Black holes, Boyle's Law and the Quark-Gluon plasma

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Outline

Temperature and entropy

- Schwarzschild black hole:
  $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2$, $f(r) = 1 - \frac{2GM}{r}$.
- Event horizon: $(r_h) = 0 \quad \Rightarrow \quad r_h = 2GM.$ $(c = 1)$
- Area: $A = 16\pi GM^2$
- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$, $(\ell_{Pl}^2 = \hbar G)$ Bekenstein (1972)
- Surface gravity: $\kappa = \frac{1}{4GM}$
- Temperature: $T = \frac{\kappa\hbar}{2\pi}$ Hawking (1974)

Hawking temperature

$$T = \frac{\hbar}{8\pi GM}$$

Solar mass black hole: $T = 6 \times 10^{-8}\ K$

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First Law

- Internal energy: identify $M = U(S)$, $T = \frac{\partial U}{\partial S} \quad \Rightarrow$

$$dM = T\,dS$$

- With $S \propto \frac{A}{\hbar} \propto \frac{16\pi M^2}{\hbar}$ and $T = \frac{\hbar}{8\pi M}$: $S = \frac{1}{4}\frac{A}{\hbar}$ $\qquad (G = 1)$

- More generally: angular momentum J, electric charge Q

First law of black hole thermodynamics

$$dM = T\,dS + \Omega dJ + \Phi dQ$$

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Homogeneous scaling

- Ordinary thermodynamics in d dimensions:
  $U(S, V, n_i)$ is a function of extensive variables

  (n_i = number of moles)

$$\lambda^d U(S,V,n) = U(\lambda^d S, \lambda^d V, \lambda^d n_i)$$

$$\Rightarrow \quad U = S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n_i \frac{\partial U}{\partial n_i} \quad \text{Euler equation}$$

$$\Rightarrow \quad U = ST - VP + n_i\mu_i \quad (\mu_i = \text{chemical potential})$$

$$\Rightarrow \quad G(T,P,n) = U + VP - ST = n_i\mu_i \quad \text{(Gibbs-Duhem relation)}$$

Lieb+Yngvason, *Phys. Rep.* **310** (1991)

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Smarr relation

- Rotating black hole (D space-time dimensions, $Q = 0$):
  $M \to \lambda^{D-3} M$, $S \to \lambda^{D-2} S$, $\mathbf{J} \to \lambda^{D-2} \mathbf{J} \quad \Rightarrow$

$$\begin{aligned} \lambda^{D-3} M(S, J) &= M(\lambda^{D-2} S, \lambda^{D-2} \mathbf{J}) \\ \Rightarrow (D-3)M &= (D-2) S \frac{\partial M}{\partial S} + (D-2) \mathbf{J} . \frac{\partial M}{\partial \mathbf{J}} \\ \Rightarrow (D-3)M &= (D-2) ST + (D-2) \mathbf{J} . \boldsymbol{\Omega} \end{aligned}$$

Smarr (1973)

- Naïve relation fails in asymptotically AdS space-time
- Cosmological constant is another dimensionful parameter

$$\Theta := \frac{\partial M}{\partial \Lambda}$$

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- $\Lambda \to \lambda^{-2} \Lambda \quad \Rightarrow$

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- Include cosmological constant Λ, contributes pressure P, energy density $\epsilon = -P = \frac{\Lambda}{8\pi}$
- Thermal energy
  $U = M + \epsilon V = M - PV \quad \Rightarrow \quad M = U + PV$
  Legendre transform $U = U(S, V)$

Enthalpy

$$M = U + PV = H(S, P)$$

(Kastor, Ray+Traschen [0904.2765])

Thermodynamic volume and the First Law

$$dU = T\,dS\ - P\,dV, \qquad V = \left.\frac{\partial M}{\partial P}\right|_S$$

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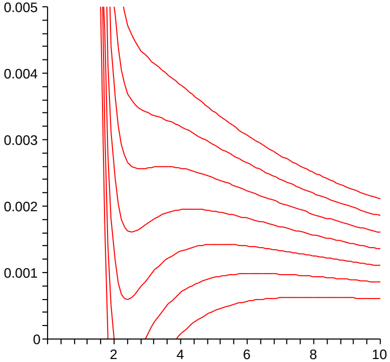
Rotating black holes: AdS Kerr



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Critical point, mean field exponents (van der Waals gas)

Caldarelli, Gognola+Klemm [hep-th/9908022];

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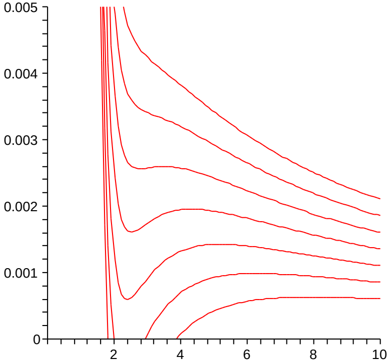

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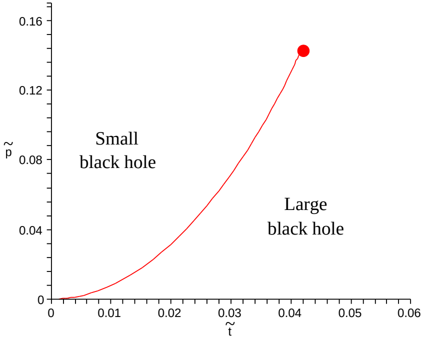
Phase diagram



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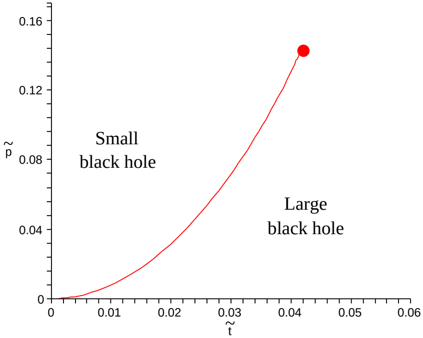

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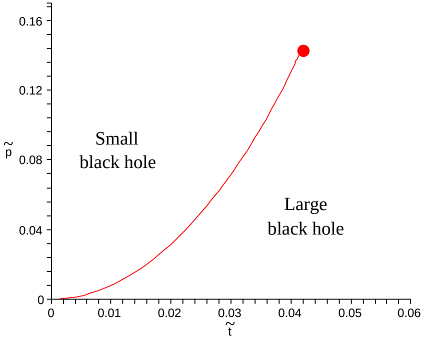

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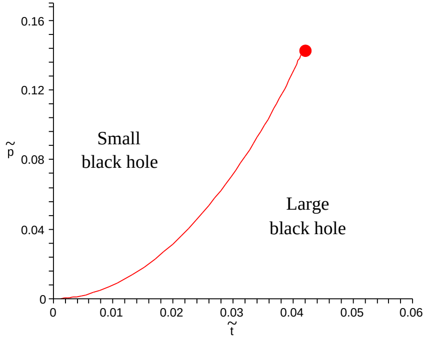

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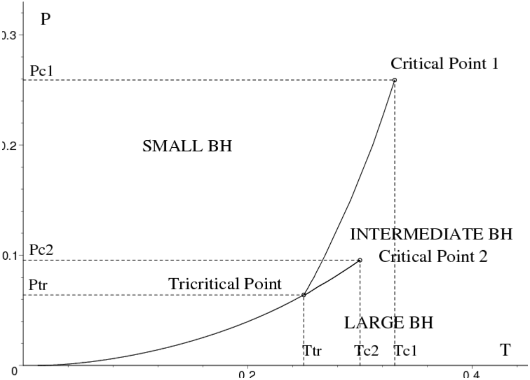
$T - P$ phase diagram
Higher dimensions

- 6-Dimensions: two angular momenta J_1 and J_2, phase diagram depends on the ratio $q = \frac{J_1}{J_2}$



Altimirano, Kubizňák, Mann+Sherkatghanad, [1401.2586]

AdS/CFT

- Weak gravity in the bulk (classical GR, $\ell_{Pl} \to 0$)
  $\Leftrightarrow$ strongly coupled CFT on the boundary.
- $10 - D$ superstring compactified on $AdS_5 \times S^5$, $\Lambda = -\frac{4}{L^2}$.
- CFT: $\mathcal{N} = 4$ SUSY $SU(N)$ Yang-Mills with $N = \frac{\pi^2 L^4}{\sqrt{2}\ell_{Pl}^4}$.
  - Vary $\Lambda \Rightarrow$ vary N;
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Yang-Mills at finite temperature

- Asymptotically AdS black hole in 5-D, with charge Q and entropy

$$S = \frac{\pi^2 r_h^3}{2G_5} = \pi N^2 \left(\frac{r_h}{L}\right)^3, \qquad \frac{1}{G_5} \sim \frac{L^5}{G_{10}} \sim \frac{N^2}{L^3}.$$

- Thermal energy of quark-gluon plasma:

$$U(S,V,Q) = M(r_h, L, Q) = \underbrace{\frac{3N^2}{4L}x^2(1+x^2)}_{\text{gluons}} + \underbrace{\frac{Q^2}{8L^2x^2}}_{\text{quarks}}$$

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QCD phase transition

- Hawking-Page phase transition in bulk
  $\Leftrightarrow$ deconfining phase transition in QCD on the boundary.
  Witten, hep-th/9802150, 9803131
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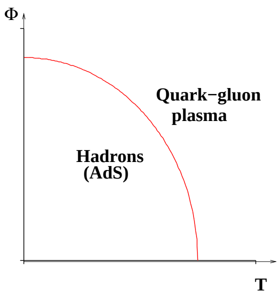
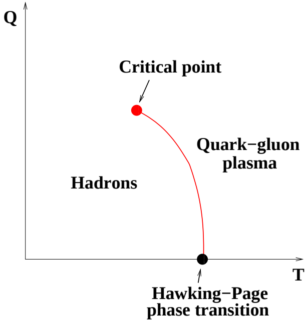
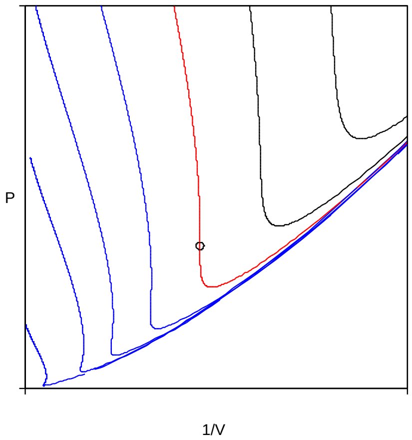




Chamblin, Emparan, Johnson and Myers, hep-th/9902170

Isotherms in $P - V$ plane



Critical exponents

$$p = \frac{P - P_{crit}}{P_{crit}}, \quad v = \frac{V - V_{crit}}{V_{crit}}, \quad t = \frac{T - T_{crit}}{T_{crit}}.$$

- $C_P = T / \left.\frac{\partial T}{\partial S}\right|_{P,Q} \propto |t|^{-\alpha}$;
- At fixed $v < 0$, $\Delta p = p_> - p_< \propto |t|^{\beta}$;
- Inverse isothermal compressibility, $-V \left(\frac{\partial P}{\partial V}\right)_{T,Q} \propto |t|^{-\gamma}$;
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$$\alpha = 0 \qquad \beta = \frac{1}{2} \qquad \gamma = 1 \qquad \delta = 3$$

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Thermodynamic Volume

- Define the thermodynamic volume $(P \geq 0,\ \Lambda \leq 0)$

$$V = \left.\frac{\partial H}{\partial P}\right|_{S,J,Q}$$

- AdS Schwarzschild: $V = \frac{4\pi}{3} r_h^3$
  Kastor, Ray+Traschen [0904.2765]
- Higher dimensions: $V = \frac{\Omega_d}{d+1} r_h^d$ BPD [1008.5023]
- AdS Myers-Perry Cvetic, Gibbons+Kubizňák [1012.2888]
- Rotating black hole in 4-D $(Q = 0)$: BPD [1106.6260]
  - with $V_0 = \frac{r_h}{3} A$,

$$V = V_0 + \frac{4\pi}{3} \frac{J^2}{M}$$

  - gives a reverse isoperimetric inequality
  - as $J \to 0$: $V \to \frac{4\pi}{3} r_h^3$, $S = \pi r_h^2$: isentropic $\Leftrightarrow$ isovolumetric

Reverse isoperimetric inequality

- Thermodynamic volume:

$$V = \left(\frac{\partial H}{\partial P}\right)_S$$

- Non-rotating black hole, $V = \frac{4\pi}{3} r_h^3$
  (Kastor, Ray+Traschen [0904.2765]; BPD [1008.5023])
- More generally, define $V_0 = \frac{r_h}{3} A$

Reverse iso-perimetric inequality

$$V = V_0 + \frac{4\pi}{3}\frac{J^2}{M} \qquad \Rightarrow \qquad \frac{A}{V} \leq \frac{A}{V_0}$$

$$\left(\frac{3V}{4\pi}\right)^{1/3} \geq \left(\frac{S}{\pi}\right)^{1/2}$$

Critical behaviour

- Critical point ($J \neq 0$):

  Caldarelli, Gognola+Klemm [hep-th/9908022]

- Define

$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4)$$

cf. van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$

Critical exponents

- $C_V = T / \left.\frac{\partial T}{\partial S}\right|_{V,J} \propto |t|^{-\alpha}$;
- At fixed $p < 0$, $v_> - v_< \propto |t|^{\beta}$;
- Isothermal compressibility, $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,J} \propto |t|^{-\gamma}$;
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Mean Field Exponents

$$\alpha = 0 \qquad \beta = \frac{1}{2} \qquad \gamma = 1 \qquad \delta = 3$$

Same as van der Waals gas

Non-zero charge, zero J

Reissner-Nordström — Anti-de Sitter ($J = 0,\ Q \neq 0$):

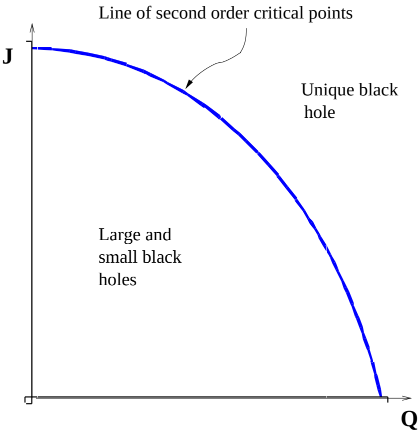
- Critical behaviour same as van der Waals
  Champlin, Emparan, Johnson+Myers:
  [hep-th/9902170]; [hep-th9904197]
- Equation of state:

$$p = \frac{8}{3}t - \frac{8}{9}tv - \frac{4}{81}v^3 + o(t^2, tv^2, v^4),$$

  Critical exponents are mean field

Kubizňák+Mann [arXiv:1205.0559]

Kerr-Reissner-Nordström-AdS



Reissner-Nordström anti-de Sitter $(J \neq 0,\ Q \neq 0)$

Caldarelli, Gognola+Klemm [hep-th/9908022]

Compressibility

BPD [arXiv:1308.5403]

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_{S,J}$
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

- $0 \leq \kappa < \infty$

Compressibility

BPD [arXiv:1109.0198]

- For $J = 0$, $\kappa = 0$ (incompressible)
- 4-D: κ is greatest for J_{max} $(T = 0)$
- e.g. $P = 0$, $\kappa_{max} = \frac{4\pi M^2}{9} = 2.6 \times 10^{-38} \left(\frac{M}{M_\odot}\right)^2 \; m\, s^2\, kg^{-1}$.
  cf. neutron star, $M \approx M_\odot$, $R \approx 10\, km$,
  degenerate Fermi gas $\Rightarrow \kappa \approx 10^{-34} m\, s^2\, kg^{-1}$
  Black holes have a very stiff equation of state!
- $\rho = \frac{M}{V}$, "speed of sound" $v_s^{-2} = \left.\frac{\partial \rho}{\partial P}\right|_{S,J}$,

"Speed of Sound"

$$v_s^{-2} = 1 + \frac{(2\pi J)^4}{\left(2S^2 + (2\pi J)^2\right)^2}$$

$\Rightarrow \quad \frac{1}{2} \le v_s^2 \le 1$, with $v_s = 1$ for $J = 0$