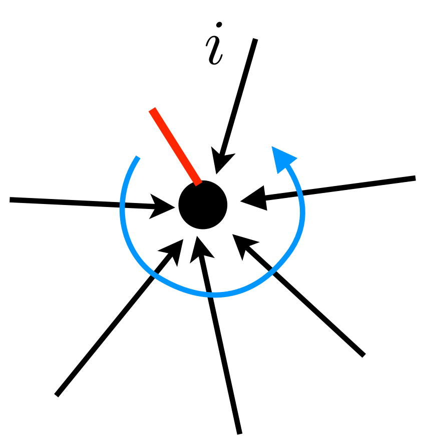


Hopf algebra gauge theory on a vertex disc



- K -module algebra structure on $K^{*\otimes |v|}$
- condition: $(\alpha)_i \lhd h = \langle \alpha_{(1)}, h\rangle\, \alpha_{(2)}$

- naively: tensor product $K^{*\otimes|v|}$ ⇨ works only for K cocommutative

$$(\alpha\otimes\beta)\lhd h = ((\alpha\otimes 1)\cdot(1\otimes\beta))\lhd h = (\alpha\lhd h_{(1)})\otimes(\beta\lhd h_{(2)})$$
$$= ((1\otimes\beta)\cdot(\alpha\otimes 1))\lhd h = (\alpha\lhd h_{(2)})\otimes(\beta\lhd h_{(1)})$$

⇨ require struture to relate $\Delta(h) = h_{(1)}\otimes h_{(2)}$ and $\Delta^{op}(h) = h_{(2)}\otimes h_{(1)}$

- K quasitriangular: $\Delta^{op} = R\cdot\Delta\cdot R^{-1}$ $\quad(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}\quad(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$

Hopf algebra gauge theory on a vertex disc

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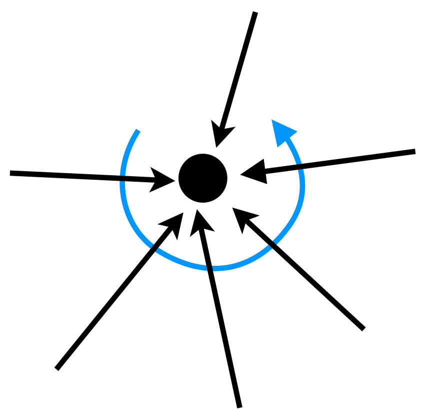

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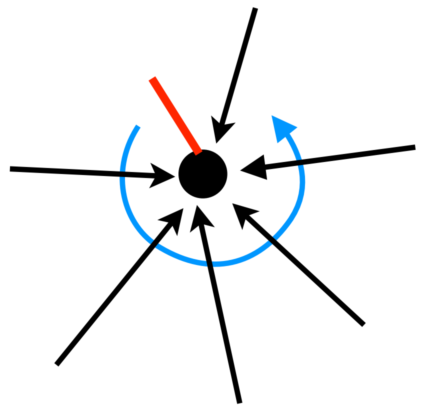

ordering of edge ends at v

Hopf algebra gauge theory on a vertex disc

$$(\alpha)_i \cdot (\beta)_j = (\alpha \otimes \beta)_{ij} \qquad i < j$$

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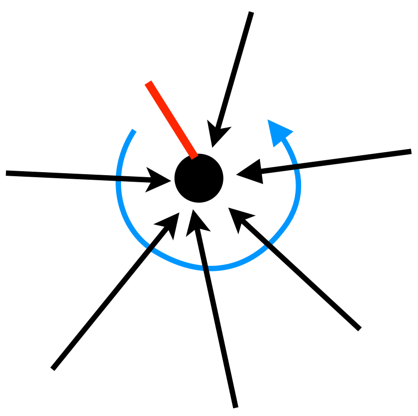

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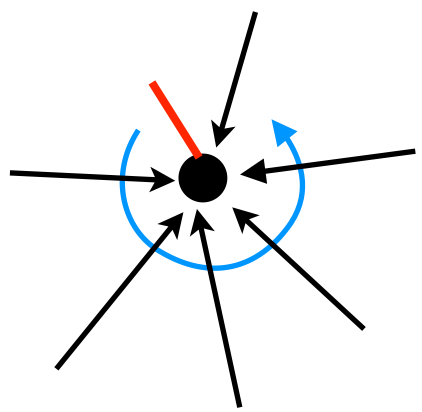

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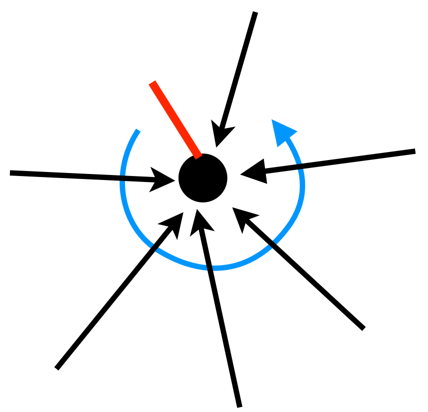

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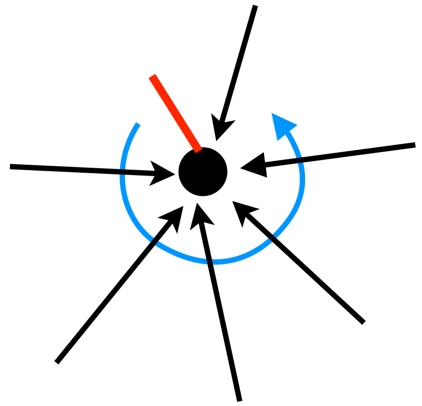

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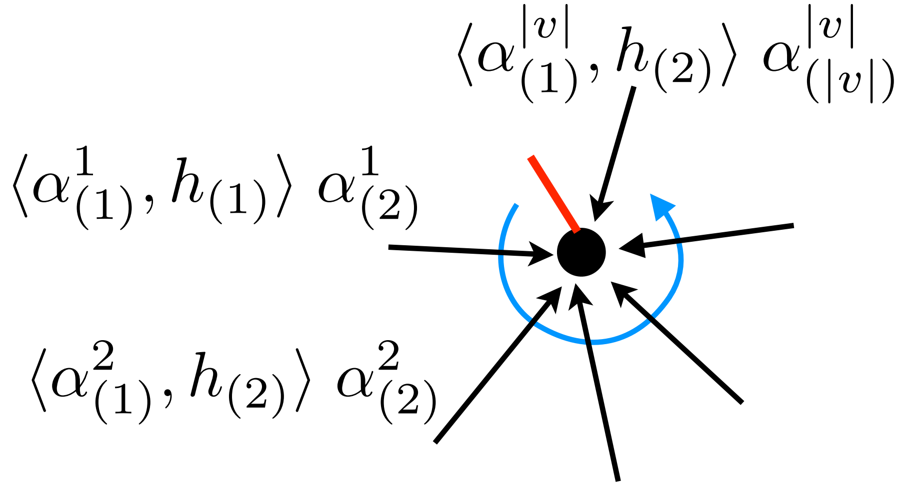

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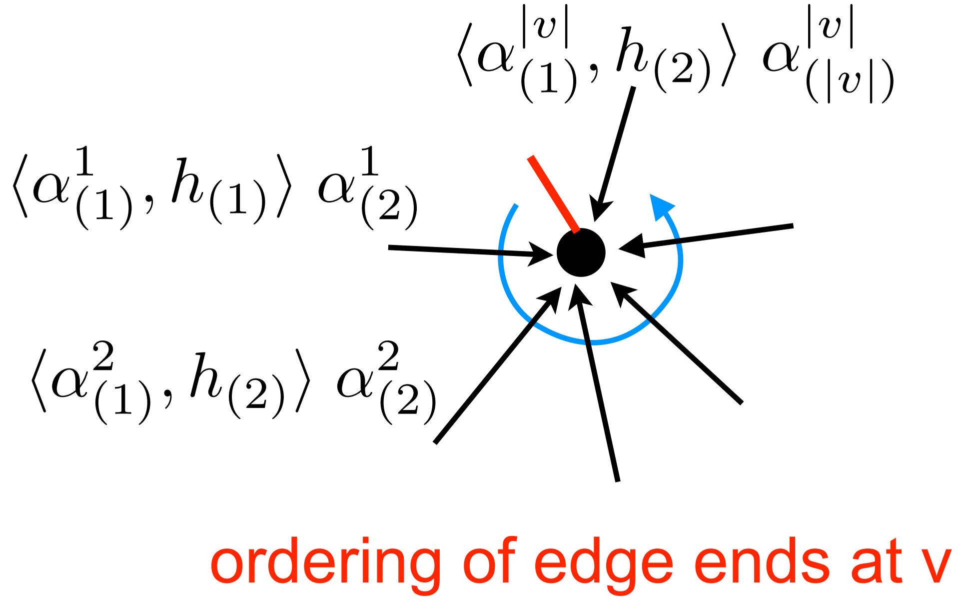

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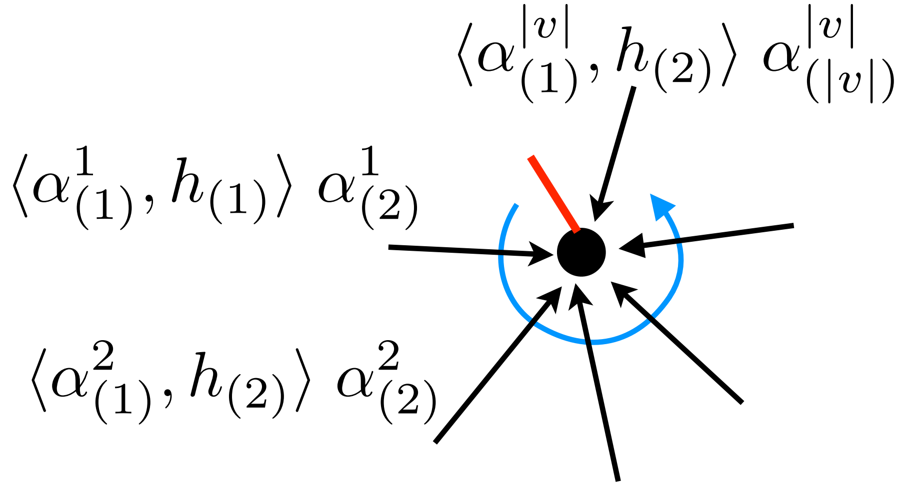

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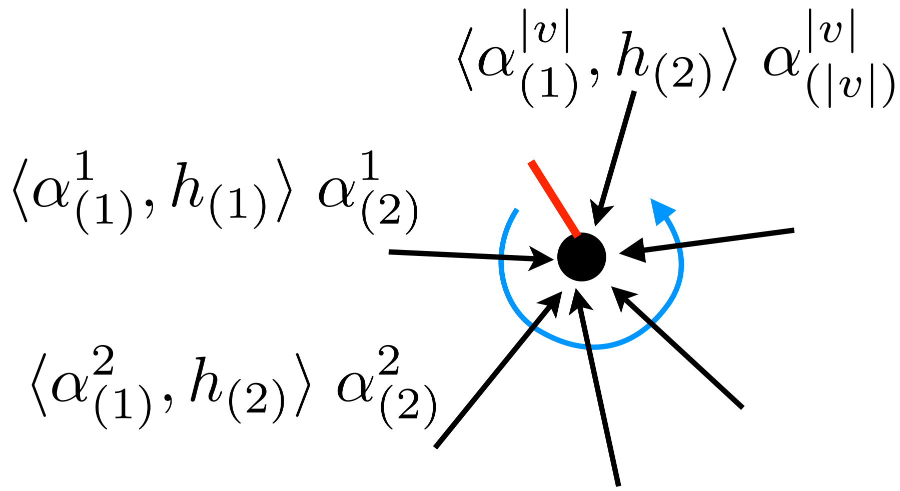

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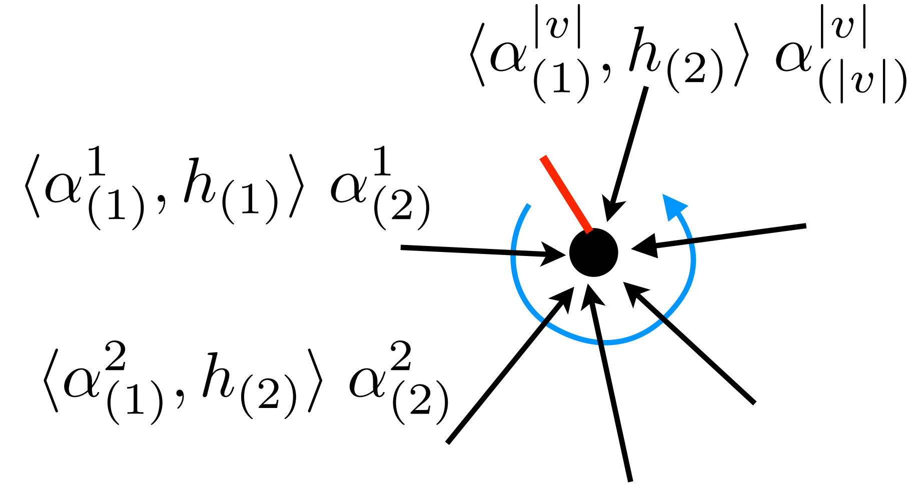

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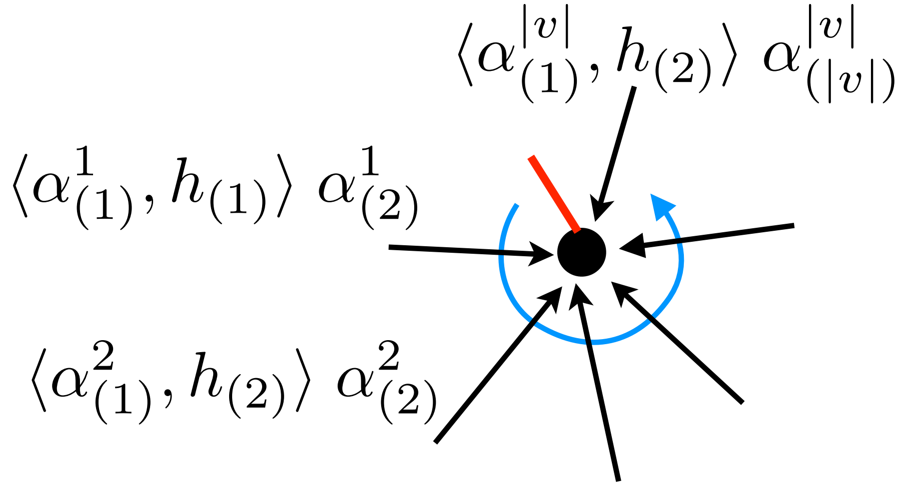

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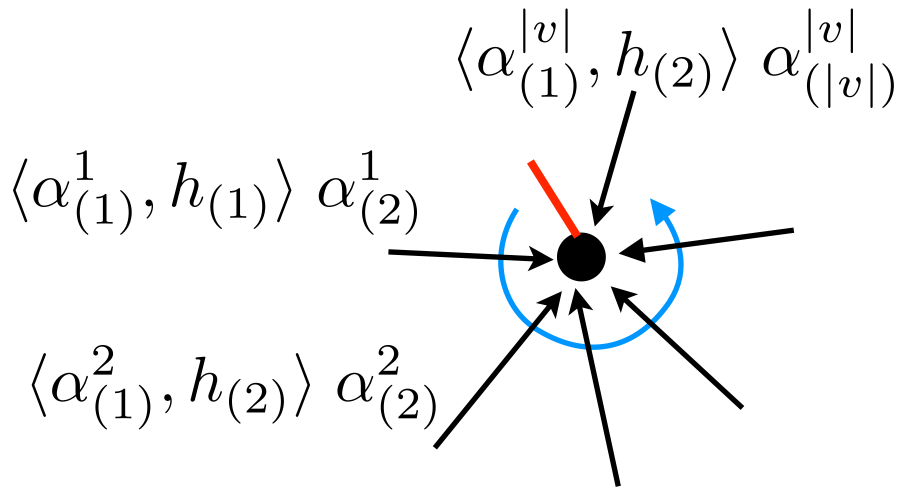

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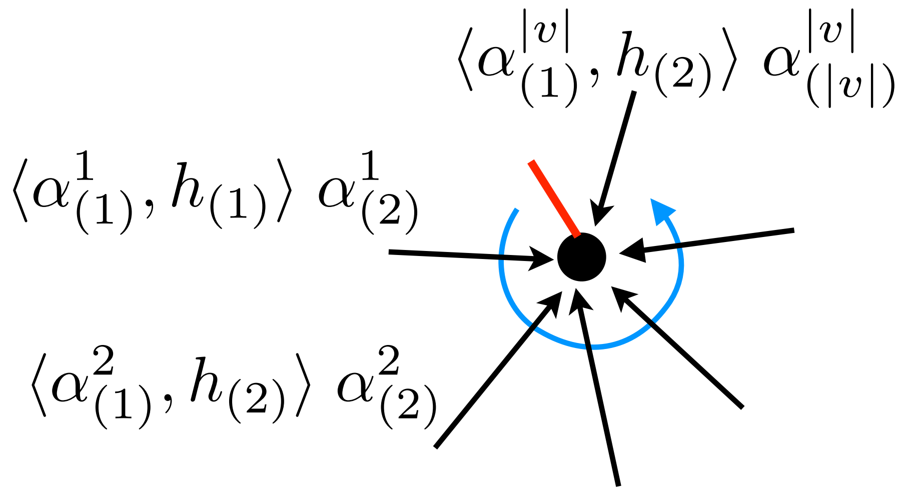

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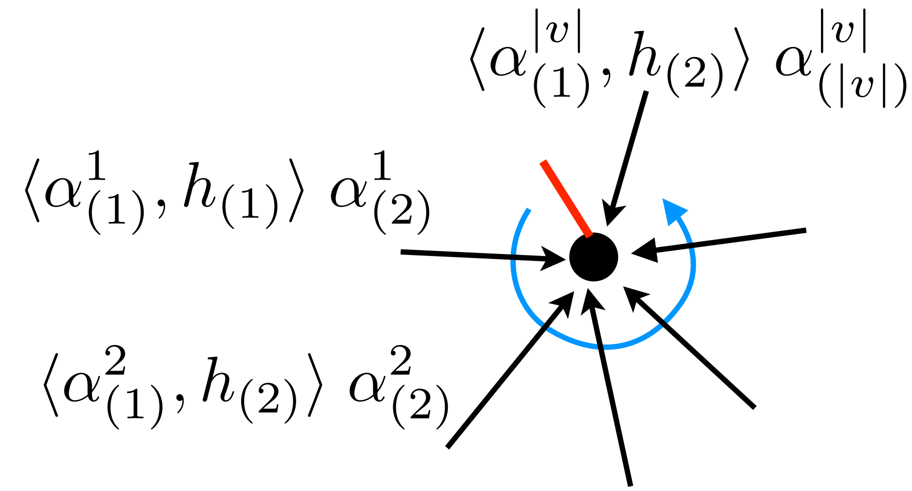

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⇨ vertex disc with arbitrary edge orientation:
require that $T^* : K^* \to K^*$ is algebra and module isomorphism

Hopf algebra gauge theory on a ribbon graph

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- select one of the ends for each edge $e \in E(\Gamma)$ → sets $I_v \subset \{1, ..., |v|\}$

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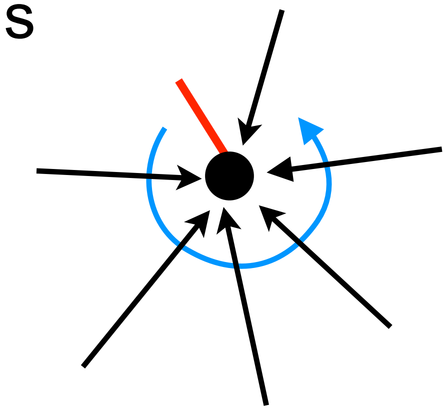

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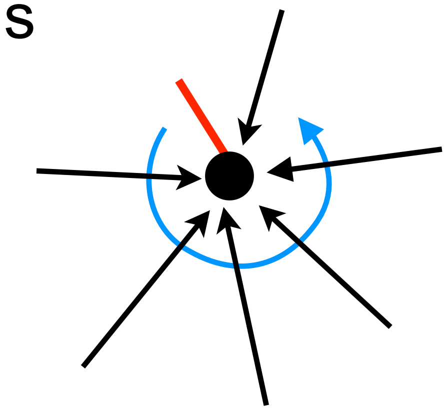

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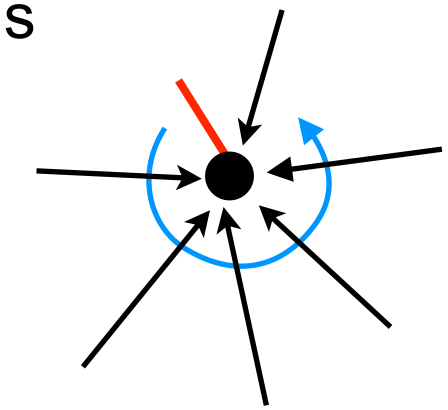

⇨ Hopf algebra gauge theory on each vertex disc:
K - module algebra $\mathcal{A}_v^*$

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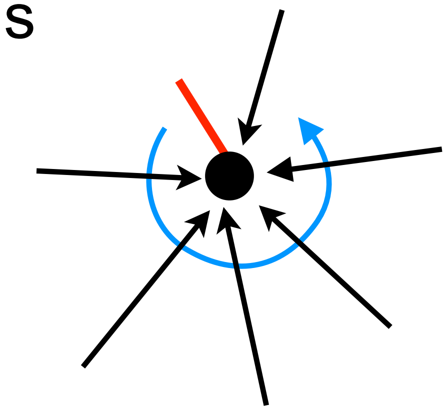

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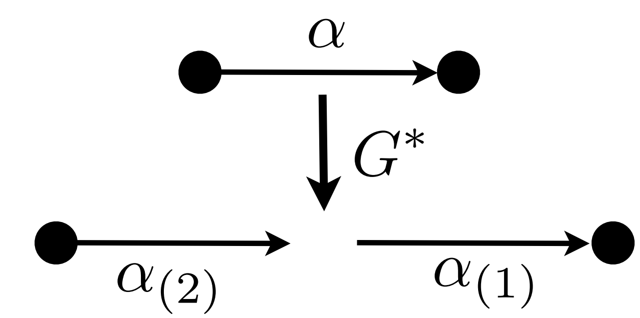

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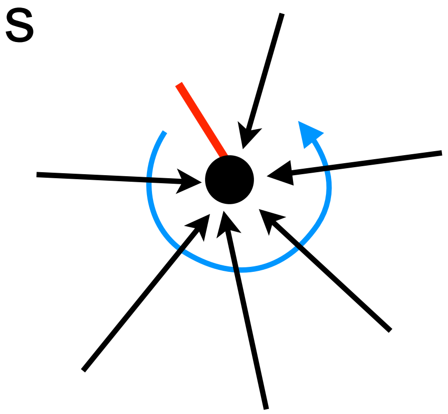

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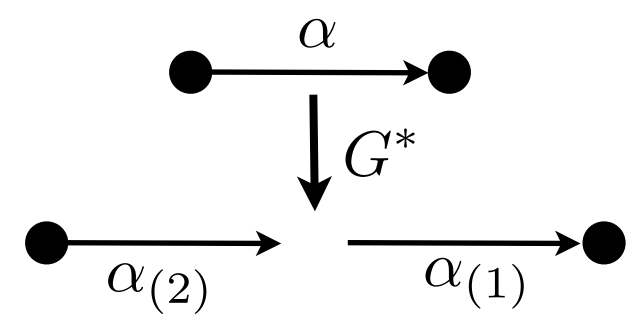

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⇨ Hopf algebra gauge theory on ribbon graph: [C.M.]

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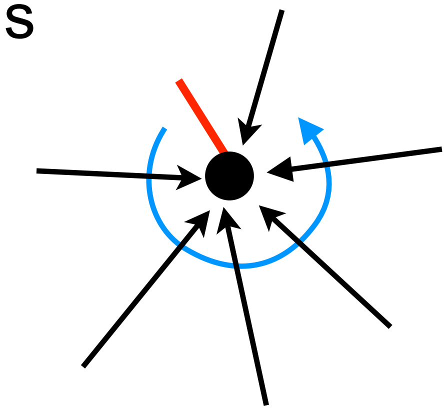

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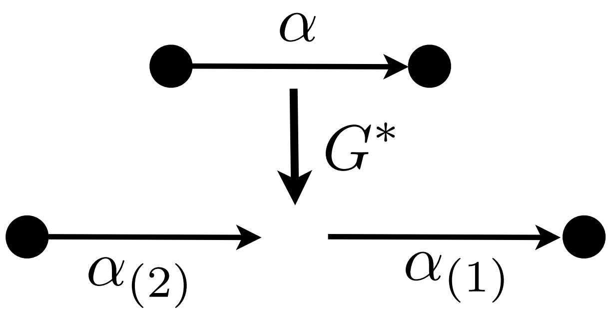

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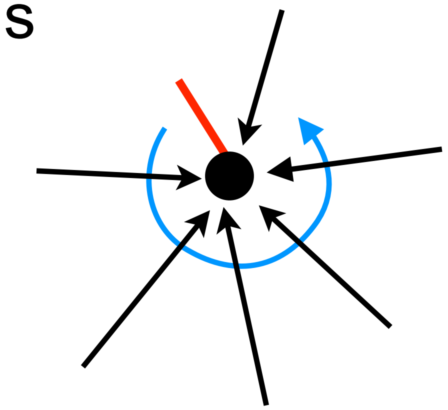

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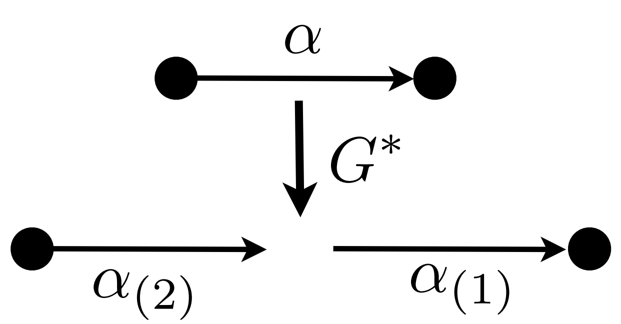

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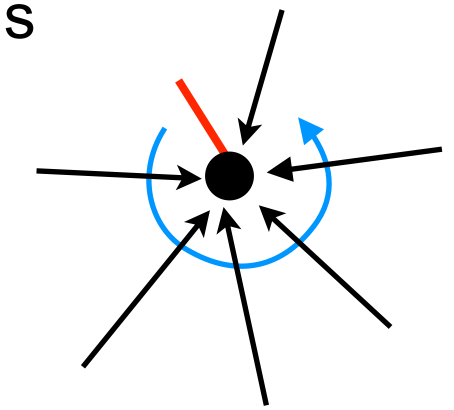

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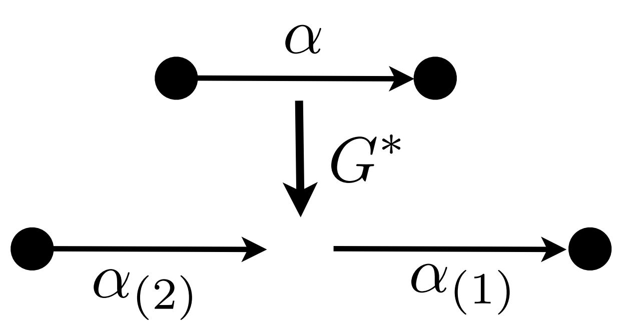

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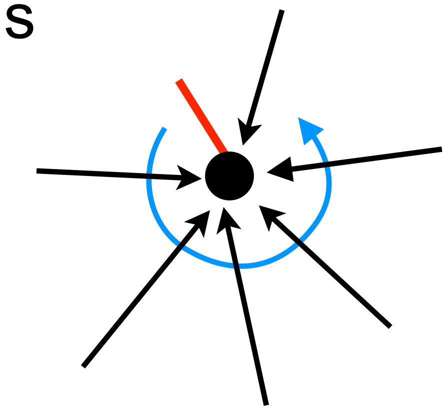

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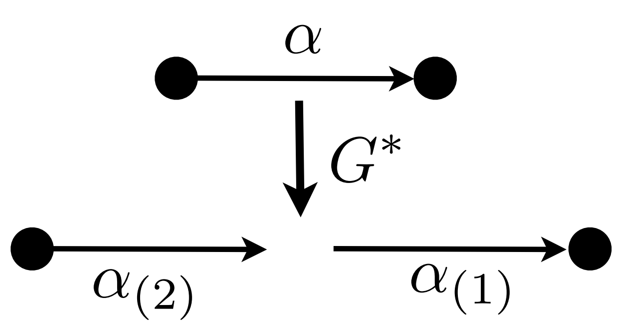

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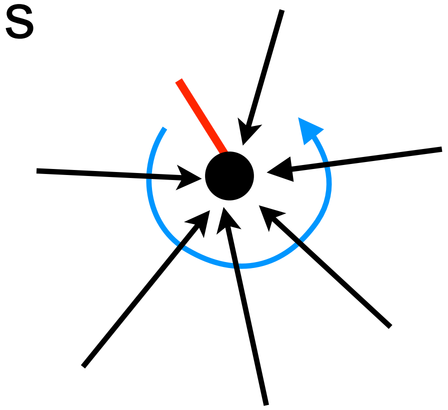

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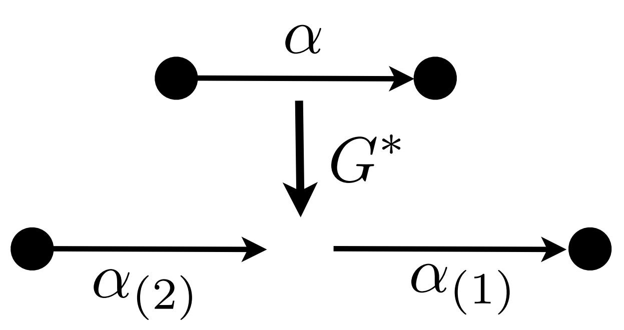

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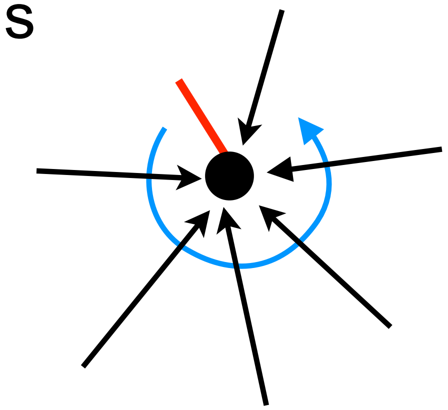

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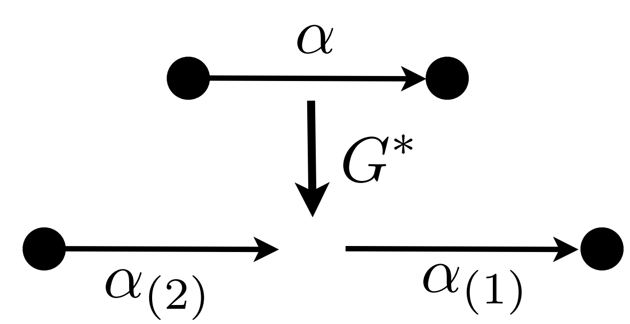

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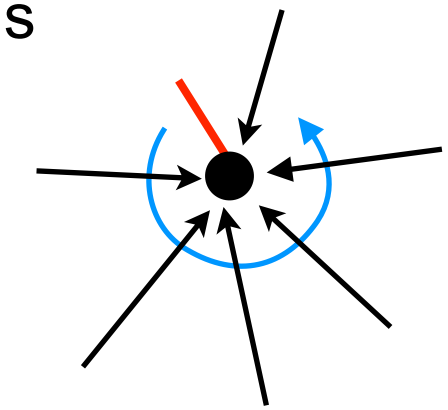

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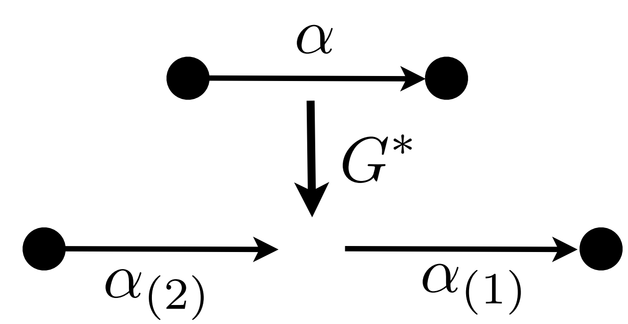

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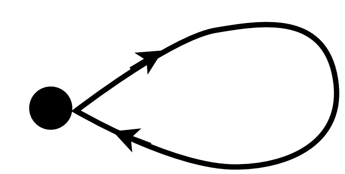

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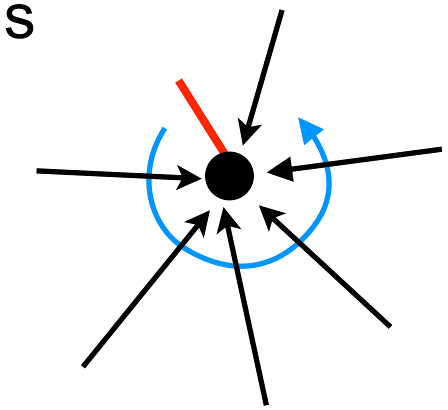

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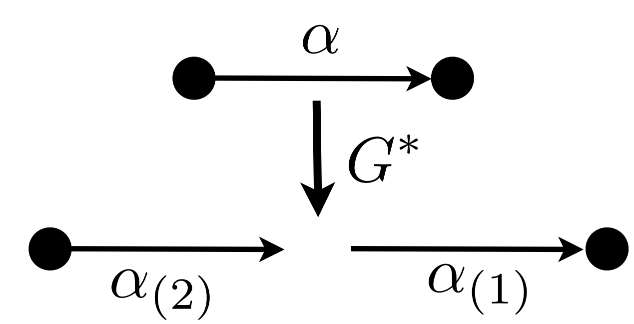

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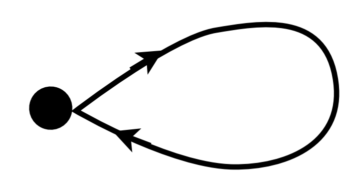

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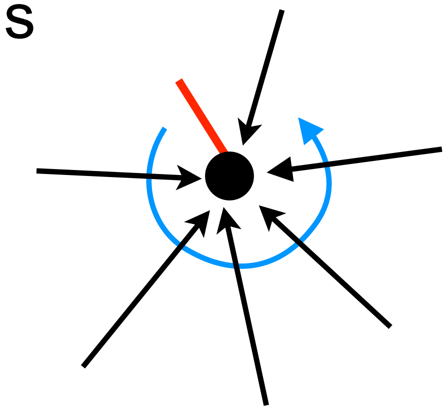

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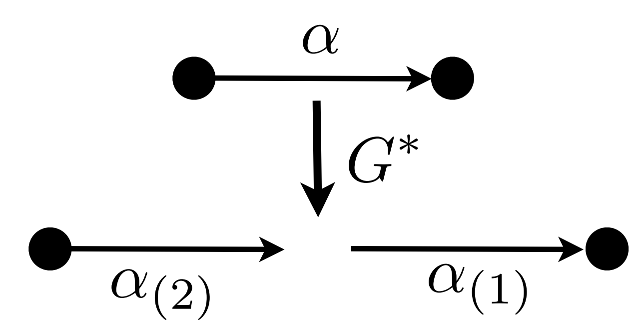

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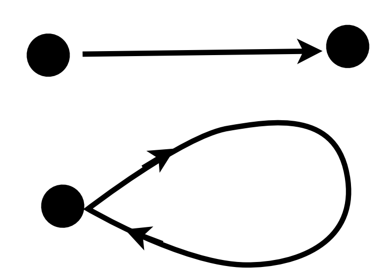

⇨ isomorphic to moduli algebra from combinatorial quantisation of Chern-Simons theory
[Alekseev, Grosse, Schomerus ’94] , [Buffenoir Roche ’95]

holonomies and curvature

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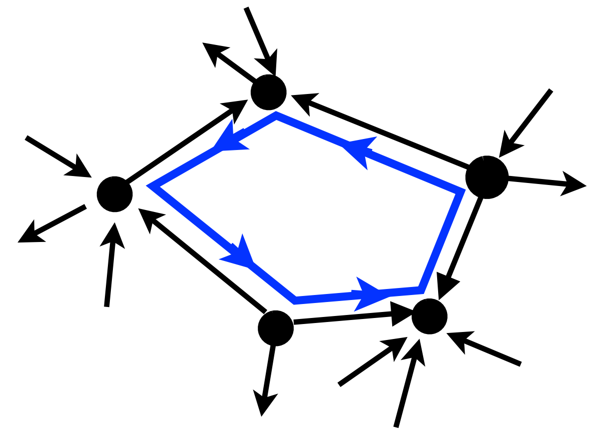

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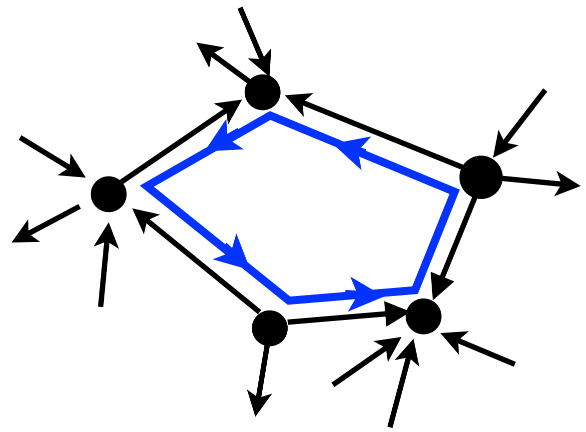

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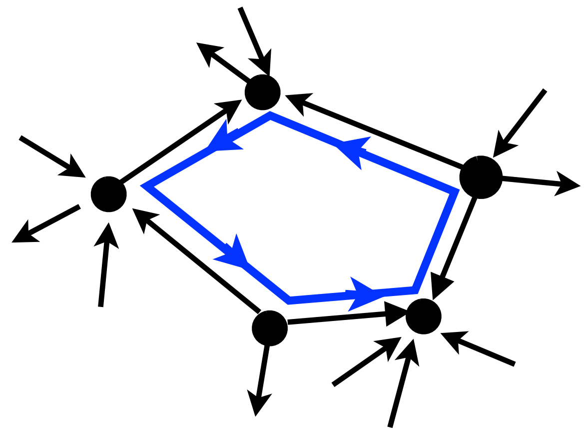

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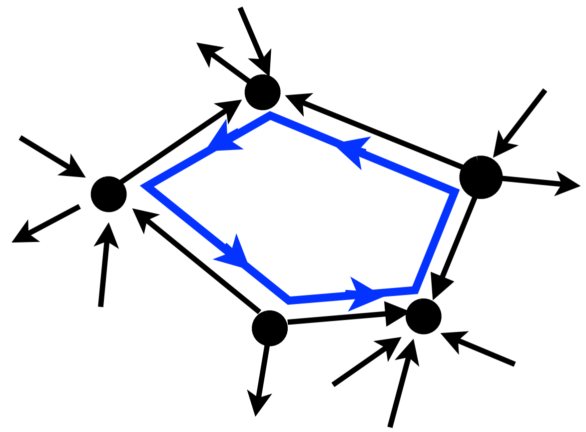

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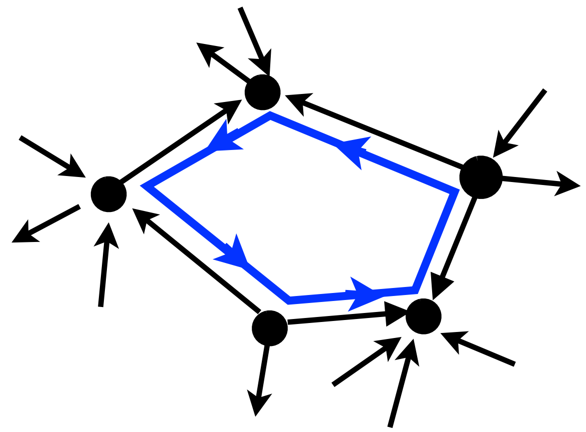

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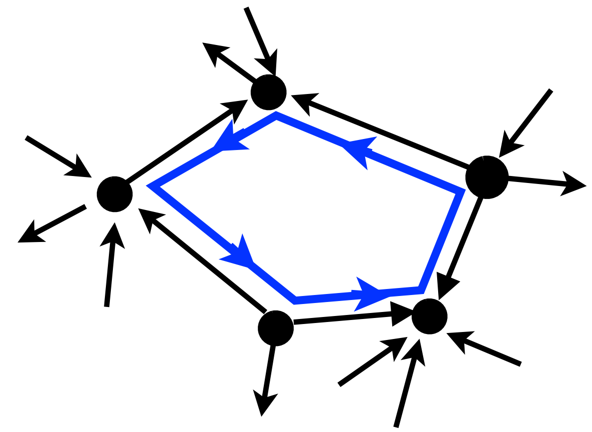

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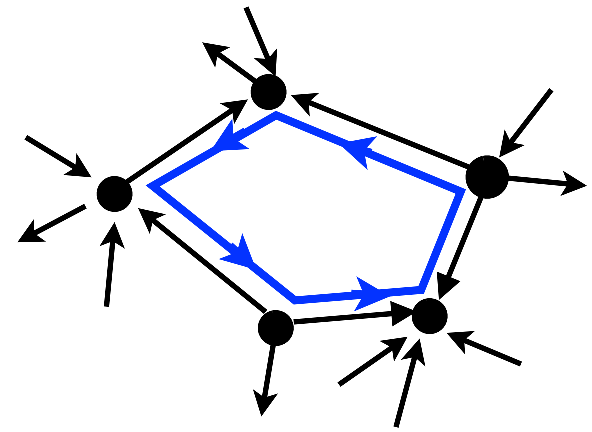

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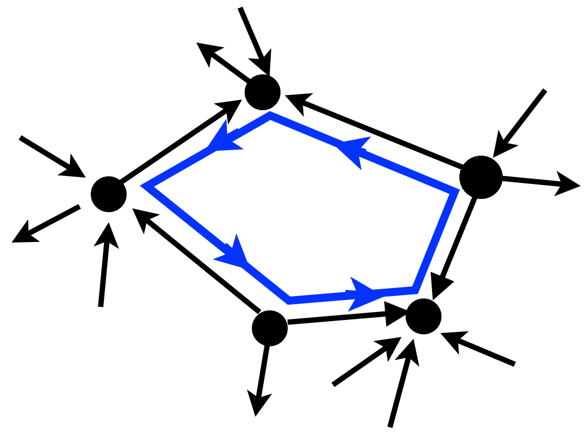

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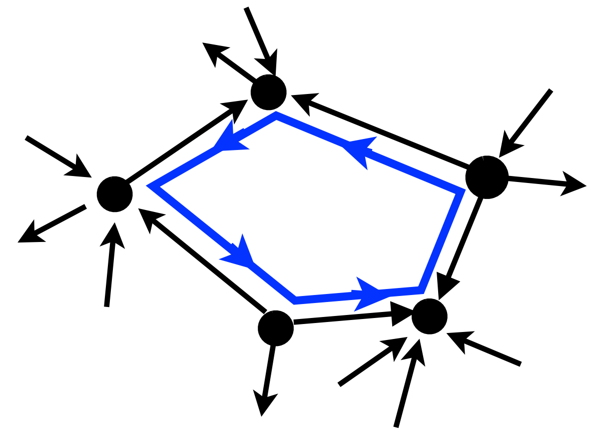

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- Hopf algebra gauge theory on fat graph determined uniquely by simple axioms:
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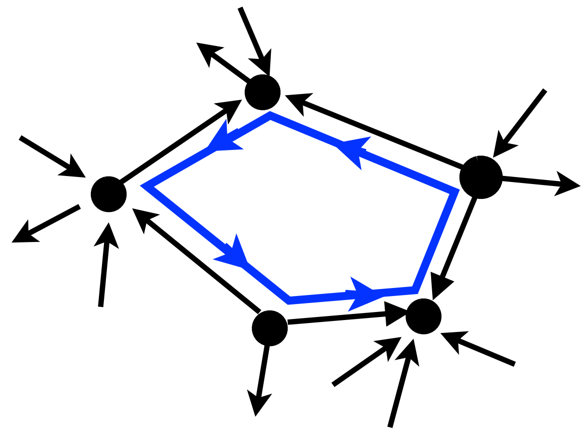

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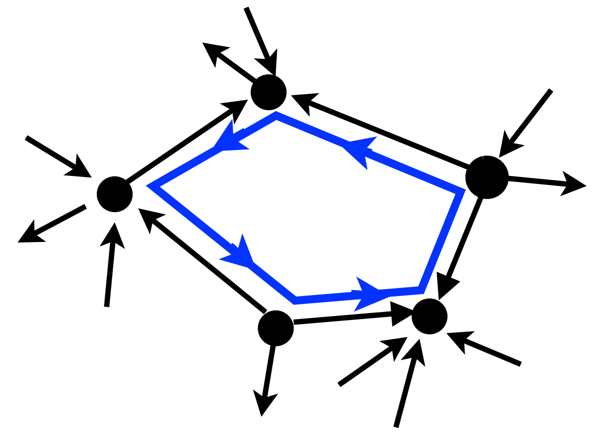

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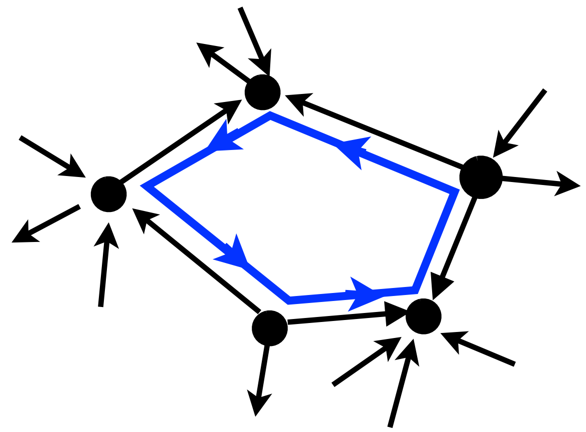

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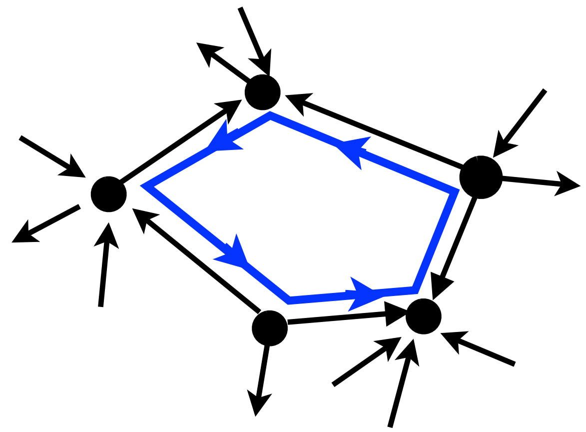

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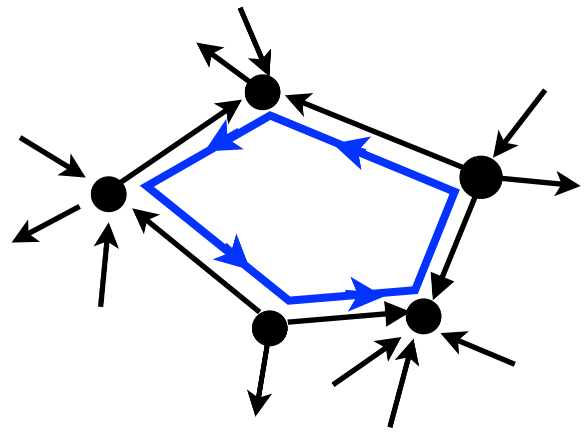

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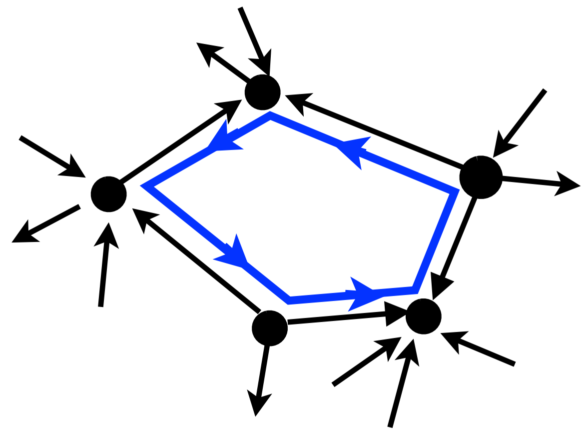

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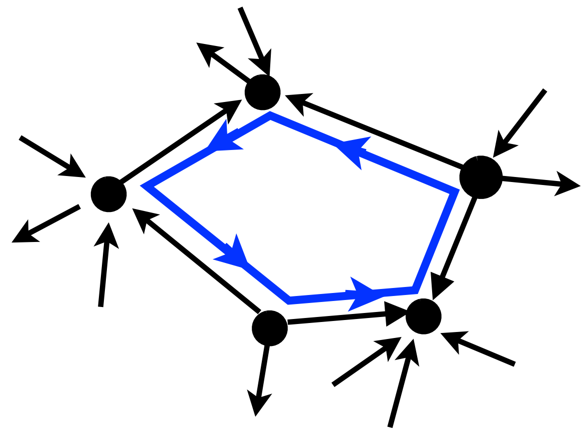

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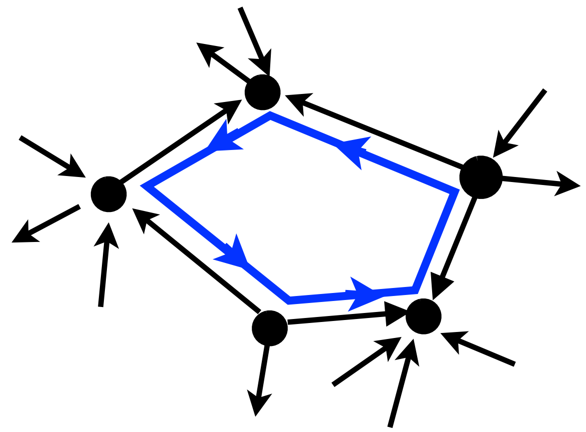

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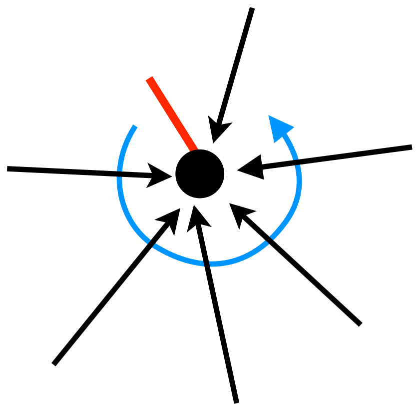

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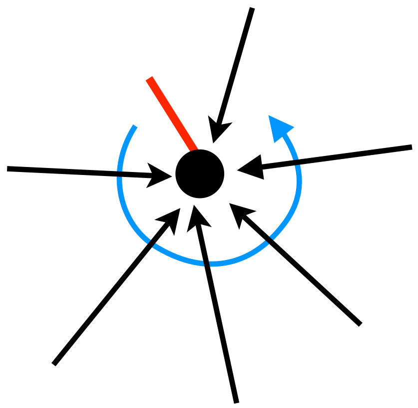

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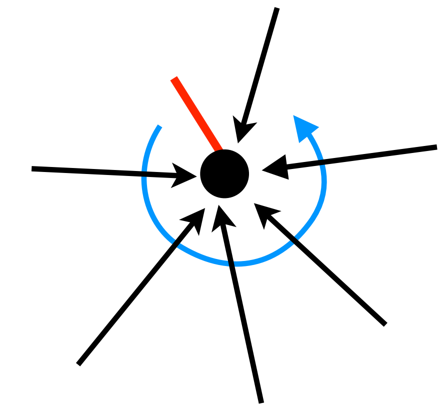

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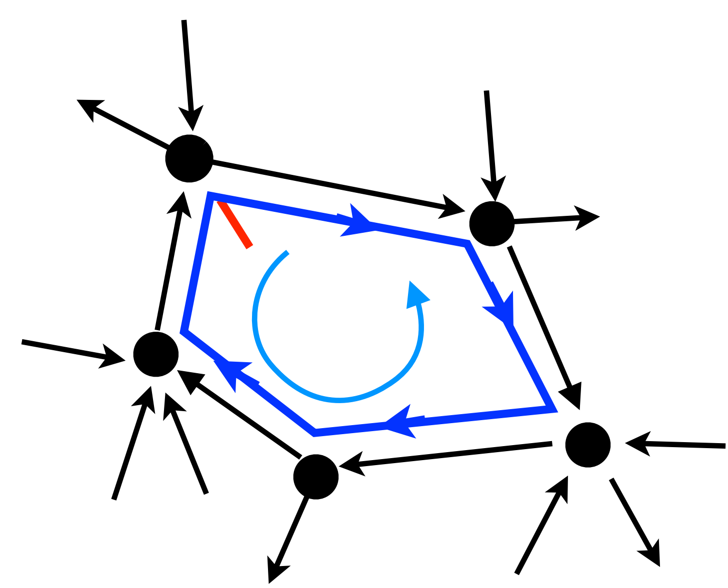

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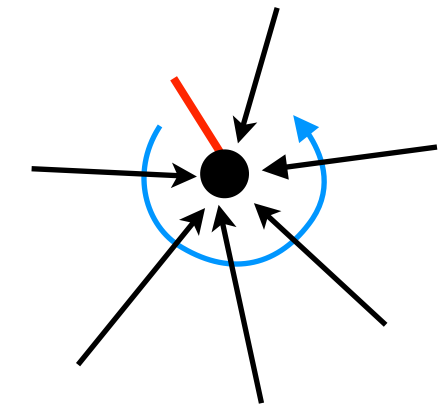

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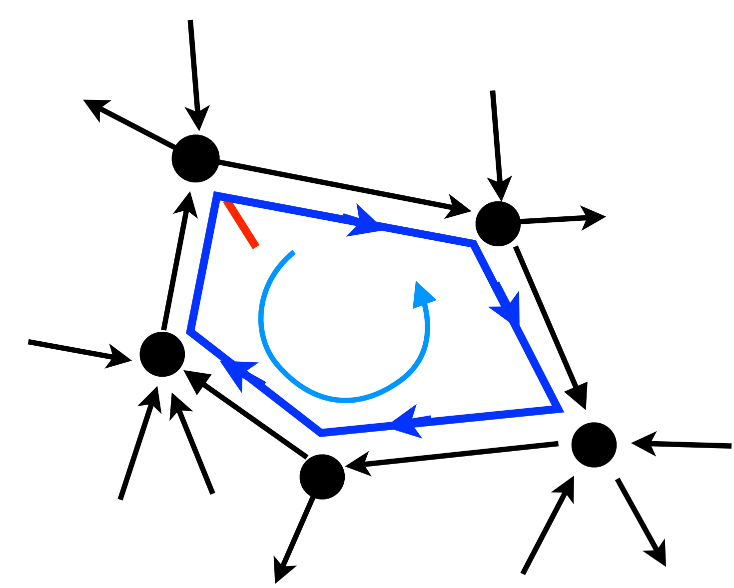

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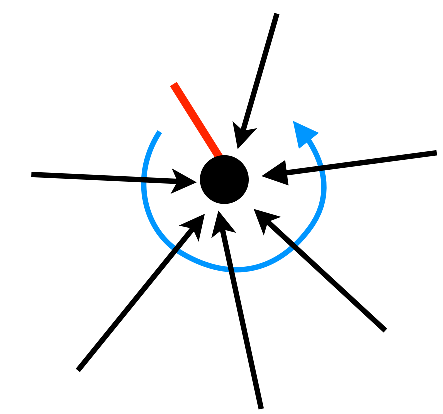

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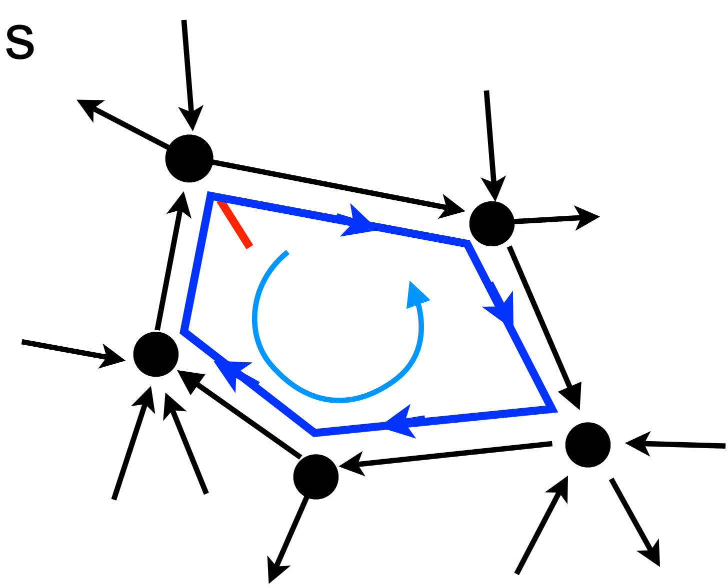

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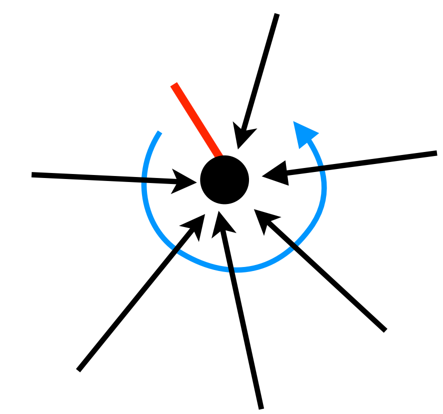

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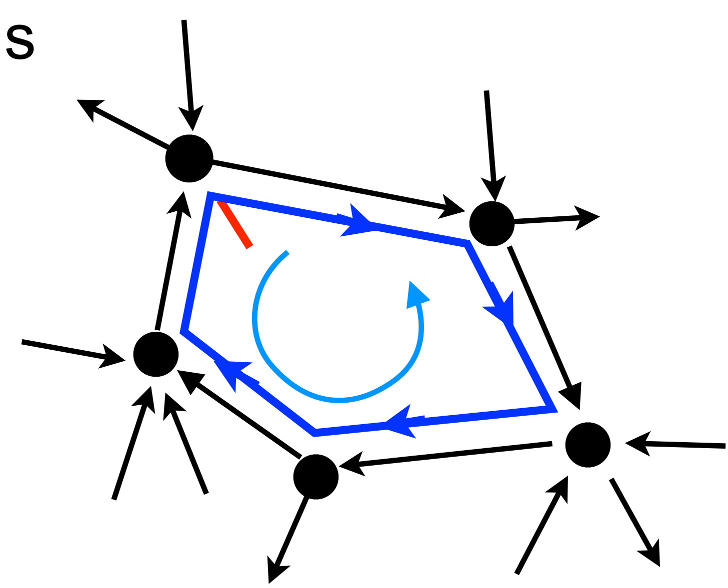

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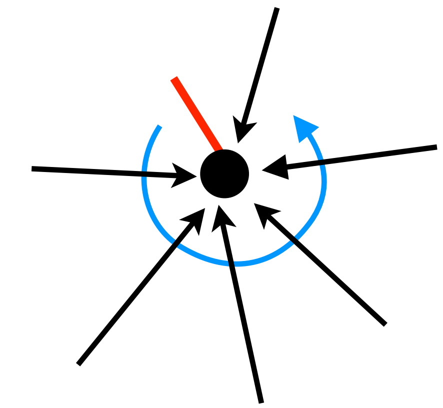
$$L^k_{e+} : (h)_e \mapsto (kh)_e \qquad T^\alpha_{e-} : (h)_e \mapsto \langle S(\alpha), h_{(1)}\rangle\, (h_{(2)})_e$$
$$L^k_{e-} : (h)_e \mapsto (hS(k))_e \qquad T^\alpha_{e+} : (h)_e \mapsto \langle \alpha, h_{(2)}\rangle\, (h_{(1)})_e$$



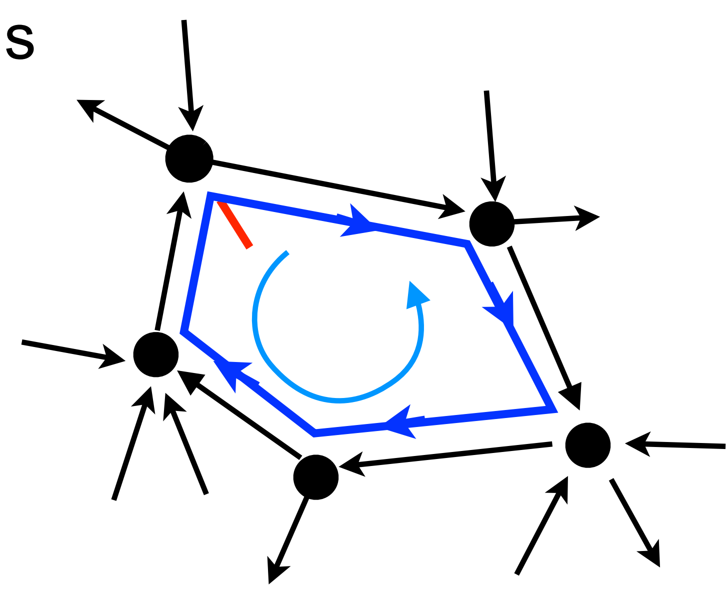
- vertex $v \in V(\Gamma)$ → vertex operator $A^k_v = \Pi_{e\in v}\, L^{k_{(i)}}_{e+} : H^{\otimes E} \to H^{\otimes E}$

$$h^1 \otimes ... \otimes h^n \mapsto k_{(1)}h^1 \otimes ... \otimes k_{(n)}h^n$$

- face $f \in F(\Gamma)$ → face operator $B^\alpha_f = \Pi_{e\in f}\, T^{\alpha_{(i)}}_{e+} : H^{\otimes E} \to H^{\otimes E}$

$$h^1 \otimes ... \otimes h^n \mapsto \langle \alpha, h^1_{(2)} \cdots h^n_{(2)}\rangle\, h^1_{(1)} \otimes ... \otimes h^n_{(1)}$$

- Hamiltonian: $H = \sum_{v\in V} A^\ell_v + \sum_{f\in F} B^\eta_f$ $\quad \ell \in K, \eta \in K^*$ Haar integrals



- `protected space' $H^{\otimes E}_{inv} = \{x \in H^{\otimes E} : H(x) = x\}$

  topological invariant, depends only on associated surface

2. Kitaev models

[Kitaev, '03] [Buerschaper, Mombelli, Christandl, Aguado, '10]

ingredients:
- finite-dimensional, semisimple Hopf algebra H
- ribbon graph Γ

- Hilbert space $H^{\otimes E}$
- edge $e \in E(\Gamma)$ → 4 edge operators $L^k_{e\pm}, T^\alpha_{e\pm} : H^{\otimes E} \to H^{\otimes E}$

$$L^k_{e+} : (h)_e \mapsto (kh)_e \qquad T^\alpha_{e-} : (h)_e \mapsto \langle S(\alpha), h_{(1)}\rangle\, (h_{(2)})_e$$
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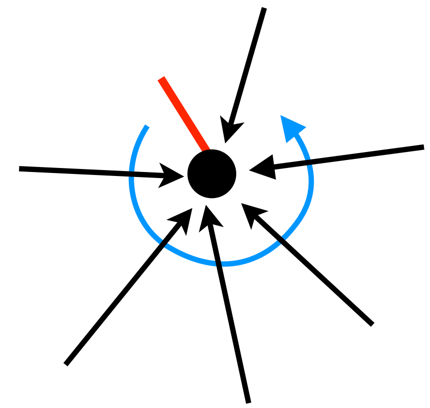

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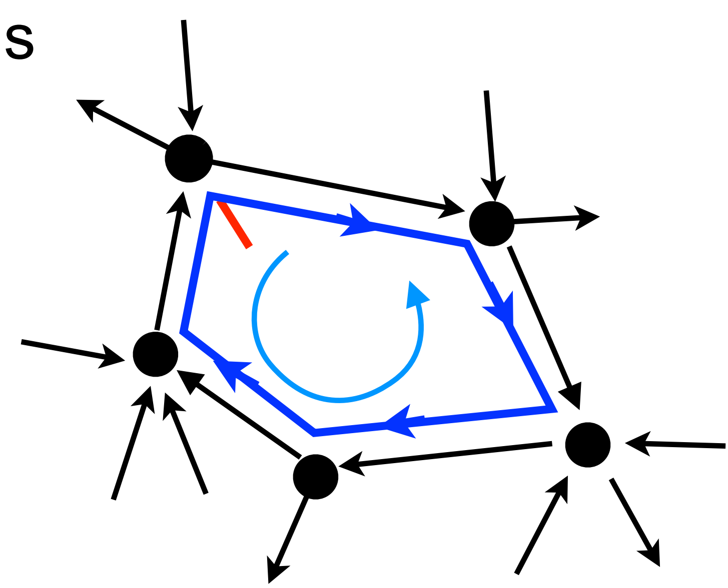
- Hamiltonian: $H = \sum_{v\in V} A^\ell_v + \sum_{f\in F} B^\eta_f$ $\quad \ell \in K, \eta \in K^*$ Haar integrals



- `protected space' $H^{\otimes E}_{inv} = \{x \in H^{\otimes E} : H(x) = x\}$

  topological invariant, depends only on associated surface

⇨ **not a Hopf algebra gauge theory but structural similarities**

Kitaev models as Hopf algebra gauge theory

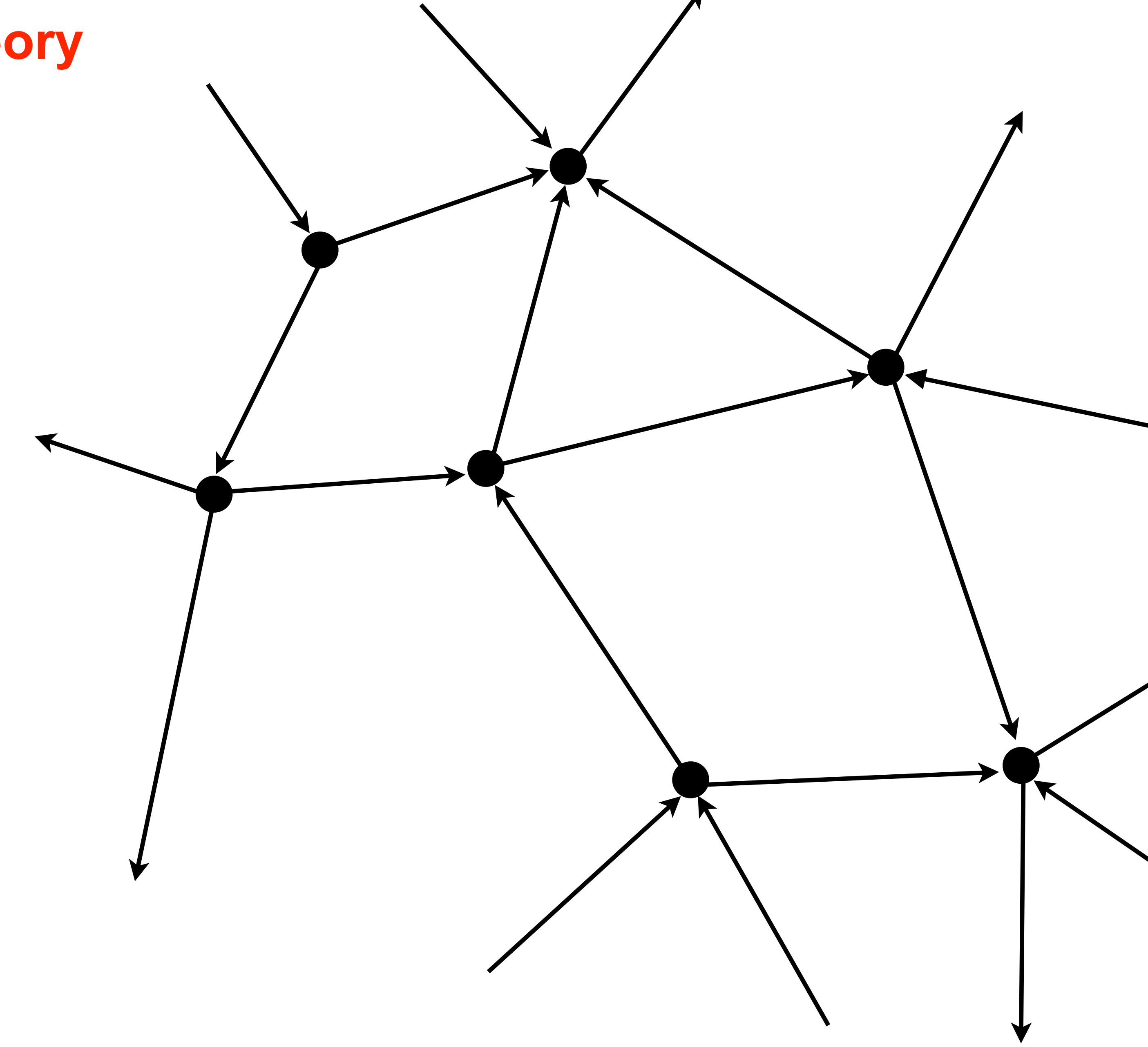
Kitaev models as Hopf algebra gauge theory

- combine ribbon graph and its dual

Kitaev models as Hopf algebra gauge theory

- combine ribbon graph and its dual

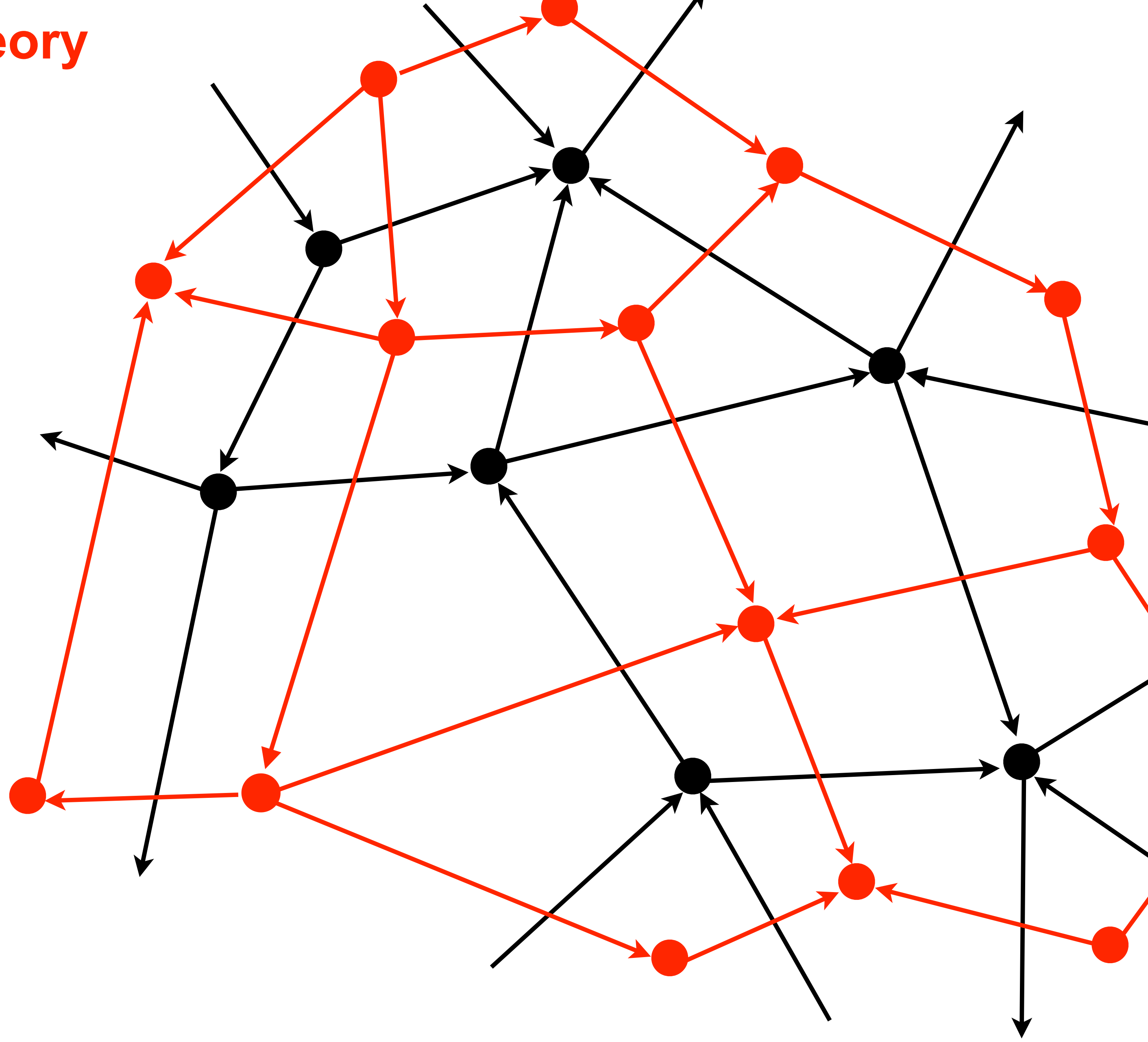
  ribbon graph Γ



Kitaev models as Hopf algebra gauge theory

- combine ribbon graph and its dual

ribbon graph Γ ⇨ dual ribbon graph $\overline{\Gamma}$



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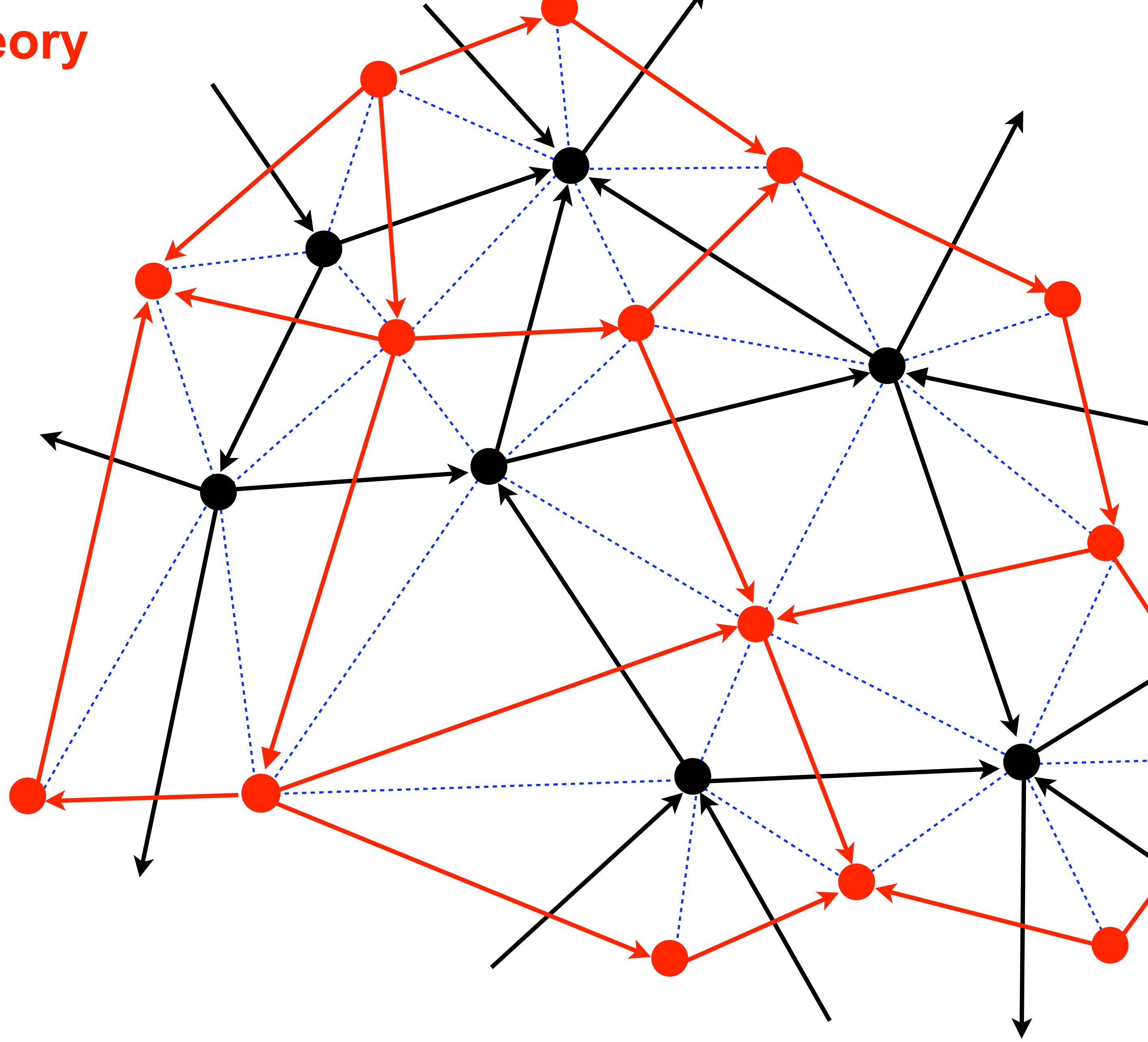

Kitaev models as Hopf algebra gauge theory

- combine ribbon graph and its dual

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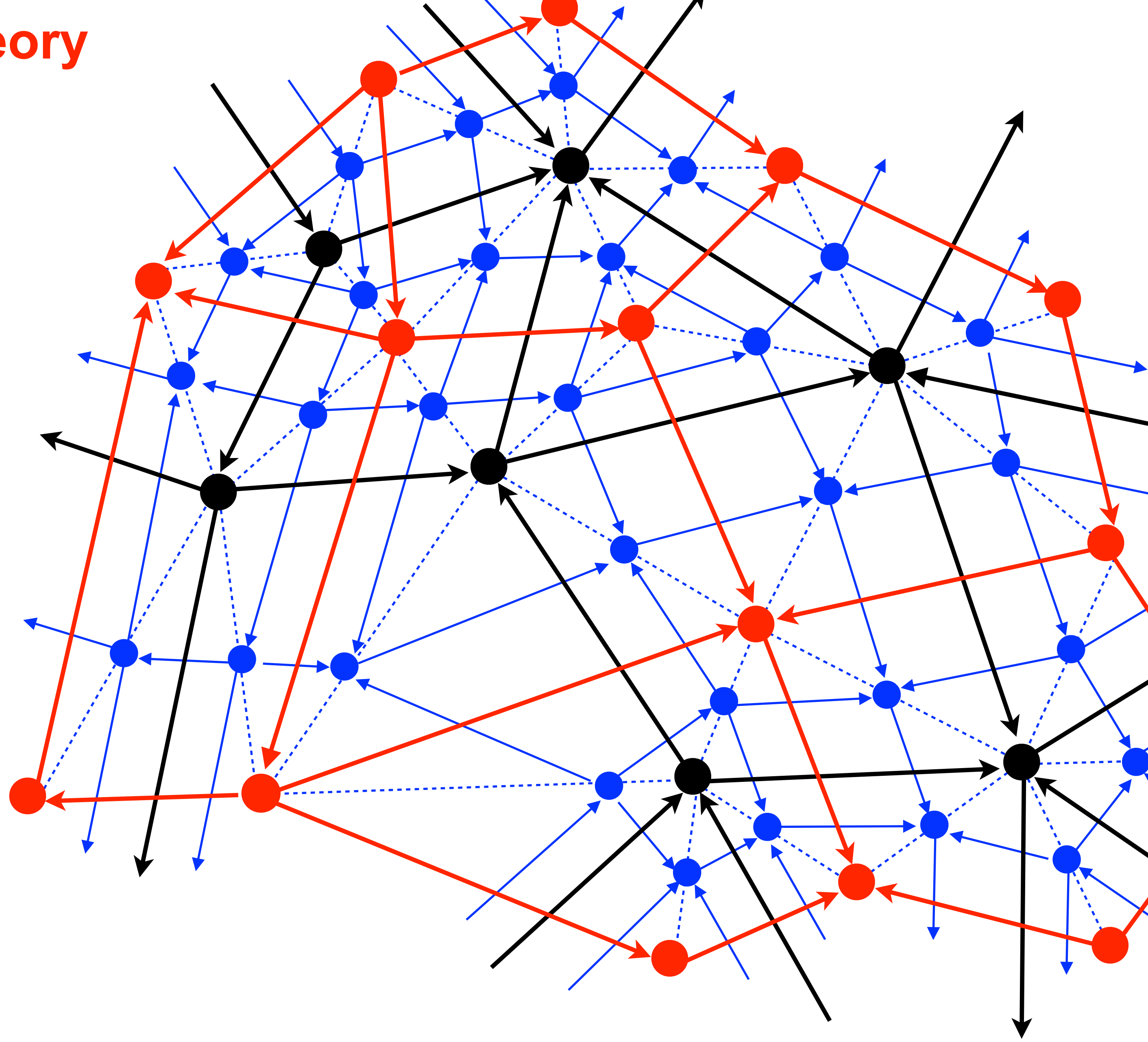
  ⇨ double ribbon graph $D(\Gamma)$



Kitaev models as Hopf algebra gauge theory

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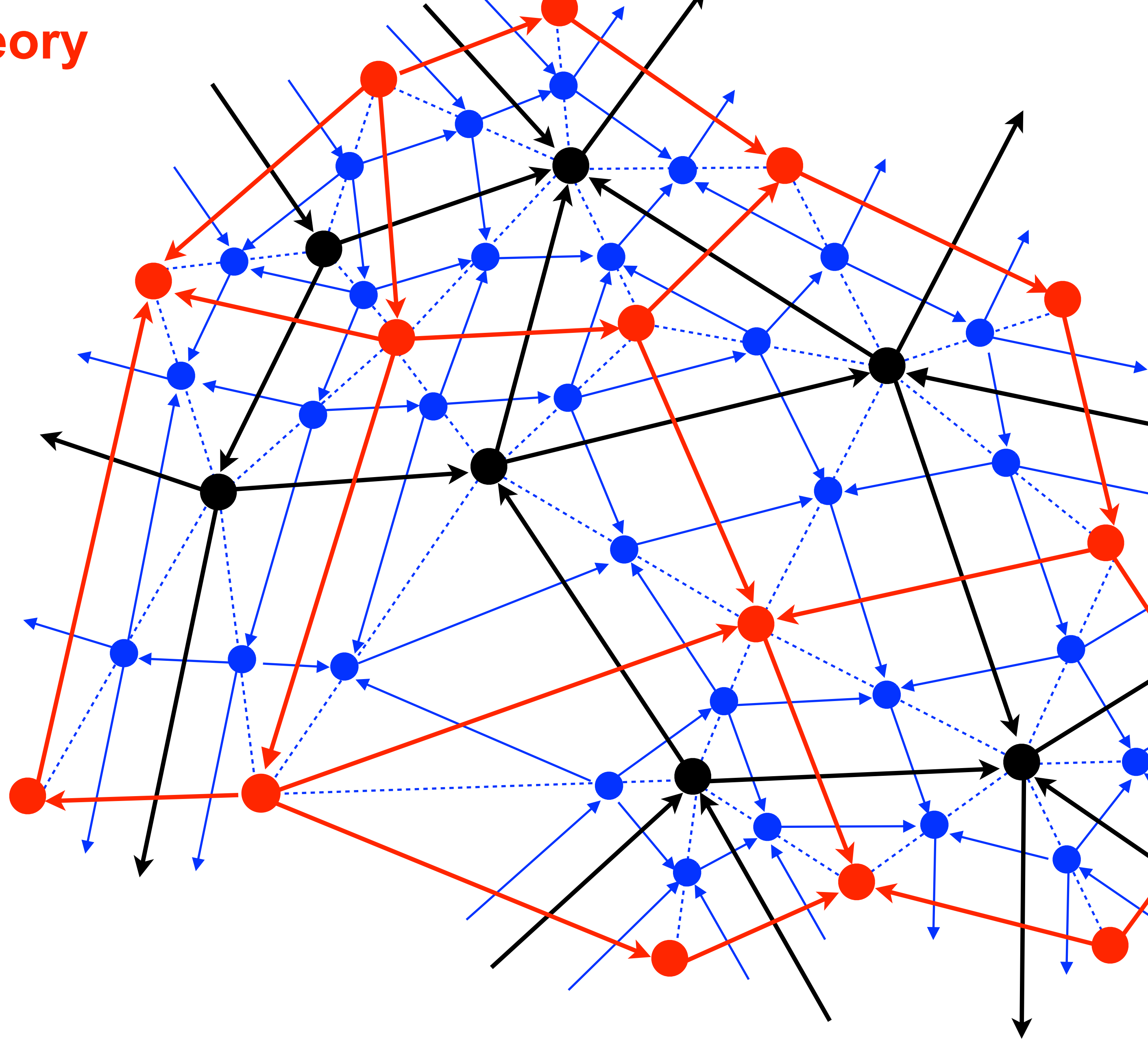
- edge $e \in E(\Gamma)$ → Heisenberg double $H\#H^*$



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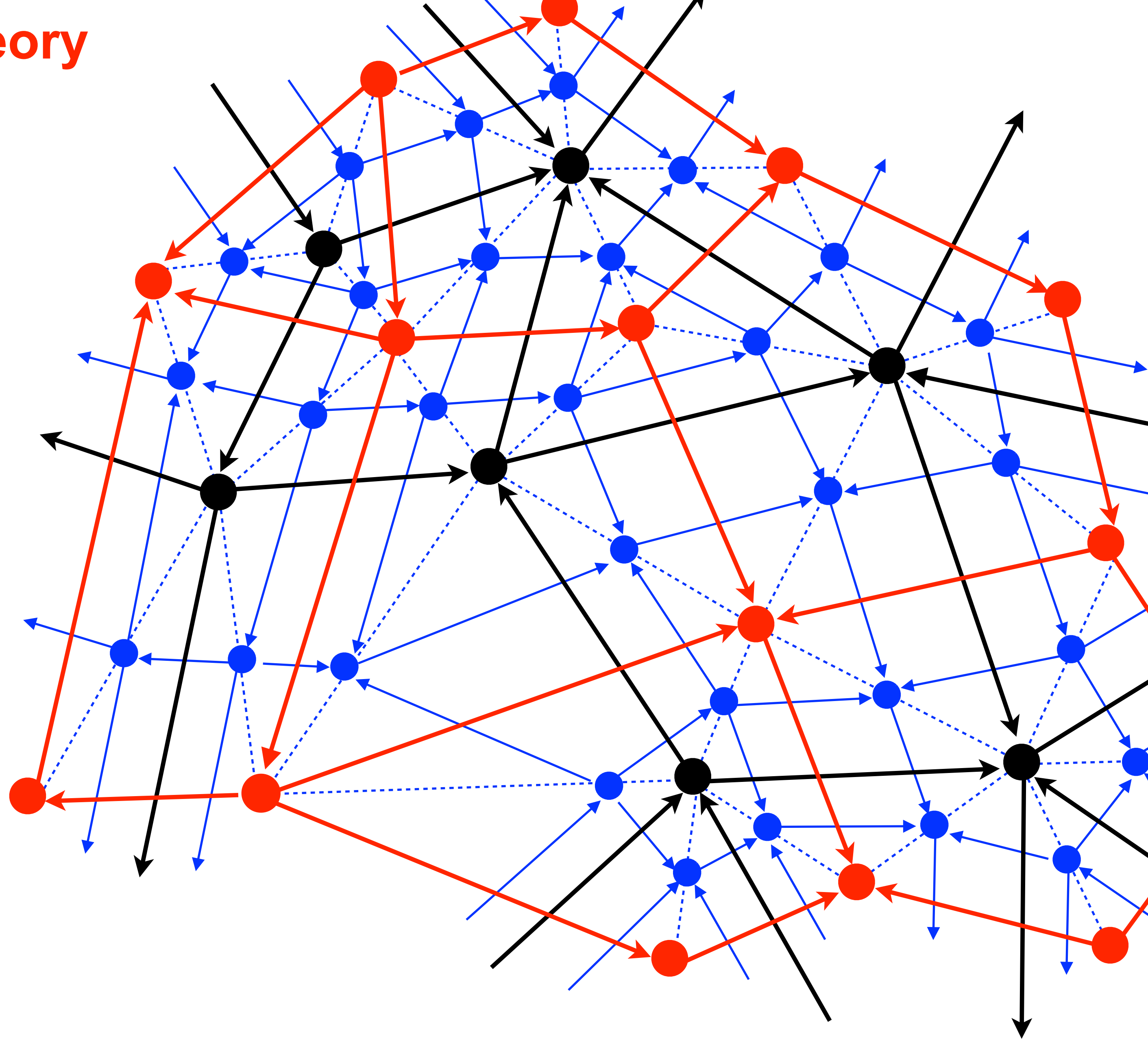
  $H \otimes H^*$ with multiplication

  $(h \otimes \alpha) \cdot (k \otimes \beta) = \langle \alpha_{(1)}, k_{(2)} \rangle \, \alpha_{(2)}\beta \otimes h_{(1)}k$



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- combine ribbon graph and its dual

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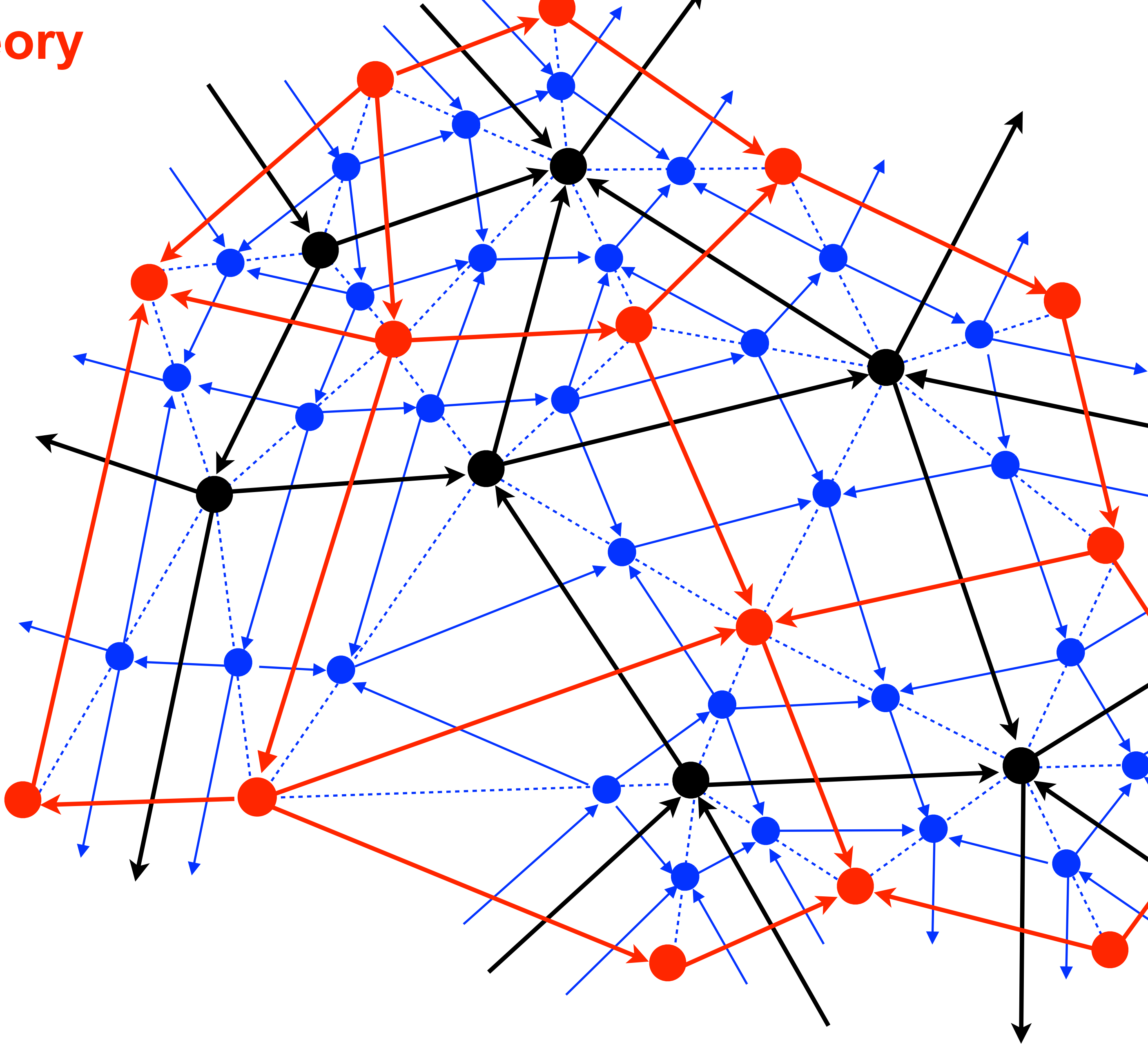
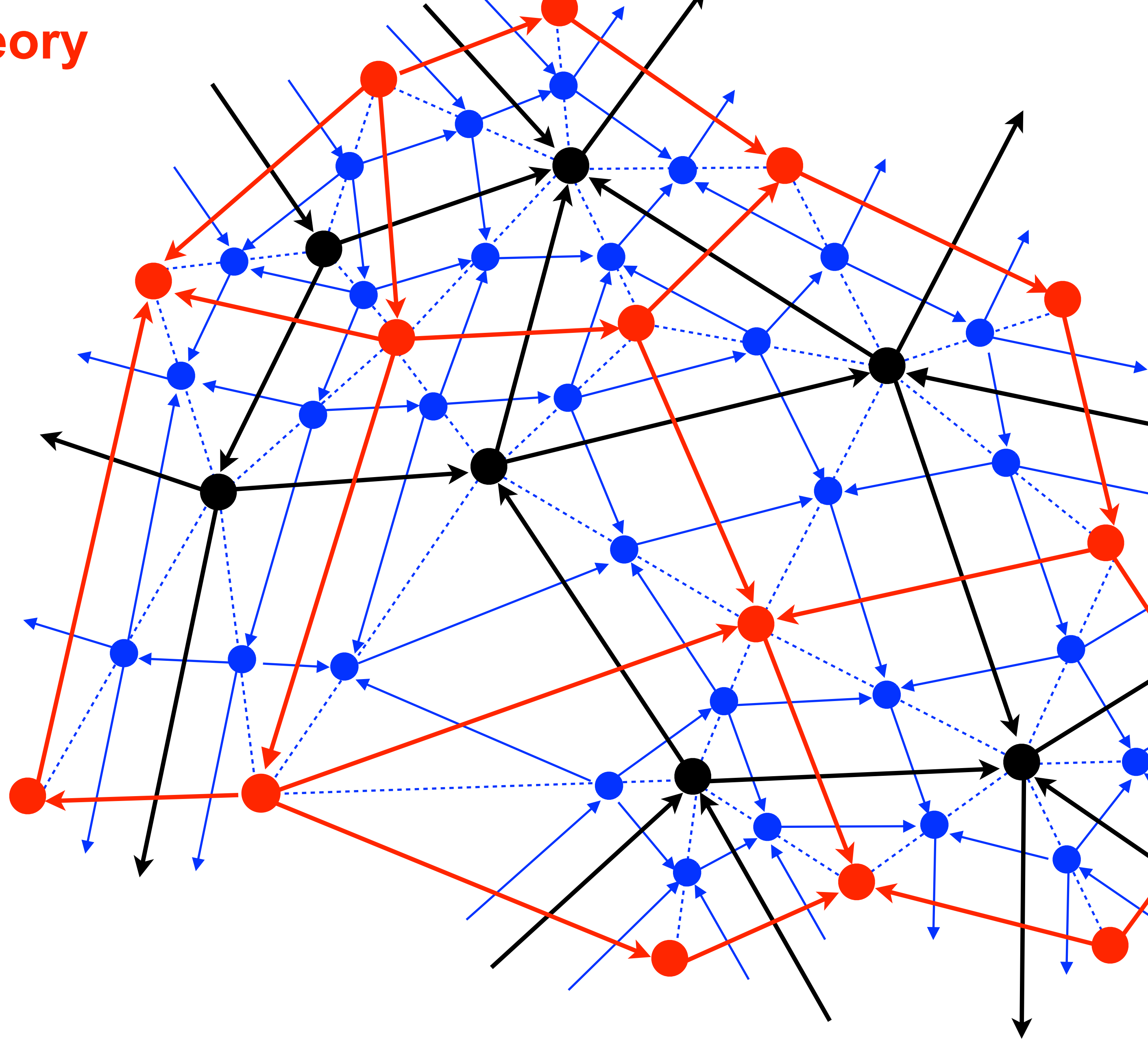
- 4 basic edge operators

  $\phi_{e^\pm}, \phi_{\overline{e}^\pm} : H \otimes H^* \to (H\#H^*)^{\otimes E}$



Kitaev models as Hopf algebra gauge theory



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  $\phi_{e+}(h \otimes \alpha) = \epsilon(h)\,(1 \otimes \alpha)_e$

Kitaev models as Hopf algebra gauge theory

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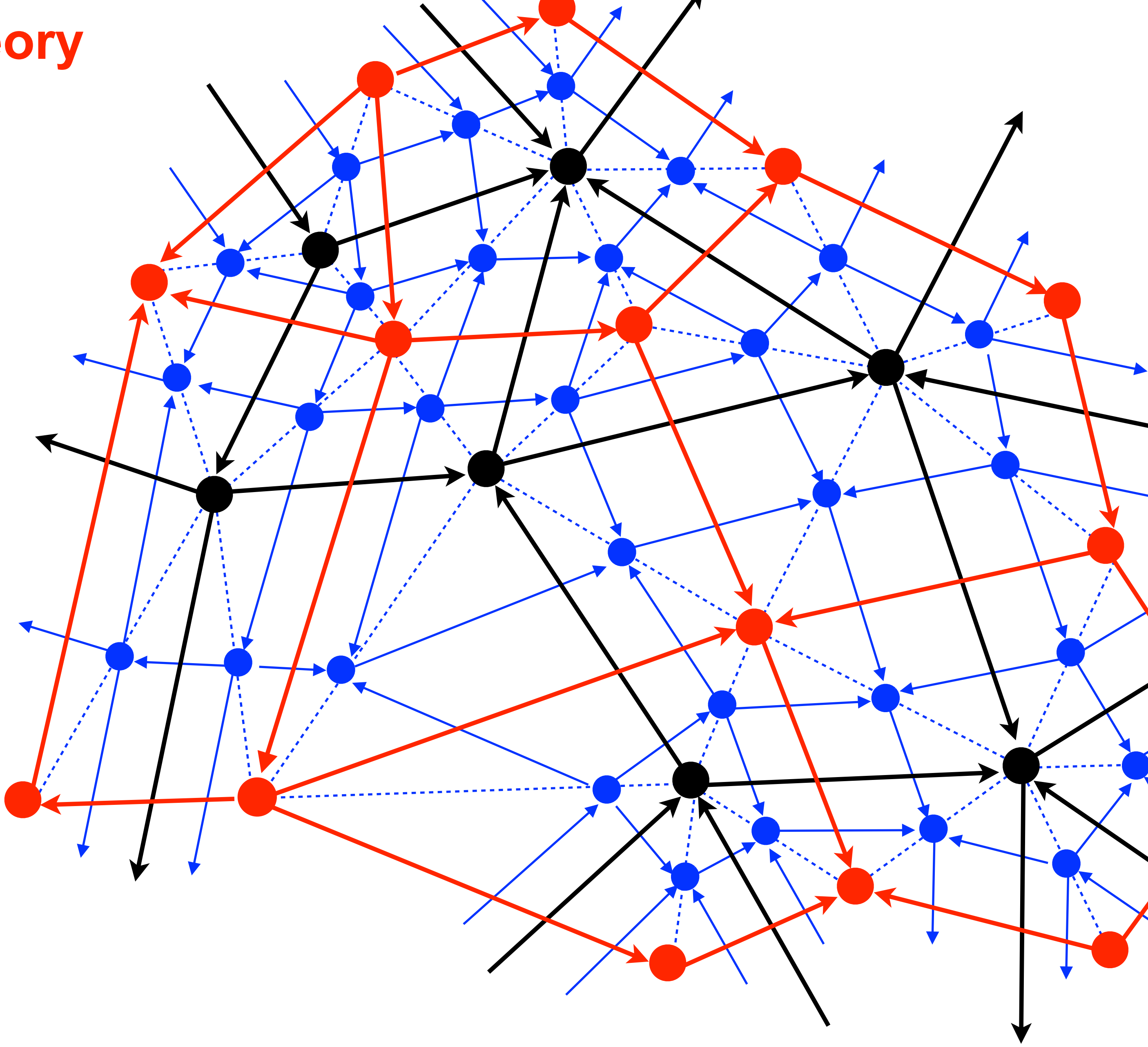
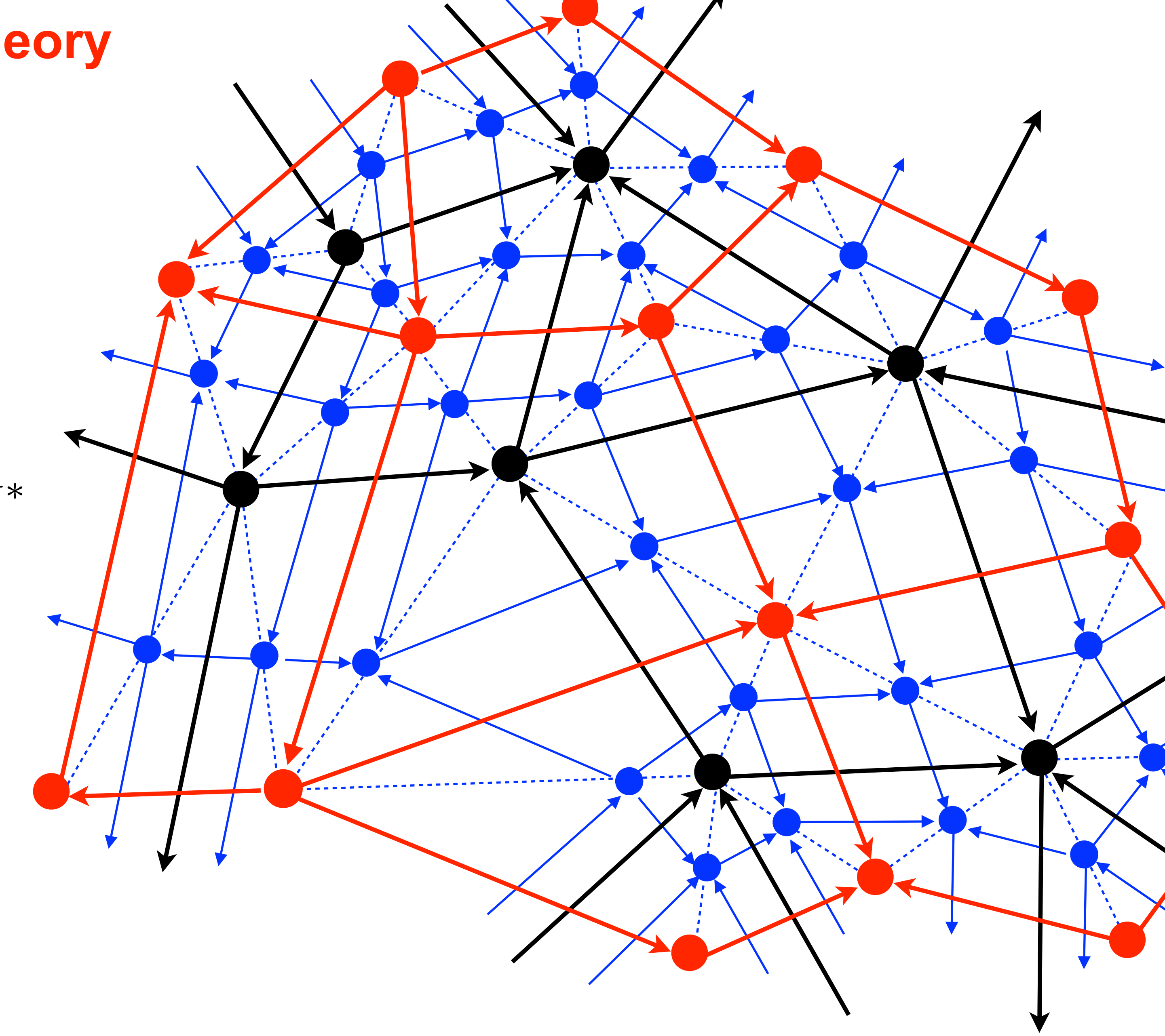


Kitaev models as Hopf algebra gauge theory



- combine ribbon graph and its dual

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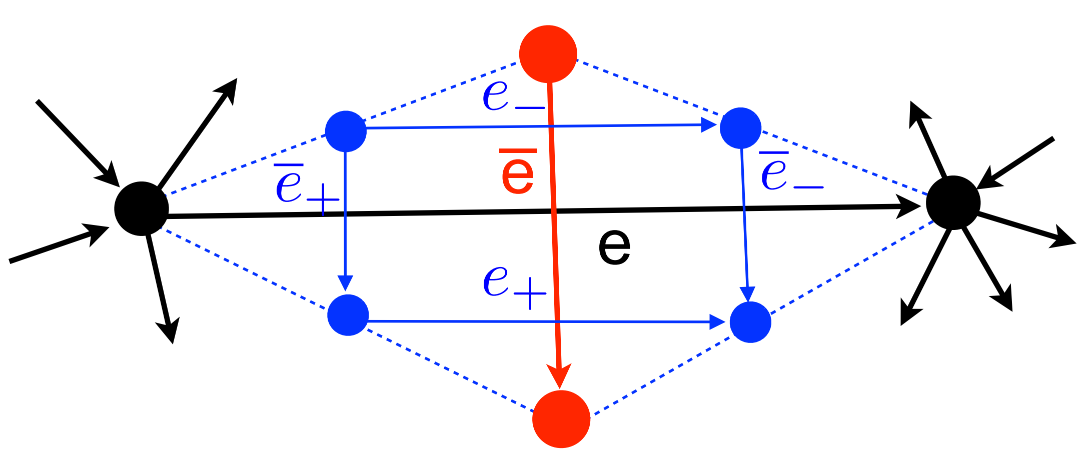
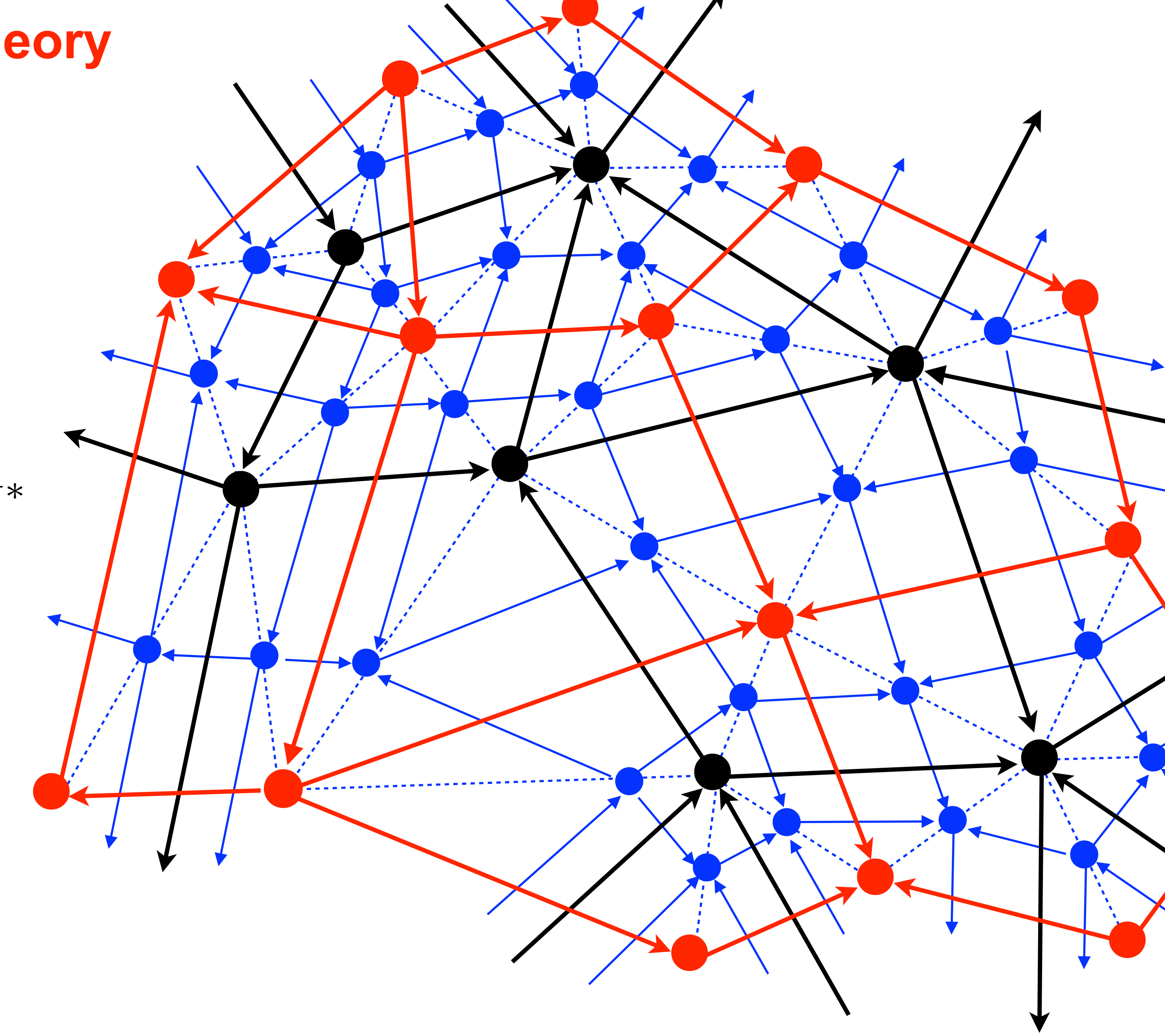
$$\phi_{e^-} = S^{\otimes E} \circ \phi_{e^-} \circ S$$



Kitaev models as Hopf algebra gauge theory



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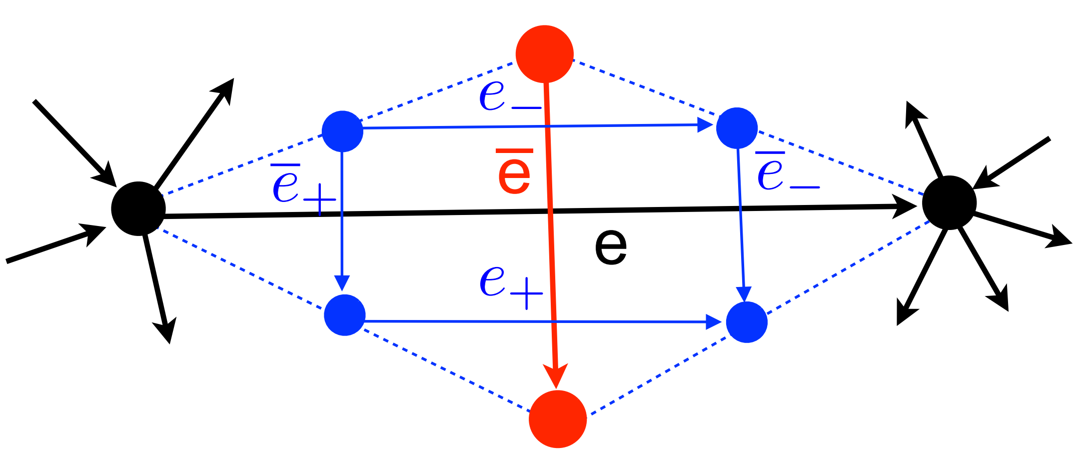

Lemma

$$\rho : L^h_{e\pm} \to \phi_{\overline{e}\pm}(h \otimes 1)$$

$$T^\alpha_{e\pm} \to \phi_{e\pm}(1 \otimes \alpha)$$

defines a faithful representation of Kitaev's edge operator algebra on $(H\#H^*)^{\otimes E}$

holonomies

holonomies path p → map $\phi_p : H \otimes H^* \to (H \# H^*)^{\otimes E}$ defined by:

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$$\phi_{e^\pm}, \phi_{\overline{e}^\pm}$$

holonomies path p → map $\phi_p : H \otimes H^* \to (H \# H^*)^{\otimes E}$ defined by:

$$\phi_{\emptyset_v} = \epsilon\,(1 \otimes 1)^{\otimes E} \qquad \phi_{e_\pm^{-1}} = \phi_{e_\pm} \circ (1 \otimes S) \quad \phi_{\overline{e}_\pm^{-1}} = \phi_{\overline{e}_\pm} \circ (S \otimes 1)$$

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$$\phi_{e^\pm}, \phi_{\overline{e}^\pm} \qquad \phi_p \circ \phi_q = m \circ (\phi_q \otimes \phi_p) \circ \Delta$$

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examples

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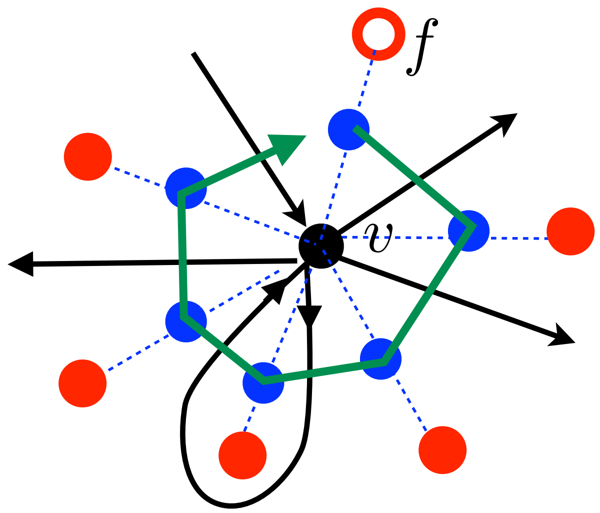

$$\phi_p(h \otimes \alpha) = \epsilon(\alpha) A^h$$

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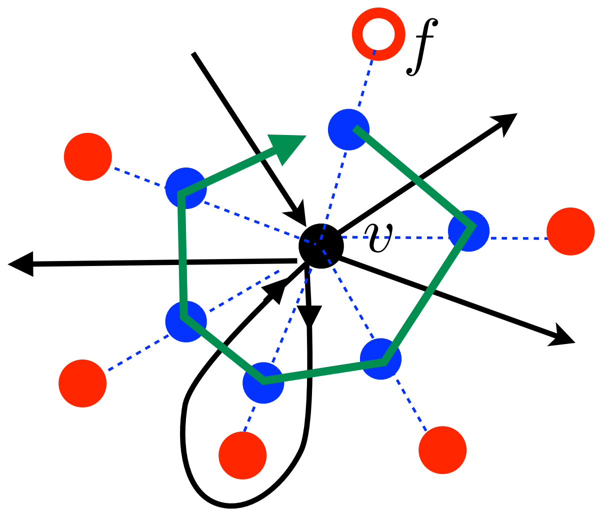
examples



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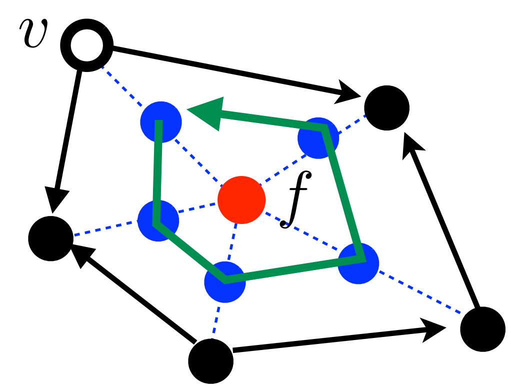

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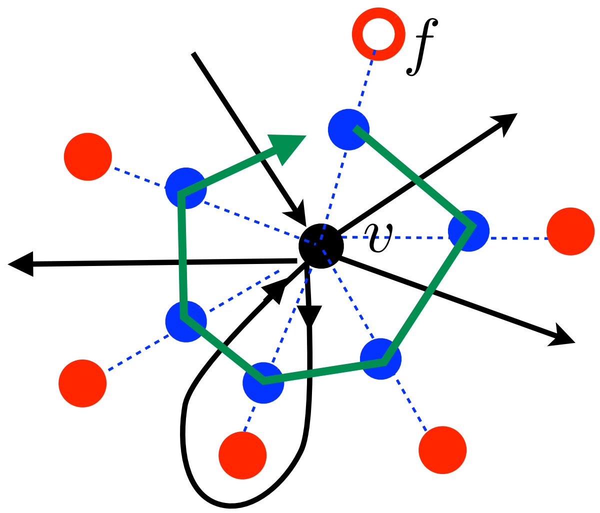

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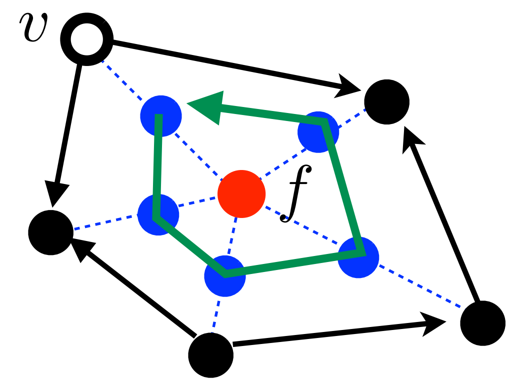

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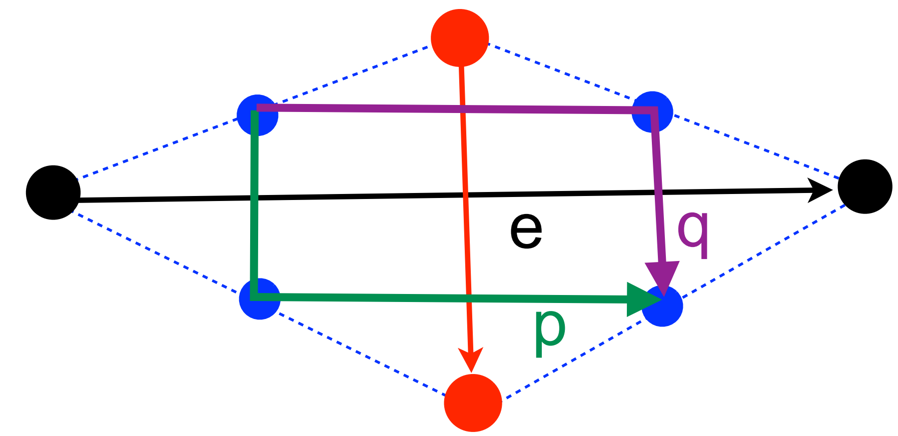

$\phi_p(h \otimes \alpha) = \phi_q(h \otimes \alpha) = h \otimes \alpha$

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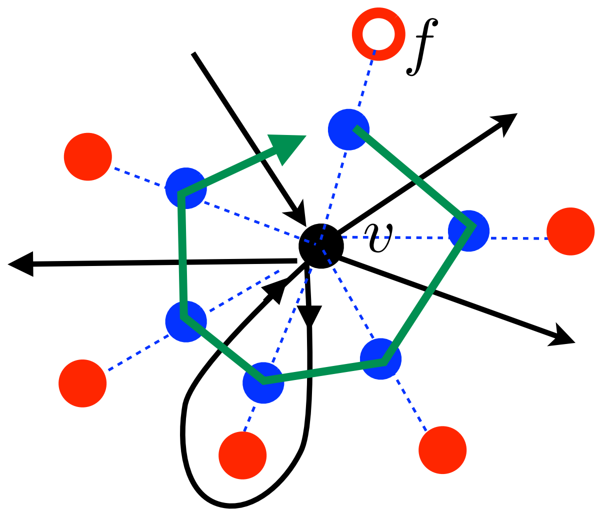

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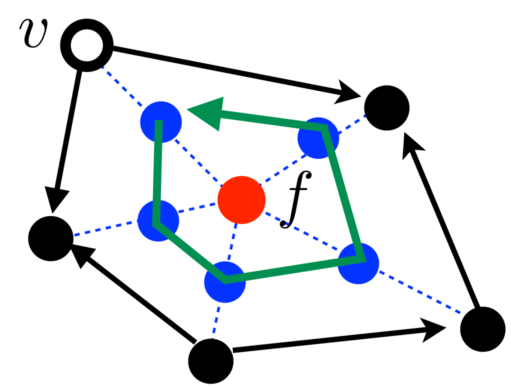

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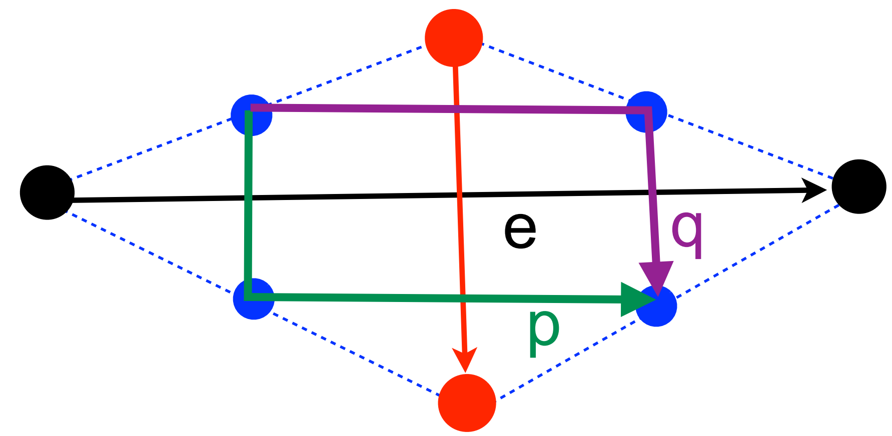

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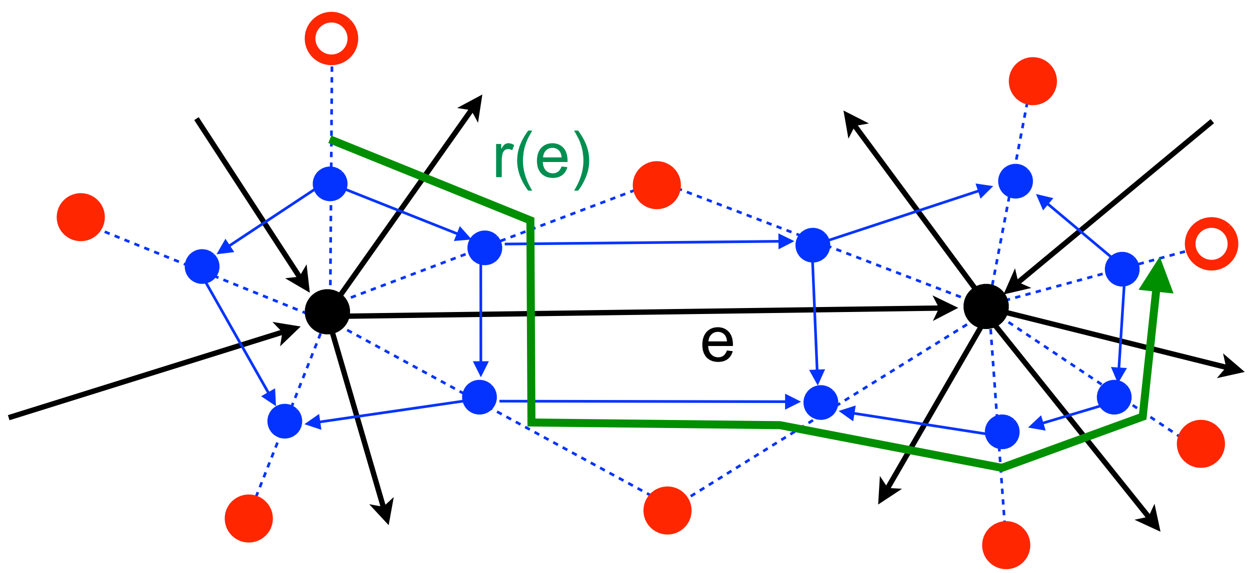

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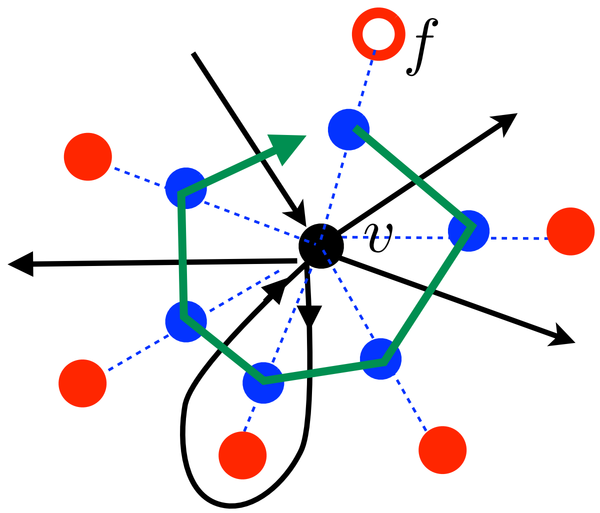
examples



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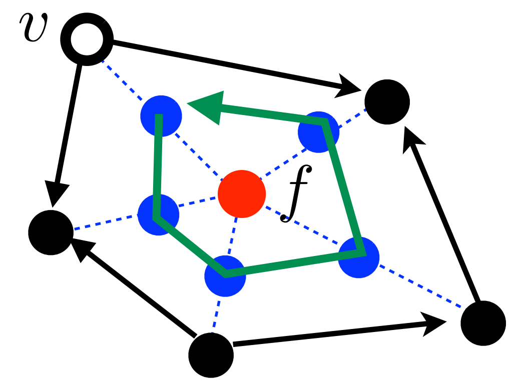

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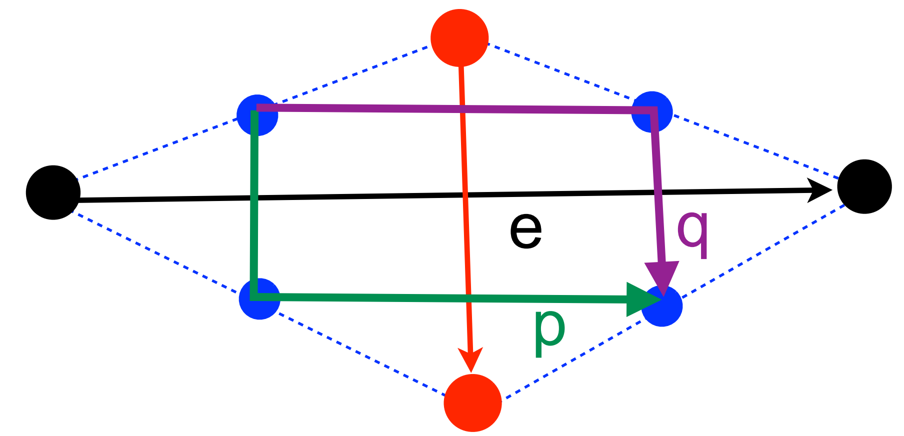

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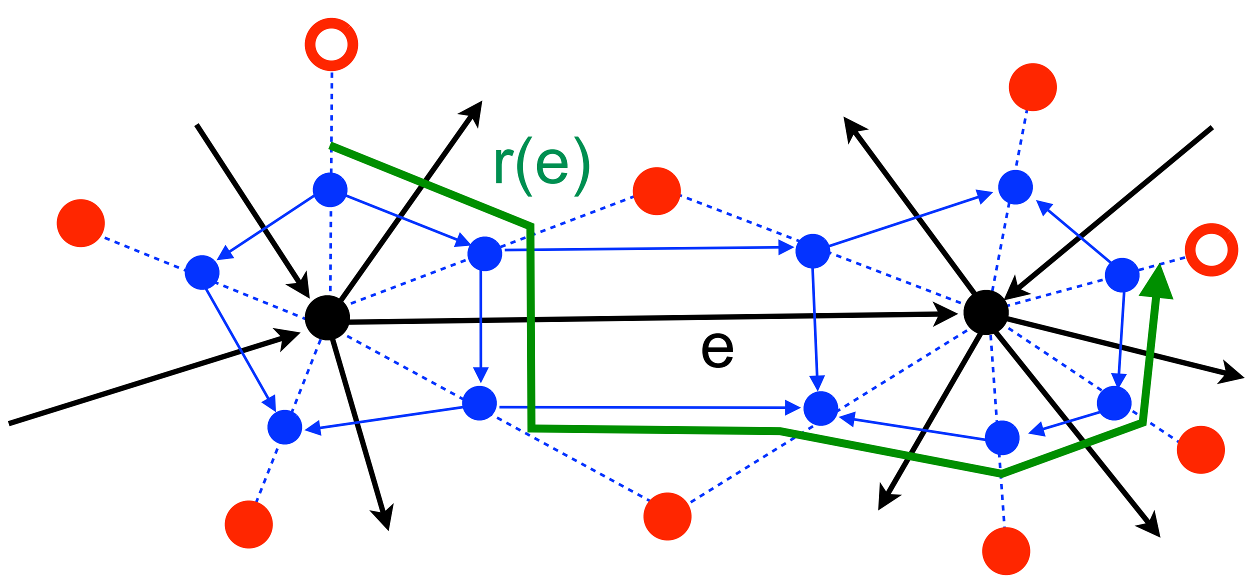

Theorem [C.M.]

holonomies path p → map $\phi_p : H \otimes H^* \to (H\#H^*)^{\otimes E}$ defined by:

$$\phi_{\emptyset_v} = \epsilon\,(1 \otimes 1)^{\otimes E} \qquad \phi_{e_\pm^{-1}} = \phi_{e_\pm} \circ (1 \otimes S) \quad \phi_{\overline{e}_\pm^{-1}} = \phi_{\overline{e}_\pm} \circ (S \otimes 1)$$

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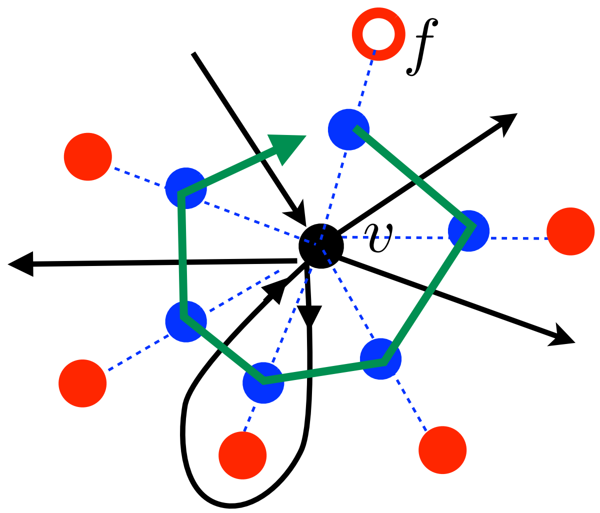
examples



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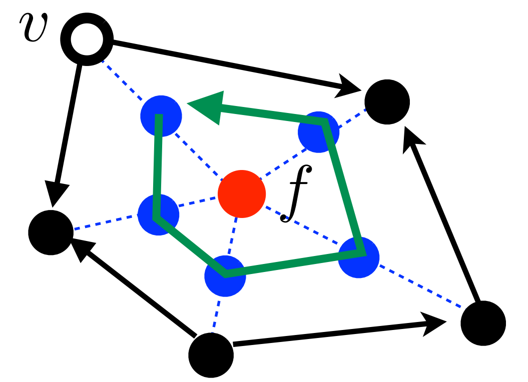

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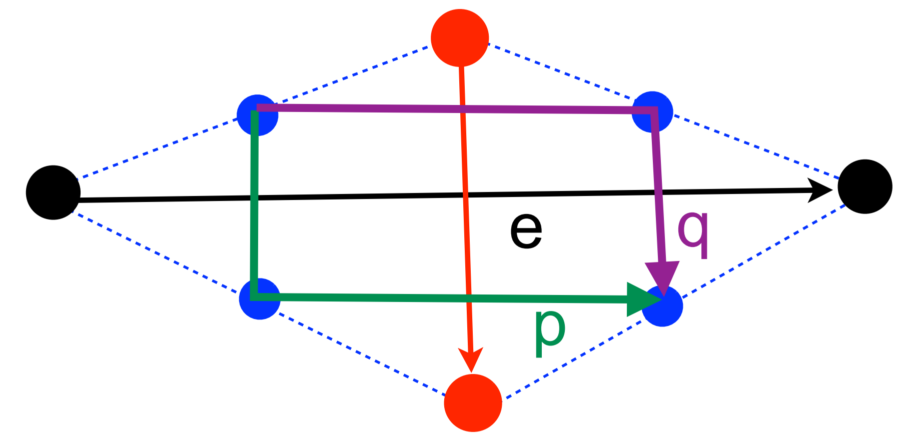

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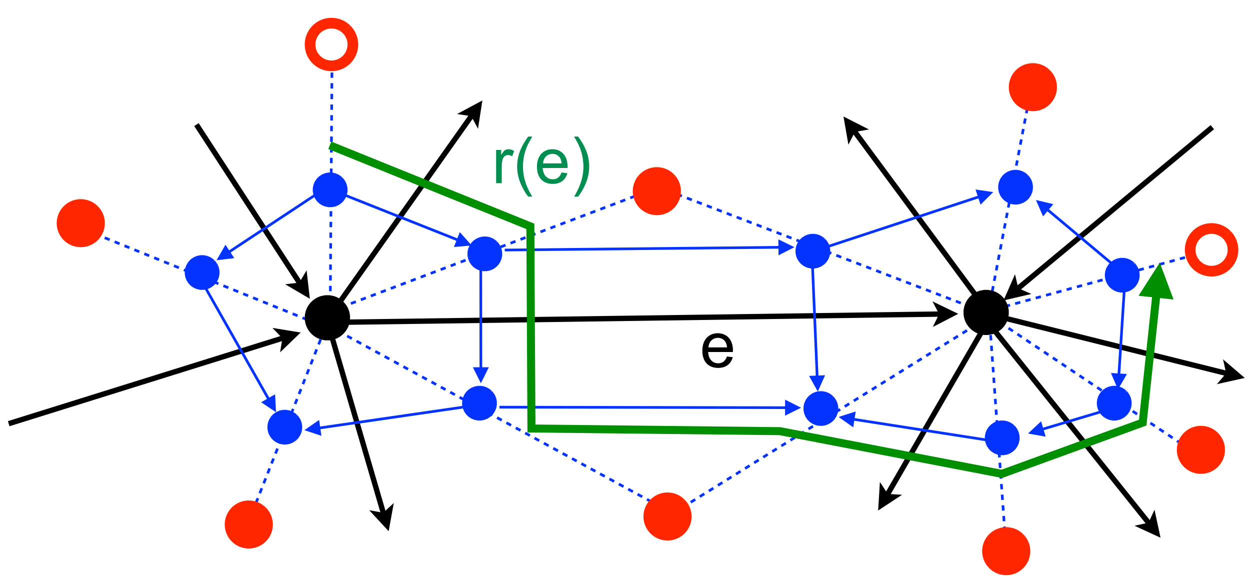


Theorem [C.M.]

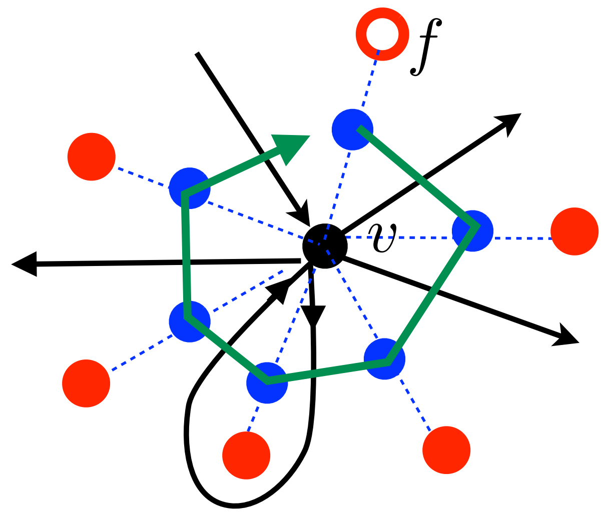
- the holonomies $\phi_{r(e)} : H \otimes H^* \to (H\#H)^{\otimes E}$ induce an algebra structure on $(H \otimes H^*)^{\otimes E}$

holonomies path p → map $\phi_p : H \otimes H^* \to (H\#H^*)^{\otimes E}$ defined by:

$$\phi_{\emptyset_v} = \epsilon\,(1\otimes 1)^{\otimes E} \qquad \phi_{e_\pm^{-1}} = \phi_{e_\pm}\circ(1\otimes S) \quad \phi_{\overline{e}_\pm^{-1}} = \phi_{\overline{e}_\pm}\circ(S\otimes 1)$$

$$\phi_{e^\pm},\phi_{\overline{e}^\pm} \qquad \phi_p\circ\phi_q = m\circ(\phi_q\otimes\phi_p)\circ\Delta$$

examples



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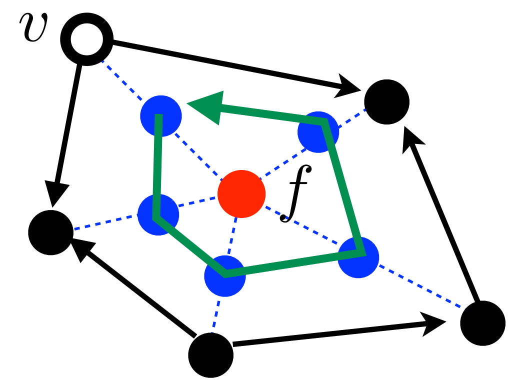

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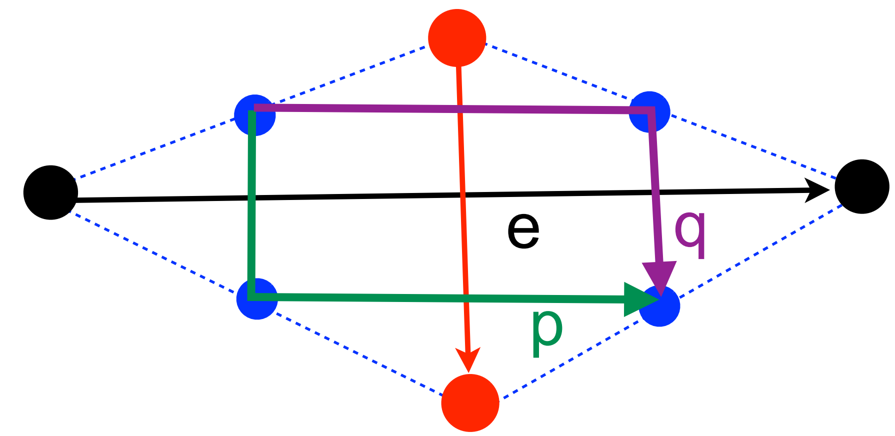

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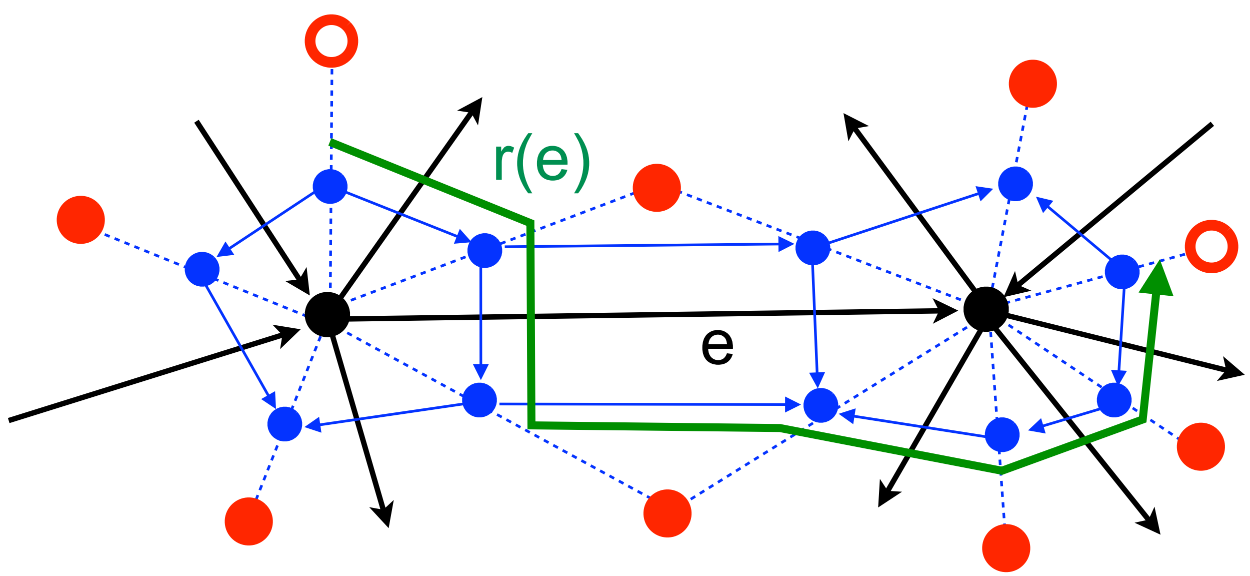

Theorem [C.M.]

- the holonomies $\phi_{r(e)} : H\otimes H^* \to (H\#H)^{\otimes E}$ induce an algebra structure on $(H\otimes H^*)^{\otimes E}$
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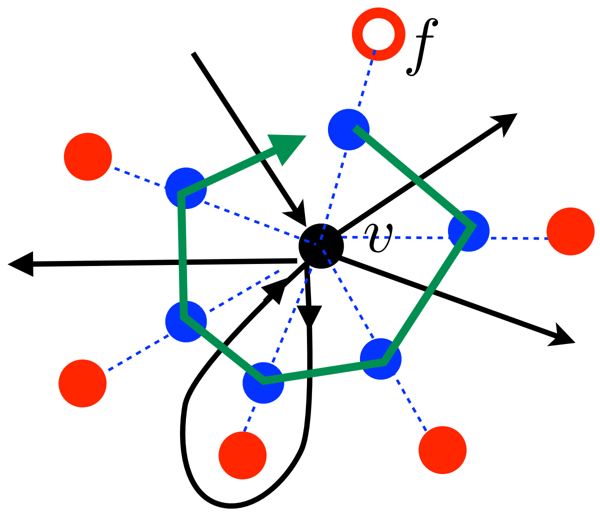

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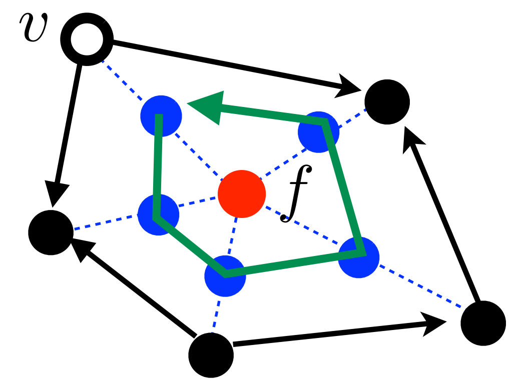

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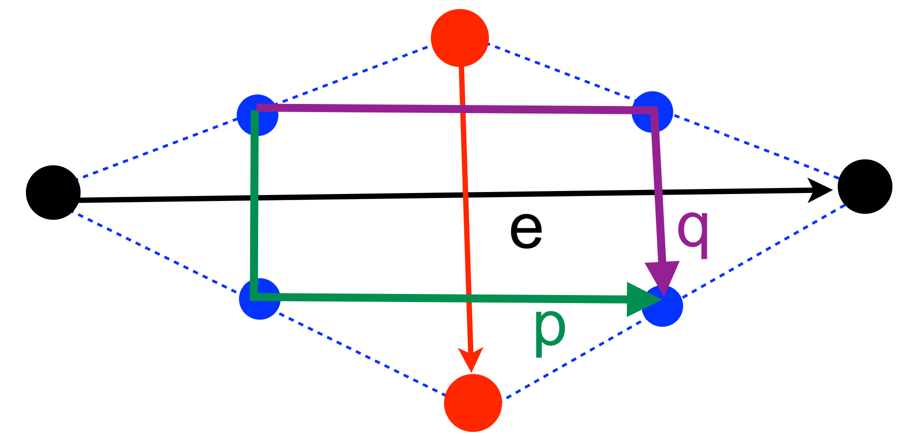

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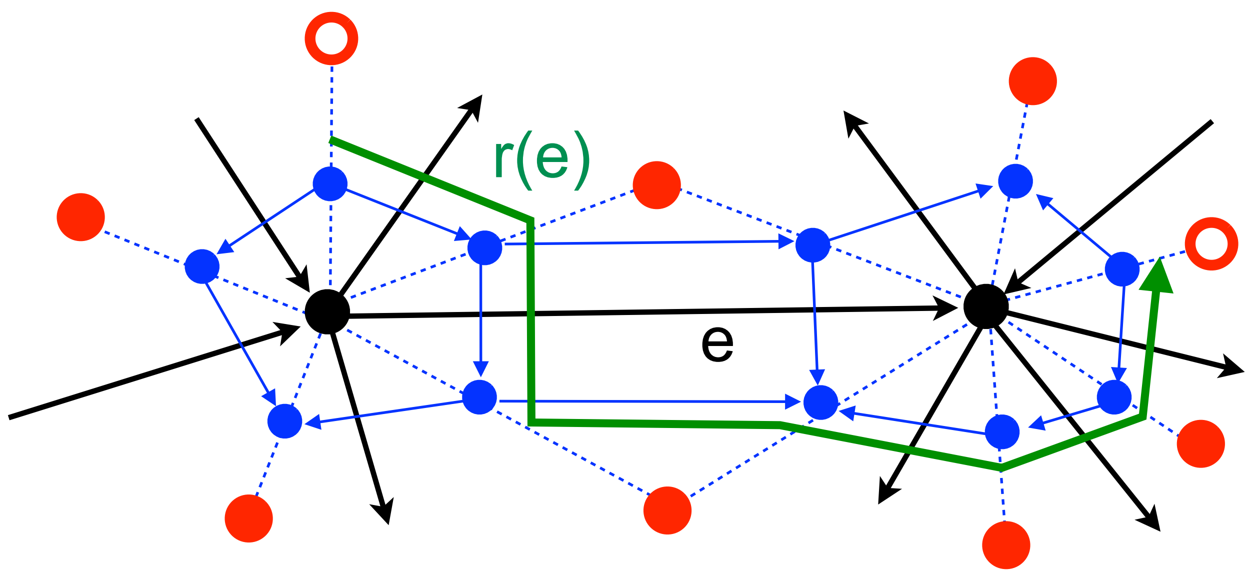

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- this is the $K^{\otimes V}$-module algebra of the Hopf algebra gauge theory for $K = D(H)$

3. Conclusions and Outlook

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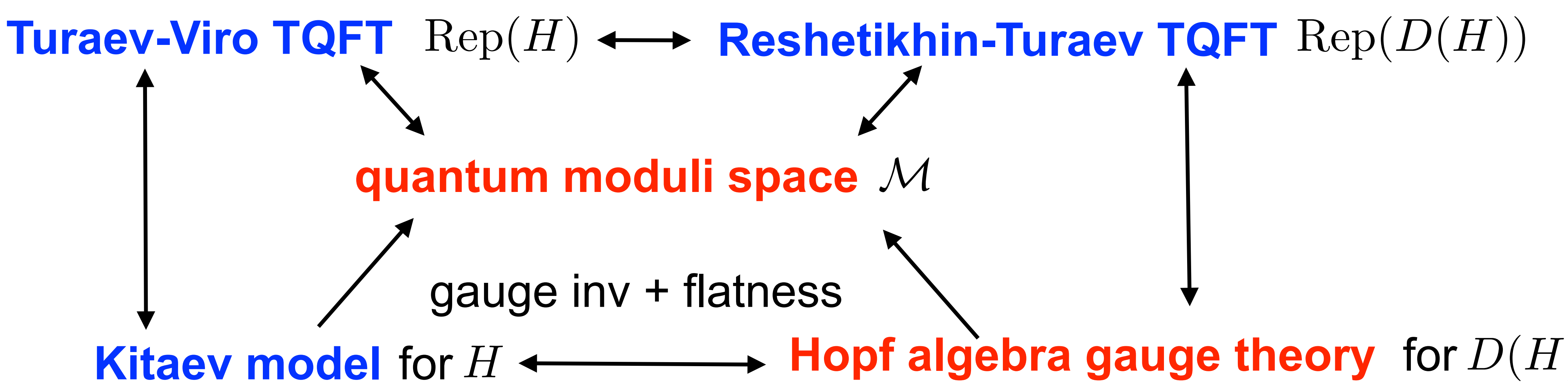

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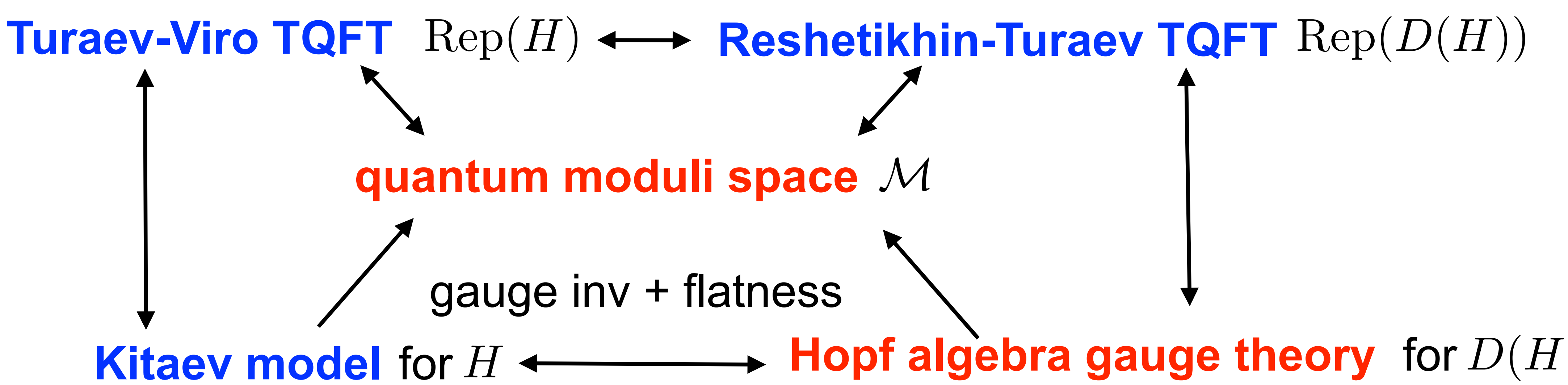

To do: • investigate defects or excitations

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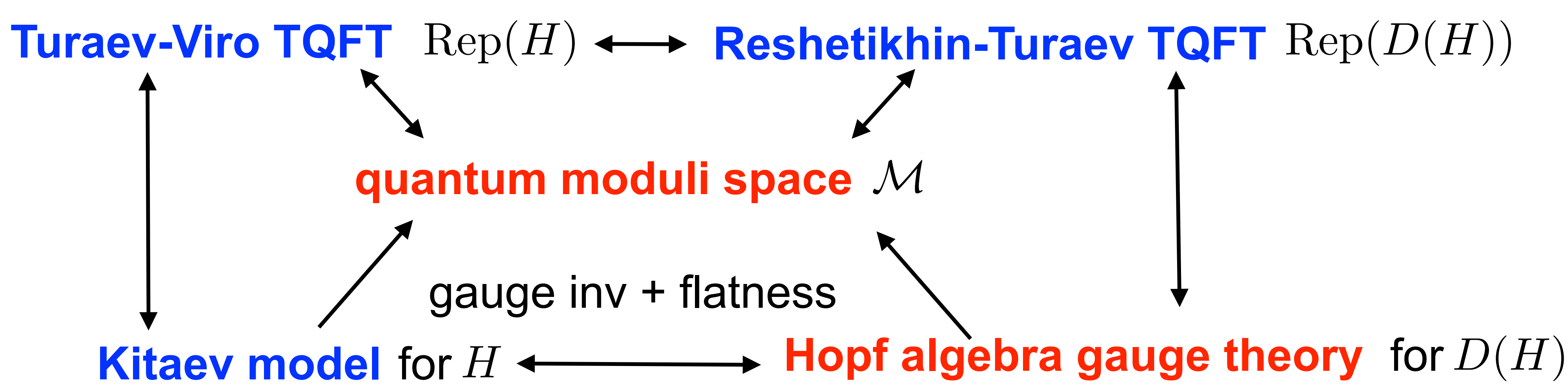

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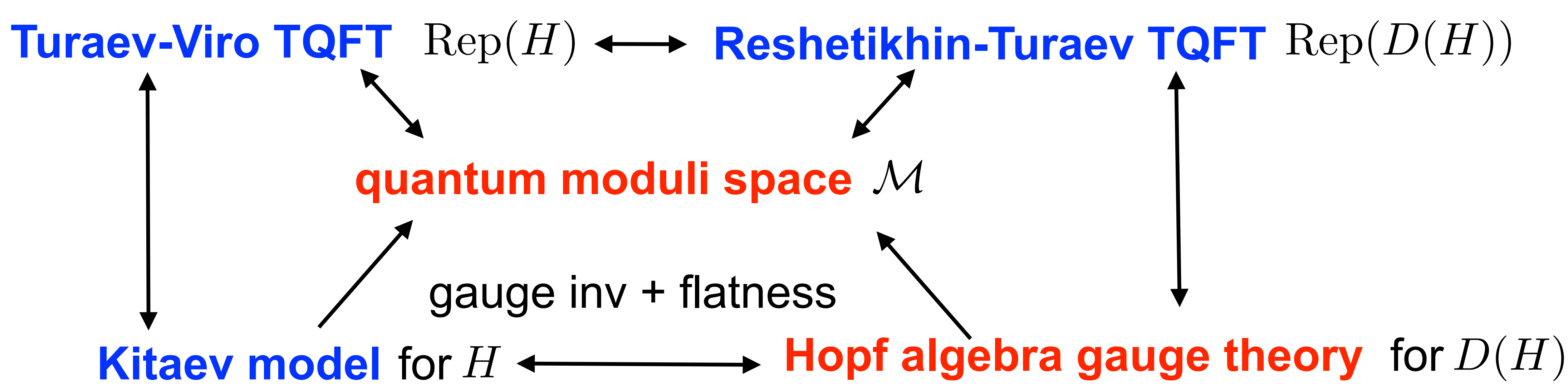

To do:
- investigate defects or excitations
- Poisson geometrical analogues
- weakening of axioms to include non-associativity → what mathematical structures?