

CORFU WORKSHOP 2015

**CERTAIN ASPECTS OF NON-ASSOCIATIVE
STRUCTURES IN PHYSICS**

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Based on joint work with Dieter Lüst [arXiv:1309.3172, 1505.04004]



Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο



ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
ΕΚΠΑΙΔΕΥΣΗ ΚΑΙ ΔΙΑ ΒΙΟΥ ΜΑΘΗΣΗ

επένδυση στην κοινωνία της γνώσης

ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ ΚΑΙ ΘΡΗΣΚΕΥΜΑΤΩΝ
ΕΙΔΙΚΗ ΥΠΗΡΕΣΙΑ ΔΙΑΧΕΙΡΙΣΗΣ

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης



OUTLINE

- Emergence of NC/NA in closed string theory vacua. [in brief]
- Magnetic field analogue of non-geometry in Maxwell-Dirac theory
 - NC/NA of momenta and the fate of angular symmetry.
[classical and quantum]
 - NC/NA is distributed, not localized, and it also closely related to breakdown of integrability of charged point-particle motion.
- Deformation theory and cohomology
 - 3-cocycles in Lie algebra and group cohomology.
 - NC/NA star-product as substitute for canonical quantization.
 - Universal characterization of obstructions.
- NC/NA structures for integrability in $2 + 1$ dimensions. [in brief]

TOROIDAL FLUX MODELS

It has been established that non-geometric closed string backgrounds exhibit non-commutativity/non-associativity among their coordinates.

The prime example is provided by backgrounds originating from torus T^3 with constant H -flux

$$H_3 = H dX^1 \wedge dX^2 \wedge dX^3$$

satisfying the standard quantization condition

$$\frac{1}{4\pi^2} \int H = k, \quad k \in \mathbb{Z}.$$

Choose the components of the anti-symmetric B -field, $H_3 = dB$, as

$$B_{12} = Hx^3, \quad B_{23} = 0 = B_{31}$$

and perform T-duality, successively, thinking of T^3 as T^n fibration over the base space T^{3-n} for $n = 0, 1, 2, 3$.

- **H -flux model**, which is the original geometric background ($n = 0$). The toroidal coordinates are commuting,

$$[x^i, x^j] = 0.$$

- **f -flux model**, which follows by performing T-duality in x^1 -direction $T_{x^1}^1$ ($n = 1$). The resulting background is the Heisenberg nilmanifold (**twisted torus**) and it is fully geometric. It exhibits the relations

$$[x^1, \tilde{x}^2] \sim f\tilde{p}^3.$$

- **Q-flux model**, which follows by performing T-duality in x^1 and x^2 directions T_{x^1, x^2}^2 ($n = 2$). The resulting background is geometric only locally but not globally, since the fibre is glued by T-duality when transported around the base. It exhibits the commutation relations

$$[x^1, x^2] \sim Q\tilde{p}^3.$$

- The geometry has already become **non-commutative**.

- **R-flux model**, which follows by performing T-duality in all directions T_{x^1, x^2, x^3}^3 ($n = 3$). The resulting background is entirely non-geometric and it exhibits the commutation relations

$$[x^1, x^2] \sim Rp^3.$$

- The geometry has become **non-associative**.

It is more convenient to work with symmetric choice of B -field

$$B_{12} = \frac{H}{3}x^3, \quad B_{23} = \frac{H}{3}x^1, \quad B_{31} = \frac{H}{3}x^2.$$

Then, the commutation relations among x^i and \tilde{x}^i take the form

$$\mathbf{H} : \quad [x^i, x^j] = 0, \quad [x^i, \tilde{x}^j] = 0, \quad [\tilde{x}^i, \tilde{x}^j] = iH\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{f} : \quad [x^i, x^j] = 0, \quad [\tilde{x}^i, \tilde{x}^j] = 0, \quad [x^i, \tilde{x}^j] = if\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{Q} : \quad [\tilde{x}^i, \tilde{x}^j] = 0, \quad [x^i, \tilde{x}^j] = 0, \quad [x^i, x^j] = iQ\epsilon^{ijk}\tilde{p}^k,$$

$$\mathbf{R} : \quad [\tilde{x}^i, \tilde{x}^j] = 0, \quad [x^i, \tilde{x}^j] = 0, \quad [x^i, x^j] = iR\epsilon^{ijk}p^k.$$

MAGNETIC FIELD ANALOGUE OF NC/NA

A spinless point-particle (\mathbf{e}, \mathbf{m}) in magnetic field background $\vec{B}(\vec{x})$ has commutation relations among its coordinates and momenta:

$$[x^i, p^j] = i\delta^{ij}, \quad [x^i, x^j] = 0, \quad [p^i, p^j] = ie \epsilon^{ijk} B_k(\vec{x})$$

leading to non-commutativity of p^i in Maxwell theory, $\vec{\nabla} \cdot \vec{B} = 0$.

In Dirac's generalization of Maxwell theory we have $\vec{\nabla} \cdot \vec{B} \neq 0$ and

$$[[p^i, p^j], p^k] + \text{cyclic} \equiv [p^i, p^j, p^k] = -e \epsilon^{ijk} \vec{\nabla} \cdot \vec{B} \neq 0$$

Associativity of momenta is lost in the presence of magnetic charges.

This provides a simple model for NC/NA of string theory with $x^i \leftrightarrow p^i$.

Consider a continuous spherically symmetric distribution of magnetic charge in space, $\rho(x)$, to study (some of) the implications of NC/NA in classical and quantum theory. Setting $x^2 = \vec{x} \cdot \vec{x}$, we have

$$\vec{\nabla} \cdot \vec{B} = \rho(x), \quad \vec{\nabla} \times \vec{B} = 0 \quad (\text{static}).$$

The particular solution of the inhomogeneous equation is expressed as

$$\vec{B}(\vec{x}) = \frac{\vec{x}}{f(x)}, \quad \rho(x) = \frac{3f(x) - xf'(x)}{f^2(x)}.$$

Some notable example are:

- $f(x) = x^3/g$ so that $\rho(x) = 4\pi g \delta(x)$ [**Dirac monopole with charge g**]
- $f(x) = 3/\rho$ so that $\rho(x) = \rho$ is constant and $\vec{B}(\vec{x}) = \rho \vec{x}/3$ [**cf R-flux**]

Study the dynamics of point-particle for general profile function $\mathbf{f}(\mathbf{x})$.

Using the Hamiltonian $H = \vec{p} \cdot \vec{p}/2m$, the Lorentz force acting on the spinless particle (e, m) in the magnetic field background is

$$\frac{d\vec{p}}{dt} = i[H, \vec{p}] = \frac{e}{2m}(\vec{p} \times \vec{B} - \vec{B} \times \vec{p})$$

which for $\vec{B}(\vec{x}) = \vec{x}/f(x)$ takes the special form

$$\frac{d^2\vec{x}}{dt^2} = -\frac{e}{mf(x)} \left(\vec{x} \times \frac{d\vec{x}}{dt} \right).$$

Lorentz force is proportional to angular momentum and does no work.

Energy conservation provides one integral of motion, $E = A/2m$.

Simple manipulation shows that $x^2(t) = At^2 + D$, setting $x^2(0) = D$.

[D provides the closest distance to the origin (perihelion of trajectory)]

Complete integrability requires three more integrals of motion.

For general choices of profile function $f(x)$, however, there are no additional integrals, since

$$\frac{d}{dt} \left(\vec{x} \times \frac{d\vec{x}}{dt} \right) = -\frac{e}{mf(x)} \vec{x} \times \left(\vec{x} \times \frac{d\vec{x}}{dt} \right) = \frac{e x^3}{mf(x)} \frac{d\hat{x}}{dt}.$$

Angular symmetry is broken in the presence of magnetic charges!

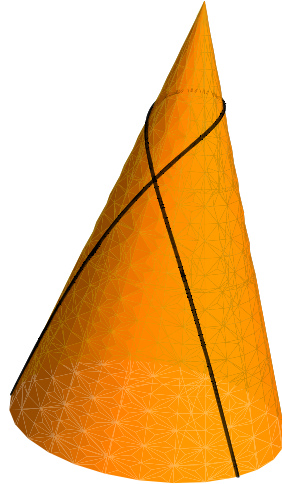
The only exception is the Dirac monopole having $f(x) = x^3/g$. In this case, the improved angular momentum $\vec{J} = m\vec{K}$ is conserved, where

$$\vec{K} \equiv \vec{x} \times \frac{d\vec{x}}{dt} - \frac{eg}{m} \hat{x}$$

is the celebrated **Poincaré vector**.

In all other case, including constant $f(x)$, angular symmetry is broken and the classical motion of the particle appears to be non-integrable.

Trajectory of a spinless particle in the field of a magnetic monopole:
the charged particle (e, m) precesses with angular velocity $\vec{K}/(At^2 + D)$



- The magnetic monopole g is located at the tip of the cone
- The Poincaré vector \vec{K} provides the axis of the cone
- $\vec{K} \cdot \hat{x} = -eg/m$ determines the opening angle of the cone

ANOTHER LOOK AT ANGULAR SYMMETRY

Try to follow as closely as possible the conventional definitions and algebraic structures of particle dynamics, without assuming particular representations nor Hilbert space [only that \vec{p} acts as derivation].

Assume that \vec{x} and \vec{p} form a complete and irreducible set of observables for the point-particle in a static magnetic field $\vec{B}(\vec{x})$.

Angular momentum \vec{J} ought to satisfy the algebraic relations

$$[J^i, x^j] = i\epsilon^{ijk} x^k, \quad [J^i, p^j] = i\epsilon^{ijk} p^k, \quad [J^i, J^j] = i\epsilon^{ijk} J^k$$

so that angular momentum is conserved, $[H, J^i] = 0$, in the background of any spherically symmetric magnetic field $\vec{B}(\vec{x})$.

Let \vec{J} be the orbital angular momentum, plus an improvement term that accounts for the angular momentum of the electromagnetic field

$$\vec{J} = \vec{x} \times \vec{p} - \vec{C}.$$

Then, we obtain the following conditions for \vec{C}

$$[x^i, C^j] = 0, \quad [p^i, C^j] = ie \left(x^i B^j - \delta^{ij} (\vec{x} \cdot \vec{B}) \right),$$

$$C^i = ex^i (\vec{x} \cdot \vec{B}) + \frac{i}{2} \epsilon^{ijk} [C^j, C^k].$$

The only consistent solution corresponds to the magnetic field of a Dirac monopole, in which case $\vec{J} = \vec{x} \times \vec{p} - eg \hat{x}$ [Poincaré vector].

Non-associativity is responsible for the violation of angular symmetry.

The apparent violation of non-associativity in a Dirac monopole field is eliminated by imposing the boundary conditions $\Psi(0) = 0$ on the wave-functions so that \vec{p} (derivations) are represented by self-adjoint operators, even though they are defined in patches as $\vec{p} = -i\nabla - e\vec{A}$.

Rotations by an angle θ around an axis \hat{n} (take $\hat{n} = \hat{x}$) are described by

$$R(\hat{n} = \hat{x}, \theta) = e^{-i\theta \hat{x} \cdot \vec{J}} = e^{-ieg \theta}.$$

Then, for a point-particle in a monopole field, single valuedness of R (up to a sign) yields Dirac's quantization condition $eg = n \in \mathbb{Z}$ ($\times \hbar/2$). Finite translations in space also associate when eg is quantized.

In all other cases, non-associativity is for real, **obstructing canonical quantization**. What can be used as substitute? \longrightarrow **star-product**.

DEFORMATIONS AND COHOMOLOGY

First, we focus on the case of constant magnetic charge density ρ for which we have the basic commutation relations

$$[p^i, p^j] = i \hbar e \frac{\rho}{3} \epsilon^{ijk} x_k, \quad [x^i, p^j] = i \hbar \delta^{ij}, \quad [x^i, x^j] = 0.$$

They should be compared to the commutation relations of the so called parabolic ***R-flux model*** whose coordinates and momenta satisfy

$$[x^i, x^j] = i \hbar \frac{1}{T^2} R \epsilon^{ijk} p_k, \quad [x^i, p^j] = i \hbar \delta^{ij}, \quad [p^i, p^j] = 0$$

where T is the string tension, $c = 1$, and R is a 3-form constant flux. Associator/Jacobiator does not vanish when \hbar and ρ are not zero

$$[p^1, p^2, p^3] = [[p^1, p^2], p^3] + \text{cycl. perm.} = -e \hbar^2 \rho.$$

Different contractions of commutation relations of constant ρ model:

$$\hbar = 0, \quad \text{any } \rho: \quad [p^i, p^j] = 0, \quad [x^i, p^j] = 0 \quad (\text{Algebra of translations } \mathfrak{t}_6)$$

$$\hbar \neq 0, \quad \rho = 0: \quad [p^i, p^j] = 0, \quad [x^i, p^j] = i\hbar\delta^{ij} \quad (\text{Heisenberg algebra } \mathfrak{g})$$

whereas in all cases the coordinates commute, $[x^i, x^j] = 0$.

The algebra of a charged point-particle moving in the magnetic field background $\vec{B} \sim \vec{x}$ is a deformation of the Lie algebras above.

Lie algebra cohomology characterizes the deformation which leads to non-associativity — Chevalley-Eilenberg cohomology

- cochains of Abelian algebra \mathfrak{t}_6 with real values — $H^*(\mathfrak{t}_6, \mathbb{R})$
- cochains of Heisenberg algebra \mathfrak{g} with values in \mathfrak{g} — $H^*(\mathfrak{g}, \mathfrak{g})$

$H^*(\mathfrak{t}_6, \mathbb{R})$: Let $T_I = x^i, p^i$ the generators of \mathfrak{t}_6 . Consider a 3-cochain with $c_3(p^1, p^2, p^3) = 1$, up to normalization, and $c_3(T_I, T_J, T_K) = 0$ for all other choices of generators (i.e., when at least one T is x). We have

$$[T_I, T_J, T_K] \sim c_3(T_I, T_J, T_K)$$

and, thus, only the associator $[p^1, p^2, p^3]$ does not vanish.

The obstruction satisfies the 3-cocycle condition $dc_3(T_I, T_J, T_K, T_L) = 0$, since for any four elements of \mathfrak{t}_6 we have

$$c_3([T_I, T_J], T_K, T_L) - c_3([T_I, T_K], T_J, T_L) + c_3([T_I, T_L], T_J, T_K) + c_3([T_J, T_K], T_I, T_L) - c_3([T_J, T_L], T_I, T_K) + c_3([T_K, T_L], T_I, T_J) = 0$$

- c_3 is not a coboundary, i.e., $c_3 \neq df_2$ in $H^*(\mathfrak{t}_6, \mathbb{R})$.

$H^*(\mathfrak{g}, \mathfrak{g})$: Let $c_2(p^i, p^j) = \epsilon^{ijk} x^k$, up to a multiplicative constant, and $c_2(x^i, p^j) = 0 = c_2(x^i, x^j)$. Acting with the coboundary operator, we obtain

$$dc_2(p^1, p^2, p^3) = -c_2([p^1, p^2], p^3) + c_2([p^1, p^3], p^2) - c_2([p^2, p^3], p^1) \\ + \pi(p^1)c_2(p^2, p^3) - \pi(p^2)c_2(p^1, p^3) + \pi(p^3)c_2(p^1, p^2)$$

where $\pi(\mathfrak{g}) = \text{Ad}_{\mathfrak{g}} = [\mathfrak{g}, \cdot]$. Then, for the Heisenberg algebra, we have

$$dc_2(p^1, p^2, p^3) = [p^1, c_2(p^2, p^3)] - [p^2, c_2(p^1, p^3)] + [p^3, c_2(p^1, p^2)]$$

leading to alternative cohomological interpretation of non-associativity

$$[p^1, p^2, p^3] \sim dc_2(p^1, p^2, p^3)$$

- The cohomological interpretation depends on the module (\mathbb{R} vs \mathfrak{g}).

LIE GROUP COHOMOLOGY

Exponentiate the action of the position and momentum generators.
The formal group elements

$$U(\vec{a}, \vec{b}) = e^{i(\vec{a}\cdot\vec{x} + \vec{b}\cdot\vec{p})}$$

satisfy the composition law, obtained by applying the **BCH** formula,

$$U(\vec{a}_1, \vec{b}_1)U(\vec{a}_2, \vec{b}_2) = e^{-\frac{i}{2}(\vec{a}_1\cdot\vec{b}_2 - \vec{a}_2\cdot\vec{b}_1)} e^{-i\frac{R}{2}(\vec{b}_1 \times \vec{b}_2)\cdot\vec{x}} U(\vec{a}_1 + \vec{a}_2, \vec{b}_1 + \vec{b}_2).$$

Successive composition of any three group elements $U_i = U(\vec{a}_i, \vec{b}_i)$ yields

$$(U_1 U_2) U_3 = e^{-i\frac{R}{2}(\vec{b}_1 \times \vec{b}_2)\cdot\vec{b}_3} U_1 (U_2 U_3)$$

which do not associate when $R \sim e\rho \neq 0$, setting $\hbar = 1$.

If R were zero, we would have a projective representation of the Abelian group of translations in phase space. The phase factor

$$\varphi_2(\vec{a}_1, \vec{b}_1; \vec{a}_2, \vec{b}_2) = \vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1$$

is a **real-valued 2-cocycle** in group cohomology, satisfying

$$d\varphi_2(\vec{b}_1, \vec{b}_2, \vec{b}_3) \equiv \varphi_2(\vec{b}_2, \vec{b}_3) - \varphi_2(\vec{b}_1 + \vec{b}_2, \vec{b}_3) + \varphi_2(\vec{b}_1, \vec{b}_2 + \vec{b}_3) - \varphi_2(\vec{b}_1, \vec{b}_2) = 0$$

and, thus, it does not show up in the associator, as in ordinary QM.

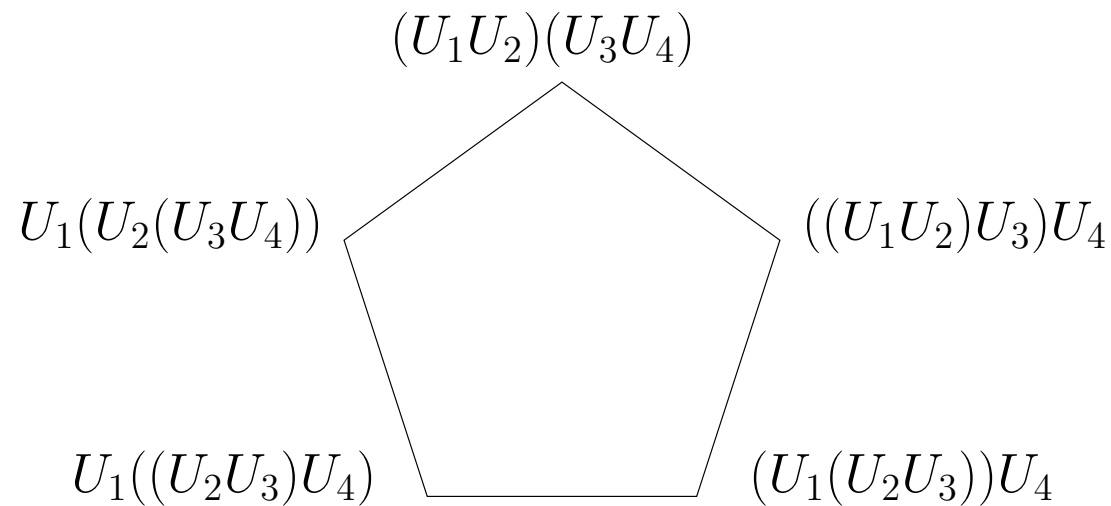
If $R \neq 0$, there is an **additional x -dependent** factor in the composition law that gives rise to a phase in the associator of three group elements

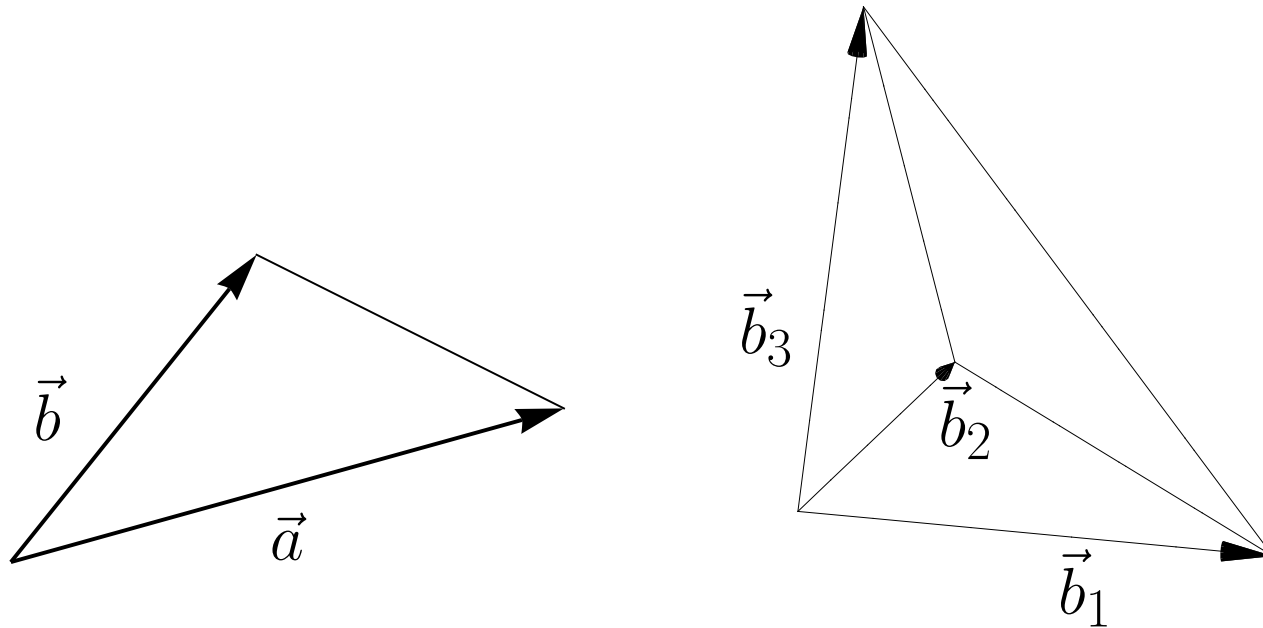
$$\varphi_3(\vec{b}_1, \vec{b}_2, \vec{b}_3) = (\vec{b}_1 \times \vec{b}_2) \cdot \vec{b}_3$$

The new phase is a **real-valued 3-cocycle** in the cohomology of the Abelian group of translations in phase space, satisfying

$$d\varphi_3(\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) \equiv \varphi_3(\vec{b}_2, \vec{b}_3, \vec{b}_4) - \varphi_3(\vec{b}_1 + \vec{b}_2, \vec{b}_3, \vec{b}_4) + \varphi_3(\vec{b}_1, \vec{b}_2 + \vec{b}_3, \vec{b}_4) - \varphi_3(\vec{b}_1, \vec{b}_2, \vec{b}_3 + \vec{b}_4) + \varphi_3(\vec{b}_1, \vec{b}_2, \vec{b}_3) = 0 .$$

A schematic representation is provided by Mac Lane's pentagon:





Geometric interpretation of the non-trivial cocycles φ_2 and φ_3 :

$$\text{Area}(\vec{a}, \vec{b}) = \frac{1}{2} |\vec{a} \times \vec{b}|$$

$$\text{Volume}(\vec{b}_1, \vec{b}_2, \vec{b}_3) = \frac{1}{6} |(\vec{b}_1 \times \vec{b}_2) \cdot \vec{b}_3|$$

ALTERNATIVE INTERPRETATION of non-associativity is provided by the cohomology of Heisenberg-Weyl group with cochains taking values in the Heisenberg algebra. Introducing the elements

$$U_W(g) = e^{i(\vec{a}\cdot\vec{x}+\vec{b}\cdot\vec{p}+c\mathbf{1})}$$

the group composition law $U(\vec{a}_1, \vec{b}_1)U(\vec{a}_2, \vec{b}_2)$ takes the form

$$U_W(g_1) U_W(g_2) = e^{-i\frac{R}{2}\varphi_2(g_1, g_2)} U_W(g_1 g_2)$$

where $\varphi_2(g_1, g_2) = (\vec{b}_1 \times \vec{b}_2) \cdot \vec{x}$ takes values in the Heisenberg algebra. Then, the obstruction to associativity assumes the **coboundary** form

$$(\vec{b}_1 \times \vec{b}_2) \cdot \vec{b}_3 = d\varphi_2(g_1, g_2, g_3).$$

Group cohomology can be employed to define a NC/NA \star -product.

THE STAR PRODUCT

When $R = 0$, all classical observables $f(x, p)$ are assigned to operators $\hat{F}(\hat{x}, \hat{p})$ acting on Hilbert space \mathcal{H} . Their product is non-commutative but associative.

An equivalent description is provided by **Moyal star-product** in phase space: Fourier analyse

$$f(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \tilde{f}(\vec{a}, \vec{b}) e^{i(\vec{a} \cdot \vec{x} + \vec{b} \cdot \vec{p})}$$

and apply Weyl's correspondence rule to assign self-adjoint operators

$$\hat{F}(\hat{x}, \hat{p}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \tilde{f}(\vec{a}, \vec{b}) \hat{U}(\vec{a}, \vec{b})$$

where

$$\hat{U}(\vec{a}, \vec{b}) = e^{i(\vec{a} \cdot \hat{x} + \vec{b} \cdot \hat{p})}.$$

The product of any two operators takes the form

$$\hat{F}_1 \cdot \hat{F}_2 = \frac{1}{(2\pi)^6} \int d^3 a_1 d^3 b_1 d^3 a_2 d^3 b_2 \tilde{f}_1(\vec{a}_1, \vec{b}_1) \tilde{f}_2(\vec{a}_2, \vec{b}_2) \hat{U}(\vec{a}_1, \vec{b}_1) \hat{U}(\vec{a}_2, \vec{b}_2)$$

and it can be worked out using the composition law

$$\hat{U}(\vec{a}_1, \vec{b}_1) \hat{U}(\vec{a}_2, \vec{b}_2) = e^{-\frac{i}{2}(\vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1)} \hat{U}(\vec{a}_1 + \vec{a}_2, \vec{b}_1 + \vec{b}_2).$$

The 2-cocycle $\varphi_2(\vec{a}_1, \vec{b}_1; \vec{a}_2, \vec{b}_2) = \vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1$ makes the product of the corresponding phase space functions non-commutative but associative.

The result turns out to be

$$(f_1 \star f_2)(\vec{x}, \vec{p}) = e^{\frac{i}{2}(\vec{\nabla}_{x_1} \cdot \vec{\nabla}_{p_2} - \vec{\nabla}_{x_2} \cdot \vec{\nabla}_{p_1})} f_1(\vec{x}_1, \vec{p}_1) f_2(\vec{x}_2, \vec{p}_2) |_{\vec{x}_1 = \vec{x}_2 = \vec{x}; \vec{p}_1 = \vec{p}_2 = \vec{p}}$$

giving rise to the series expansion

$$(f_1 \star f_2)(\vec{x}, \vec{p}) = (f_1 \cdot f_2)(\vec{x}, \vec{p}) + \frac{i}{2} \{f_1, f_2\} + \dots$$

Non-commutative geometry: the notion of point becomes fuzzy.

Quantum dynamics is equivalently described by the Moyal bracket

$$\{\{f_1, f_2\}\} \equiv -i(f_1 \star f_2 - f_2 \star f_1) = \{f_1, f_2\} + \text{higher derivatives}$$

acting as derivation

$$\{\{f_1, f_2 \star f_3\}\} = f_2 \star \{\{f_1, f_3\}\} + \{\{f_1, f_2\}\} \star f_3.$$

When $R \neq 0$, the rules of canonical quantization do not apply, but it is still possible to define a star-product **non-commutative/non-associative**.

We follow the same line of thought as before, assigning to $f(\vec{x}, \vec{p})$

$$F(\vec{x}, \vec{p}) = \frac{1}{(2\pi)^3} \int d^3a d^3b \tilde{f}(\vec{a}, \vec{b}) U(\vec{a}, \vec{b}),$$

and using the generalized composition law,

$$U(\vec{a}_1, \vec{b}_1) U(\vec{a}_2, \vec{b}_2) = e^{-\frac{i}{2}(\vec{a}_1 \cdot \vec{b}_2 - \vec{a}_2 \cdot \vec{b}_1)} e^{-i\frac{R}{2}(\vec{b}_1 \times \vec{b}_2) \cdot \vec{x}} U(\vec{a}_1 + \vec{a}_2, \vec{b}_1 + \vec{b}_2).$$

The result is the NC/NA **x -dependent** star-product

$$(f_1 \star_x f_2)(\vec{x}, \vec{p}) = e^{i\frac{R}{2} \vec{x} \cdot (\vec{\nabla}_{p_1} \times \vec{\nabla}_{p_2})} e^{\frac{i}{2} (\vec{\nabla}_{x_1} \cdot \vec{\nabla}_{p_2} - \vec{\nabla}_{x_2} \cdot \vec{\nabla}_{p_1})} f_1(\vec{x}_1, \vec{p}_1) f_2(\vec{x}_2, \vec{p}_2) |_{\vec{x}_1 = \vec{x}_2 = \vec{x}; \vec{p}_1 = \vec{p}_2 = \vec{p}}.$$

The substitute for quantum dynamics is provided by the bracket

$$\{\{f_1, f_2\}\}_x \equiv -i(f_1 \star_x f_2 - f_2 \star_x f_1)$$

which **does not** act as derivation, i.e.,

$$\{\{f_1, f_2 \star_x f_3\}\}_x \neq f_2 \star_x \{\{f_1, f_3\}\}_x + \{\{f_1, f_2\}\}_x \star_x f_3.$$

A related result is that the associator/Jacobiator does not vanish

$$\{\{f_1(p), f_2(p), f_3(p)\}\}_x \neq 0.$$

- Symmetries appear to be broken as consequence of non-associativity like the breakdown of angular symmetry discussed earlier.

FURTHER GENERALIZATIONS

In the more general case, $[p^1, p^2, p^3] = -e\hbar^2 \vec{\nabla} \cdot \vec{B}$ is not constant. Static spherically symmetric fields $\vec{B} = \vec{x}/f(x)$, with $x^2 = \vec{x} \cdot \vec{x}$, yield

$$\vec{\nabla} \cdot \vec{B} = \rho(x) = \frac{3f(x) - xf'(x)}{f^2(x)}.$$

Since $[p^i, p^j] = i\hbar e \epsilon^{ijk} B_k(\vec{x})$, one can still think of the basic commutation relations as deformation of the Heisenberg algebra \mathfrak{g} by a 2-cochain

$$c_2(p^i, p^j) \sim \epsilon^{ijk} B_k(\vec{x})$$

taking values in the space of (say) local smooth functions of x , which is a \mathfrak{g} -module, and let c_2 be zero otherwise. As before, we have

$$\begin{aligned}
dc_2(p^1, p^2, p^3) &= -c_2([p^1, p^2], p^3) + c_2([p^1, p^3], p^2) - c_2([p^2, p^3], p^1) \\
&\quad + \pi(p^1)c_2(p^2, p^3) - \pi(p^2)c_2(p^1, p^3) + \pi(p^3)c_2(p^1, p^2).
\end{aligned}$$

For the Heisenberg algebra the momenta act as derivation and they commute, so the computation results to

$$\begin{aligned}
dc_2(p^1, p^2, p^3) &= [p^1, c_2(p^2, p^3)] - [p^2, c_2(p^1, p^3)] + [p^3, c_2(p^1, p^2)] = \\
&= -i\frac{\partial}{\partial x^1}c_2(p^2, p^3) + i\frac{\partial}{\partial x^2}c_2(p^1, p^3) - i\frac{\partial}{\partial x^3}c_2(p^1, p^2) = -i\vec{\nabla} \cdot \vec{B},
\end{aligned}$$

while dc_2 vanishes identically for all other entries with one or more x .

This provides a **universal** cohomological interpretation of NC/NA, as

$$[p^1, p^2, p^3] \sim dc_2(p^1, p^2, p^3).$$

The cohomological characterization of NC/NA can also be considered at group level and use it to define the star-product for more general distribution of magnetic charge, $\rho(x)$.

Question: Is there a one-to-one correspondence between magnetic field backgrounds and non-geometric closed string models? **[under $x \leftrightarrow p$]**

- A way to go about it is to find the closed string analogue of a Dirac monopole and use it to engineer more general string models for given distribution ρ .

NON-ASSOCIATIVITY FOR INTEGRABILITY

Let me end with a brief discussion of yet another (less well known) aspect of NC/NA that works for rather than against integrability.

A number of **two-dimensional** models in statistical mechanics have been exactly solved using the so called **triangle** (or **Yang-Baxter**) relations. Schematically, they are of the following form,

$$R_{ab} R_{ac} R_{bc} = R_{bc} R_{ac} R_{ab}$$

and they can be regarded as the associativity condition for an L -operator algebra

$$L_{1,a} L_{1,b} R_{ab} = R_{ab} L_{1,b} L_{1,a} .$$

The triangle relations are the conditions for the row-to-row transfer matrices to commute. Alternatively, these models can be put into field theoretic form by considering the transfer matrix that adds a single face to the lattice and regarding this as an S -matrix. Then, the triangle relations become the condition for the S -matrix to factorize.

These structures can be generalized to **three-dimensional models** of statistical mechanics and field theory. Then, one has the so called **tetrahedron** (or **Zamolodchikov**) equations

$$R_{abc} R_{ade} R_{bdf} R_{cef} = R_{cef} R_{bdf} R_{ade} R_{abc}$$

regarded as compatibility condition of the L -operator algebra

$$L_{12,a} L_{13,b} L_{23,c} R_{abc} = R_{abc} L_{23,c} L_{13,b} L_{12,a} .$$

The tetrahedron equations are the integrability conditions for the transfer matrices of statistical models to commute and the S -matrix of field theory models to be factorizable in three dimensions.

[there are also higher simplex equations in higher dimensions]

Theorem (Baez-Crans, 2003): Any semi-strict Lie 2-algebra provides a solution of the tetrahedron equations, just as any Lie algebra provides a solution of the triangle relations. [arXiv:0307263 [math.QA]]

Bottom line: These diverse results provide common ground for using **homotopy associative algebras** in string theory, Maxwell-Dirac theory, as well in higher dimensional integrable systems that is worth exploring further by putting them all together.

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THANK YOU!