Extended κ-deformations and extended kappa-Minkowski spacetimes

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- One of the approaches to the theory describing Planck scale is to consider deformations of relativistic symmetries and noncommutative spacetimes.
- The κ-Poincaré algebra was obtained by the contraction of q-deformed Anti de Sitter Hopf algebra $U_q(o(3,2))$ [J. Lukierski, A. Nowicki, H. Ruegg, V. N. Tolstoy '91];
- The κ-Minkowski space was introduced as a quantum space covariant under the action of κ-Poincaré Hopf algebra[S. Majid, H. Ruegg '94; S. Zakrzewski '94];
- Quantum deformations for Lorentz and Poincaré symmetries have been classified in terms of classical r-matrices [S. Zakrzewski '96-7];
- Parallel to Zakrzewski the dual matrix quantum group version of Poincaré Hopf algebras were classified [P. Podles, S. Woronowicz '96];

- Various physical applications of this mathematical framework have been broadly investigated [G. Amelino-Camelia, M. Arzano, P. Aschieri, M. Dimitrijevic, G. Fiore, L. Jonke, J.Kowalski-Glikman, J. Lukierski, S. Majid, S. Meljanac, J. Wess, M. Woronowicz].

I will focuse on

- real forms
- general covarince
- unfied description
- specialization
- universal form of $\kappa-$Minkowski quantum space
- twisted extensions and new quantum Minkowski spaces

Preliminaries

- Let V be a D-dimensional (real) vector space, with a metric tensor g of arbitrary signature.
  For an arbitrary basis $\{e_\mu\}_{\mu=0}^{D-1}$ the metric components are $g_{\mu\nu} = g\left(e_\mu, e_\nu\right)$.
  In the dual basis $\{e^\mu\}_{\mu=0}^{D-1}$ in the dual vector space $V^\#$ one can write $g = g_{\alpha\beta} e^\alpha \otimes e^\beta$.
- The (special) orthogonal group of g we denote as $SO\left(g\right) = \{\Lambda \in GL(V) : \Lambda^T g \Lambda = g, \det \Lambda = 1\}$ and the corresponding inhomogeneous orthogonal group as $ISO\left(g\right)$ - Poincaré group.

- The Lie algebra of $ISO(g)$ consists of $\frac{1}{2}D(D+1)$ generators $(M_{\mu\nu}, P_\alpha)$ satisfying the standard commutation relations

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i(g_{\mu\lambda}M_{\nu\rho} - g_{\nu\lambda}M_{\mu\rho} + g_{\nu\rho}M_{\mu\lambda} - g_{\mu\rho}M_{\nu\lambda}), \\
[M_{\mu\nu}, P_\rho] &= i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \quad , \qquad [P_\mu, P_\lambda] = 0.
\end{aligned}
$$

- The relation with the basis $\{e_\mu\}_{\mu=0}^{D-1}$ of V is through the (complex) vector representation

$$M_{\mu\nu} \mapsto -i(g_{\mu\alpha}e_\nu - g_{\nu\alpha}e_\mu) \otimes e^\alpha \in EndV \otimes \mathbb{C}$$

- The generators $(M_{\mu\nu}, P_\alpha)$ can be treated as determining the real form of the complex Lie algebra $iso(g)$.

- The original κ-deformation of Poincaré algebra corresponds to the following classical r-matrix $r = M_{0i} \wedge P^i$.
- The more general family was found (for any non-zero vector τ and any metric tensor $g_{\mu\nu}$) by Zakrzewski:

$$r_{(\tau,g)} = \tau^\alpha M_{\alpha\mu} \wedge P^\mu \equiv \tau^\alpha g^{\beta\sigma} M_{\alpha\beta} \wedge P_\sigma \equiv \frac{1}{2}\tau \llcorner \Omega_g \in \wedge^2 iso(g)$$

  where $\Omega_g = M_{\mu\nu} \wedge P^\mu \wedge P^\nu$ is the only invariant element in $\wedge^3 iso(g)$.
- Additional vector field parametrizes classical $r-$matrices. Non-equivalent deformations are labelled by the corresponding type of stability subgroups of τ.
- We are considering the Drinfeld-type quantization of inhomogeneous orthogonal groups determined by a metric tensor of an arbitrary signature living in a spacetime of arbitrary dimension.

The Schouten bracket reads: $\left[\left[r_{(\tau,g)}, r_{(\tau,g)}\right]\right] = -\tau_g^2\,\Omega_g$ where $\tau_g^2 \equiv \tau^\mu \tau_\mu \equiv g_{\mu\nu}\tau^\mu\tau^\nu$ is the scalar square of τ w.r.t. the metric g. This implies two possibilities:

1. $\tau_g^2 \neq 0$ - the r-matrix satisfies MYBE (Modified Yang-Baxter Equation) $<->$ providing the standard (a.k.a. Drinfeld-Jimbo) quantization with the quasi-triangular quantum $R-$matrix.
   In this case there exists a basis providing $1 + (D-1)$ **orthogonal decomposition**.
   The stability subgroup of τ_g is a homogeneous orthogonal group $SO(g_{ij})$ in $D-1$ dimensions.
   The signature of the metric g_{ij} indicates the orbit type.
2. $\tau_g^2 = 0$ (provided non-Euclidean signature) with r_τ satisfying CYBE (Classical Yang-Baxter equation) $<->$ the non-standard (a.k.a. twisted) triangular deformation.
   Light-cone basis provides $2 + (D-2)$ **orthogonal decomposition**.
   The signature of the metric g_{ab} indicates the orbit type.
   The stability subgroup is an inhomogeneous orthogonal group $ISO(g_{ab})$ in $D-2$ dim in this case.

$\kappa\,(\tau)$-deformed (inhomogeneous) orthogonal Lie algebra

$U\,(iso\,(g))\,[[\frac{1}{\kappa}]]$ denoted as $U_{\kappa,\tau}\,(iso\,(g))$- besides the standard orthogonal Lie algebra structure, has deformed coalgebraic sector:

$$\begin{aligned}
\Delta_\tau\,(P_\mu) &= P_\mu \otimes \Pi_\tau + 1 \otimes P_\mu - \frac{\tau_\mu}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes P_\alpha - \frac{\tau_\mu}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes P_\tau \\
\Delta_\tau\,(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} + \frac{1}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes (\tau_\nu M_{\alpha\mu} - \tau_\mu M_{\alpha\nu}) \\
&- \frac{1}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes (\tau_\mu M_{\tau\nu} - \tau_\nu M_{\tau\mu})
\end{aligned}$$

where $P_\tau = \tau^\mu P_\mu, M_{\tau\lambda} = \tau^\alpha M_{\alpha\lambda}$, $\tau_\mu = g_{\alpha\mu}\tau^\mu$. where

$$\begin{aligned}
\Pi_\tau &= \frac{1}{\kappa} P_\tau + \sqrt{1 + \frac{\tau^2}{\kappa^2} C} \quad , \quad \Pi_\tau^{-1} = \frac{\sqrt{1 + \frac{\tau^2}{\kappa^2} C} - \frac{1}{\kappa} P_\tau}{1 + \frac{1}{\kappa^2}\,(\tau^2 C - P_\tau^2)} \\
\tau^2 C_\tau &= \kappa^2 \left(\Pi_\tau + \Pi_\tau^{-1} - 2 + \frac{1}{\kappa^2} \left(\tau^2 C - P_\tau^2 \right) \Pi_\tau^{-1} \right) = 2\kappa^2 \left(\sqrt{1 + \frac{\tau^2}{\kappa^2} C} - 1 \right)
\end{aligned}$$

as formal power series in $\frac{1}{\kappa}$.
$C \equiv P^\alpha P_\alpha = g^{\alpha\beta} P_\alpha P_\beta$ is the Casimir element.
For the case $\tau^2 = 0$ one should take $C_\tau = C$.

- Simultaneous re-scaling of τ and κ by the same factor does not change coproducts involving these symbols, so it can be treated as an isomorphism of the corresponding Hopf algebras, i.e.

$$U_{\kappa,\tau}\left(iso\left(g\right)\right) \cong U_{\lambda\kappa,\lambda\tau}\left(iso\left(g\right)\right)$$

Thus one finds that the vector τ can be normalized to the values $\tau^2 = \pm 1, 0$.

This unified description has the **general covariance** manifested via tensorial character of all defining formulas.

- Consider a change of basis in V: $e_\mu \mapsto \tilde{e}_\mu = A^\alpha_\mu e_\alpha$ by a non-degenerate matrix $A^\beta_\alpha \in GL(D, \mathbb{R})$.
- Considering the new generators

$$\tilde{P}_\alpha = A^\mu_\alpha P_\mu, \qquad \tilde{M}_{\alpha\beta} = A^\mu_\alpha A^\nu_\beta M_{\mu\nu}$$

  together with $\tilde{g}_{\alpha\beta} = A^\mu_\alpha A^\nu_\beta g_{\mu\nu}$ and $\tilde{\tau}_\alpha = A^\mu_\alpha \tau_\mu$.
  Note that $\tilde{\tau}^\alpha = (A^{-1})^\alpha_\mu \tau^\mu$ and therefore $P_{\tilde{\tau}} = P_\tau$.
- $U_{\kappa,\tau}(iso\,(g)) \cong U_{\kappa,\tilde{\tau}}(iso\,(\tilde{g}))$ as Hopf algebras.
- In particular, if $A^\beta_\alpha \in O(g)$ then $g_{\alpha\beta} = \tilde{g}_{\alpha\beta}$ (internal authomorphism).
  This fact is important for possible physical applications and interpretations.
- This transformation does not change the metric signature.

- The universal formulas describe κ-Poincare Hopf algebra not only in different Lie algebra basis induced by different basis in the underlying vector space V but also **provide the different types of deformations**.

- This can be seen from

$$\lim_{\kappa\to\infty} \kappa(\Delta_\tau - \Delta_\tau^{op})(X) = [\Delta_0(X)\, , r_\tau]$$

relating deformed coproducts with the corresponding classical $r-$matrices, where $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ denotes primitive (undeformed) coproduct for $X \in iso(g)$ and Δ^{op} stands for the opposite coproduct with flipped legs.

- The right hand side defines cobracket determining Lie bialgebra structures on $iso(g)$.
Therefore our coproducts can be considered as their quantization.

The orthogonal $D = 1 + (D-1)$ decomposition

- For $\tau^2 \neq 0$, one can assume $\tau^\mu = (1, 0, \ldots, 0)$, without a loss of generality, by the choice of the suitable basis $(e_\mu)_{\mu=0}^{D-1}$ in the vector space V with $e_0 = \tau$ and $(e_i)_{i=1}^{D-1}$ being orthogonal to τ : $g_{00} = \tau^2$; $\quad g_{0i} = g(e_0, e_i) = 0$.
- This provides the orthogonal decomposition $(V, g_{\mu\nu}) \cong (\mathbb{R}, g_{00}) \times (V^{D-1}, g_{ij})$.
- The (D-1) dimensional metric g_{ij} does not need to be in the diagonal form.

Contracting the universal formula for coproducts with τ^μ yields

$$\begin{aligned}
\Delta(P_\tau) &= P_\tau \otimes \Pi_\tau + \Pi_\tau^{-1} \otimes P_\tau - \frac{\tau^2}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes P_\alpha - \frac{\tau^2}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes P_\tau \\
\Delta_\tau (M_{\tau\nu}) &= M_{\tau\nu} \otimes 1 + 1 \otimes M_{\tau\nu} + \frac{1}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes \left(\tau_\nu M_{\alpha\tau} - \tau^2 M_{\alpha\nu} \right) - \\
&- \frac{\tau^2}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes M_{\tau\nu}
\end{aligned}$$

In the corresponding Lie algebra basis $\{P_\tau, P_i, M_{\tau i}, M_{ij}\}$ one can choose the new system of generators $\rightarrow \{\tilde{P}_\tau, \tilde{P}_i, M_{\tau i}, M_{ij}\}$ with

$$\tilde{P}_\tau \doteq \kappa \ln \Pi_\tau, \qquad \tilde{P}_i \doteq P_i \Pi_\tau^{-1} \quad \Rightarrow \quad \Pi_\tau = e^{\frac{\tilde{P}_\tau}{\kappa}}$$

which provides the deformed coproducts in the bicrossproduct (a.k.a. Majid-Ruegg) basis

$$\Delta_\kappa\left(\tilde{P}_\tau\right) = 1 \otimes \tilde{P}_\tau + \tilde{P}_\tau \otimes 1, \qquad \Delta_\kappa(M_{ij}) = 1 \otimes M_{ij} + M_{ij} \otimes 1 \quad (1)$$

$$\Delta_\kappa\left(\tilde{P}_i\right) = \exp(-\frac{\tilde{P}_\tau}{\kappa}) \otimes \tilde{P}_i + \tilde{P}_i \otimes 1 \quad (2)$$

$$\Delta_\kappa(M_{\tau j}) = M_{\tau j} \otimes 1 + \exp(-\frac{\tilde{P}_\tau}{\kappa}) \otimes M_{\tau j} - \frac{1}{\kappa}\tau^2 \tilde{P}^k \otimes M_{kj} \quad (3)$$

The algebraic relations in this basis are

$$[M_{\tau i}, \tilde{P}_\tau] = -i\tau^2 \tilde{P}_i, [M_{ij}, \tilde{P}_k] = i(g_{jk}\tilde{P}_i - g_{ik}\tilde{P}_j), [M_{ij}, \tilde{P}_\tau] = 0 \quad (4)$$

$$[M_{\tau i}, \tilde{P}_j] = \frac{i}{2}\kappa g_{ij}\left(1 - \exp(-\frac{2\tilde{P}_\tau}{\kappa}) - \frac{\tau^2}{\kappa^2}\tilde{P}_i\tilde{P}^i\right) + \frac{i\tau^2}{\kappa}\tilde{P}_j\tilde{P}_i \quad (5)$$

Comments:

$->$ The Lie algebra of the stability group G_τ consist of the elements $\{M_{ij}\}$ for which the coproduct remains primitive.

$->$ The expressions (1)-(5) cover all the standard $\kappa-$deformations; for the Lorentzian signature they describe both the time-like and the space-like quantizations.

The light-like deformation and the 2+(D-2) decomposition

[A. Ballesteros, F.J. Herranz, M.A. Olmo, M. Santander (1995)]

- In this case, i.e. when $\tau^2 = 0$, one deals with the non-Euclidean geometry $ISO(p,q)$; $p, q \neq 0$.
  We use the "light-cone" Poincaré generators:

  $$P_\mu = (P_+ \, , P_- \, , P_a) \, , \; M_{\mu\nu} = (M_{+-} \, , M_{+\,a} \, , M_{-\,a} \, , M_{ab}) \, , a, b = 1, 2$$

  as a basis in the Lie algebra $iso(g_{p,q})$.

- We have to decompose the space $V^D = V^2 \times V^{D-2}$, by a suitable choice of basic vectors, into the orthogonal product of the two-dimensional Lorentzian space $\{V^2, g_{AB}\}$ with a $D-2$ dimensional one $\{V^{D-2}, g_{ab}\}$: $(A, B = +, -)$, $(a, b = 1, 2 \ldots D-2)$.

- The total metric $g_{\mu\nu} = g_{AB} \times g_{ab}$ becomes a product metric. We choose $g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in its anti-diagonal (light-cone) form as well as two null-vectors $\tau^\mu \equiv \tau_+^\mu = (1, 0, \ldots 0)$, $\tilde{\tau}^\mu \equiv \tau_-^\mu = (0, 1, 0 \ldots 0)$: $\tau_+ \tau_- = 1$ in order to obtain the convenient light-cone basis in the space of the Lie algebra generators.

- The total metric $g_{\mu\nu} = g_{AB} \times g_{ab}$ becomes a product metric. We choose $g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in its anti-diagonal (light-cone) form as well as two null-vectors $\tau^\mu \equiv \tau^\mu_+ = (1, 0, \ldots 0)$, $\tilde{\tau}^\mu \equiv \tau^\mu_- = (0, 1, 0 \ldots 0)$: $\tau_+ \tau_- = 1$ in order to obtain the convenient light-cone basis in the space of the Lie algebra generators.

This algebra consists of the following (non-vanishing) commutators:

$$
\begin{aligned}
[M_{+\,a}\,, M_{-\,b}] &= -i\,(M_{ab} + g_{ab} M_{+\,-}) \quad , \qquad [M_{\pm\,a}\,, M_{\pm\,b}] = 0 \\
[M_{\pm\,a}\,, M_{b\,c}] &= i\,(g_{ab} M_{\pm\,c} - g_{a\,c} M_{\pm\,b}) \quad , \qquad [M_{+\,-}\,, M_{\pm\,a}] = \pm i M_{\pm\,a} \\
[M_{+\,-}, P_\pm] &= \pm i\, P_\pm \quad , \qquad [M_{\pm\,a}\,, P_b] = i g_{ab} P_\pm \\
[M_{\pm\,a}\,, P_\pm] &= [M_{+\,-}\,, P_a] = 0 \quad , \qquad [M_{\pm\,a}\,, P_\mp] = -\, i P_a
\end{aligned}
$$

together with the standard commutation relations within the $D-2$ dimensional sector $(M_{a\,b}\,, P_a\,, g_{ab})$.

The universal formula for the coalgebra structure, in this case, reduces to

$$\begin{aligned}
\Delta_\tau(M) &= M\otimes 1 + 1\otimes M \qquad \text{for} \qquad M\in\{M_{+\,a}\,, M_{ab}\}\\
\Delta_\tau(P) &= P\otimes \Pi_+ + 1\otimes P \qquad \text{for} \qquad P\in\{P_+\,, P_a\}\\
\Delta_\tau(P_-) &= P_-\otimes\Pi_+ + \Pi_+^{-1}\otimes P_- - \frac{1}{\kappa}\left(P_- + \frac{1}{2\kappa}C_+\right)\Pi_+^{-1}\otimes P_+\\
&- \frac{1}{\kappa}P^a\Pi_+^{-1}\otimes P_a\\
\Delta_\tau(M_{+\,-}) &= M_{+\,-}\otimes 1 + \Pi_+^{-1}\otimes M_{+\,-} - \frac{1}{\kappa}P^a\,\Pi_+^{-1}\otimes M_{+\,a}\\
\Delta_\tau(M_{-\,a}) &= M_{-\,a}\otimes 1 + \Pi_+^{-1}\otimes M_{-\,a} - \frac{1}{\kappa}\left(P_- + \frac{1}{2\kappa}C_+\right)\Pi_+^{-1}\otimes M_{+\,a}\\
&- \frac{1}{\kappa}P^b\Pi_+^{-1}\otimes M_{ba}
\end{aligned}$$

where $\Pi_+ \doteq 1 + \frac{1}{\kappa}P_+$ and $\left(1 - \frac{1}{\kappa}P_+\Pi_+^{-1}\right) = \left(\Pi_+ - \frac{1}{\kappa}P_+\right)\Pi_+^{-1} = \Pi_+^{-1}$ and C_+ is still to be determined.

- The **Lie subalgebra corresponding to the stability group** of τ_+ consists of $iso(p-1, q-1) = gen\{M_{a\,b}, M_{+\,b}\}$, i.e. the generators with the primitive coproducts.

- The classical $r-$matrix corresponding to the vector τ_+ reads

$$r_{LC} = M_{+\,-} \wedge P_+ + M_{+\,a} \wedge P^a$$

  Since $\tau_+^2 = 0$ it satisfies the CYB equation and generates **the non-standard (triangular) deformation**.

- Its construction involves two Abelian $D-1$ dimensional subalgebras $\Gamma_+ = gen\{M_{+\,-}\ , P^a\}$ and $\Gamma_- = gen\{P_+\ , M_{+\,a}\}$ satisfying certain cross-commutation relations.

The corresponding twisting element (extended Jordanian twist) has the following form:

$$\begin{aligned} \mathcal{F} &= \exp\left(-i \ln \Pi_+ \otimes M_{+\,-}\right) \exp\left(-\frac{i}{\kappa} P^a \Pi_+^{-1} \otimes M_{+\,a}\right) \\ &= \exp\left(-\frac{i}{\kappa} P^a \otimes M_{+\,a}\right) \exp\left(-i \ln \Pi_+ \otimes M_{+\,-}\right) \end{aligned}$$

Calculating coproducts directly from the twist:

$$\Delta_{LC}(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1} = \Delta_\tau\,(X)$$

And $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ is a triangular quantum $R-$matrix.

Specialization of the deformation parameter κ to a numerical value

One can consider sub-Hopf algebra generated by elements $\left(M_{ij}, P_i, M_{\tau i}, \Pi_\tau, \Pi_\tau^{-1}\right)$

$$\begin{aligned}
\Pi_\tau \Pi_\tau^{-1} &= 1 = \Pi_\tau \Pi_\tau^{-1} \quad , \\
[P_i, \Pi_\tau] &= [M_{ij}, \Pi_\tau] = 0 \quad , \quad [M_{\tau i}, \Pi_\tau] = -\frac{i}{\kappa} P_i \quad , \\
[M_{\tau i}, P_j] &= i g_{ij} \frac{\kappa}{2} \left(\Pi_\tau - \Pi_\tau^{-1} \left(1 + \frac{\tau^2}{\kappa^2} P^m P_m \right) \right).
\end{aligned}$$

Commutators with Π_τ^{-1} can be easily calculated from the above (e.g. $\left[M_{\tau i}, \Pi_\tau^{-1}\right] = \frac{\imath}{\kappa} P_i \Pi_\tau^{-2}$).

$$\begin{aligned}
\Delta_\tau(\Pi_\tau) &= \Pi_\tau \otimes \Pi_\tau \quad , \quad \Delta_\tau(\Pi_\tau^{-1}) = \Pi_\tau^{-1} \otimes \Pi_\tau^{-1} \\
\Delta_\tau(P_i) &= P_i \otimes \Pi_\tau + 1 \otimes P_i \quad , \qquad i, j = 1, \ldots, D-1 \\
\Delta_\tau(M_{ij}) &= M_{ij} \otimes 1 + 1 \otimes M_{ij} \\
\Delta_\tau(M_{\tau i}) &= M_{\tau i} \otimes 1 + \Pi_\tau^{-1} \otimes M_{\tau i} + \frac{\tau^2}{\kappa} P^j \Pi_\tau^{-1} \otimes M_{ij}
\end{aligned}$$

Further we can eliminate κ by setting $\kappa = 1$ or re-scaling $P_i \rightarrow \frac{1}{\kappa} P_i$.

κ-Minkowski algebra

$\mathcal{M}_{\kappa,\tau}$- unital associative algebra generated by the noncommutative spacetime coordinate generators $\hat{x}^\mu$ modulo the following relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa}\left(\tau^\mu \hat{x}^\nu - \tau^\nu \hat{x}^\mu\right)$$

where τ^μ is a fixed four-vector from V; $\mu, \nu = 0, 1, \ldots, D-1$.

- This algebra becomes a Hopf module algebra with respect to the $\kappa-$deformed Hopf algebra structure, i.e. under the module action $\triangleright$ of the quantum $U_{\kappa,\tau}\left(iso(g)\right)$

$$P_\mu \triangleright \hat{x}^\nu = -\imath \delta^\nu_\mu \qquad , \qquad M_{\mu\nu} \triangleright \hat{x}^\rho = i\left(g_{\mu\alpha}\delta^\rho_\nu - g_{\nu\alpha}\delta^\rho_\mu\right)\hat{x}^\alpha$$

- Under this action the algebra becomes a covariant quantum space in a sense of the compatibility condition (a.k.a. generalized Leibniz rule):

$$L \triangleright \left(\hat{x}^\mu \cdot \hat{x}^\nu\right) = \left(L_{(1)} \triangleright \hat{x}^\mu\right) \cdot \left(L_{(2)} \triangleright \hat{x}^\nu\right)$$

Remarks:

- the metric components are not involved in the definition of κ, τ-Minkowski algebra, so the algebra is **independent of the metric** itself and the metric signature in particular.
- It is **generally covariant**, i.e. introducing new generators $\tilde{x}^\alpha = (A^{-1})^\alpha_\mu \hat{x}^\mu$ and new components $\tilde{\tau}^\alpha = (A^{-1})^\alpha_\mu \tau^\mu$ $(\tilde{e}_\alpha = A^\mu_\alpha e_\mu, A^\mu_\alpha \in GL(D))$ one preserves the form of κ, τ-Minkowski algebra.
- It shows that the κ, τ-Minkowski algebra is, in fact, independent of the components of the vector $\tau \neq 0$ (for $\tau = 0$ one obtains undeformed Abelian algebra).
- One can always reach the well-known standard form of the κ-Minkowski spacetime algebra - by taking any basis with $e_0 = \tau$:

$$\left[\hat{x}^0, \hat{x}^i\right] = \frac{i}{\kappa}\hat{x}^i, \qquad \left[\hat{x}^i, \hat{x}^j\right] = 0, \quad i, j = 1, \ldots, D-1$$

One can conclude that up to the isomorphism mentioned above, for any dimension there is only one $\kappa-$Minkowski spacetime algebra $\mathcal{M}_{\kappa,\tau}$.

Extended $\kappa-$deformations

Zakrzewski already proposed a list of Abelian extensions of r_τ which we can be used in the time-, light- and space- like cases of the vector τ:

$$r_{\tau,ext} = r_\tau + \xi P_\tau \wedge X \quad , \quad [P_\tau, X] = 0$$

where X belongs to a Lie algebra for the stability subgroup G_τ of τ (rememberthat for time-like case, $G_\tau = SO\,(3)$; for light-like case $G_\tau = E\,(2) = ISO\,(2)$; for space-like case $G_\tau = SO(2,1)$. Here ξ is a new (independent) deformation parameter. Later on Lyakhovsky found more sophisticated extensions of a time-like $\kappa-$Poincaré case (11 subcases) showing at the same time that the list presented by Zakrzewski is incomplete (as already mentioned by Zakrzewski himself). According to our best knowledge the problem of final classification is still open. It what follows we are using twist (star-product type) technique in order to deform κ-Minkowski algebraic relation as well as the corresponding quantum group coproducts.

type	r-matrix	algebra type
light-like case with $\tau^{+} = (1,0,0,0)$		metric in $2+2$ decomposition
$L1$	$\xi P_{+} \wedge M_{+1}$	$M^{3}_{a=1}$
$L2$	$\xi P_{+} \wedge M_{3}$	$M^{6}_{a,b}$ with $a = -\frac{\left(3+(2\kappa\xi)^2\right)}{9}$; $b = \frac{\left(1+(2\kappa\xi)^2\right)}{27}$
space-like case with $\tau^{\mu} = (0,1,0,0)$ and $\eta_{\mu\nu} = (+,-,-,-)$		
$S1$	$\xi P_{1} \wedge M_{1}$	$\mathrm{M}^{6}_{a,b}$ with $a = -\frac{\left(3+(\kappa\xi)^2\right)}{9}$; $b = \frac{\left(1+(\kappa\xi)^2\right)}{27}$
$S2$	$\xi P_{1} \wedge (M_{1} + N_{3})$	the same as $S1$
$S3$	$\xi P_{1} \wedge N_{3}$	the same as $S1$
time-like case with $\tau = (1,0,0,0)$ and $\eta_{\mu\nu} = (-,+,+,+)$		
$T1$	$\xi M_{3} \wedge P_{0}$	$M^{6}_{a,b}$ with $a = -\frac{\left(3+4(\alpha\kappa)^2\right)}{9}$, $b = \frac{\left(1+4(\alpha\kappa)^2\right)}{27}$
$T3$	$\pm\frac{1}{2\kappa} M_{3} \wedge P_{0} + \xi \tilde{M}_{\pm} \wedge \tilde{P}_{\pm}$	$M^{13}_{b=-\frac{2}{9}}$ (as complex algebra)
$T4$	$\pm\frac{1}{\kappa} M_{3} \wedge P_{0} \pm \xi(P_{3} \wedge \tilde{M}_{\pm} + M_{3} \wedge \tilde{P}_{\pm})$	M_{8} (as complex algebra)

Solvable Lie algebras

For given Lie algebra $\mathfrak{g}$ we define a sequence of subalgebras of $\mathfrak{g}$ (the so-called lower derived series) by setting $\mathfrak{g}_{(0)} = \mathfrak{g}$, $\mathfrak{g}_{(1)} = [\mathfrak{g}_{(0)}, \mathfrak{g}_{(0)}], ..., \mathfrak{g}_{(i)} = [\mathfrak{g}_{(i-l)}, \mathfrak{g}_{(i-l)}]$. We call $\mathfrak{g}$ solvable if $\mathfrak{g}_{(n)} = 0$ for some finite n. In a similar manner, the upper sequence $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}^{(0)}], ..., \mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-l)}]$ determines nilpotent Lie algebras. For example, nilpotent (e.g. Abelian) algebras are solvable, whereas semisimple algebras are definitely nonsolvable. Moreover, a finite dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero is solvable if and only if $\mathfrak{g}_{(1)} \equiv \mathfrak{g}^{(1)}$ is nilpotent. Classification of all three dimensional real Lie algebras is well known for a long time since Bianchi 1898 (Lie himself had earlier classified the complex ones).
Four dimensional case has been solved by J.Patera 1976.

Some four-dimensional solvable Lie algebras according to W. A. de Graaf, Experiment. Math. Volume 14, Issue 1, 15-25 (2005)

- M^2 where $\left[x^0, x^1\right] = x^1$; $\left[x^0, x^2\right] = x^2$; $\left[x^0, x^3\right] = x^3$ (four-dimensional $\kappa-$Minkowski spacetime algebra); In dim=3 corresponds to Bianchi V.
- M_a^3:
  $\left[x^0, x^1\right] = x^1$; $\left[x^0, x^2\right] = x^3$; $\left[x^0, x^3\right] = -ax^2 + (a+1)\,x^3$,
  where $a \in \mathbb{R}$ (or $\mathbb{C}$)
- $M_{a,b}^6$: $\left[x^0, x^1\right] = x^3$; $\left[x^0, x^2\right] = x^1$; $\left[x^0, x^3\right] = x^3 + ax^2 + bx^1$,
  where $a, b \in \mathbb{R}$ (or $\mathbb{C}$)
- M^8 : $\left[x^1, x^2\right] = x^2$; $\left[x^0, x^3\right] = x^3$
- M_b^{13}:
  $\left[x^0, x^1\right] = x^1 + bx^3$; $\left[x^0, x^2\right] = x^2 = \left[x^3, x^1\right]$; $\left[x^0, x^3\right] = x^1$,
  where $b \in \mathbb{R}$ (or $\mathbb{C}$)

L1) From twist one obtains the $\star$-commutators:

$$\left[x^{+}, x^{1}\right]_{\star} = \frac{i}{\kappa}x^{1} + 2i\xi x^{-}; \quad \left[x^{+}, x^{2}\right]_{\star} = \frac{i}{\kappa}x^{2}; \quad \left[x^{+}, x^{-}\right]_{\star} = \frac{i}{\kappa}x^{-} \tag{6}$$

Check that for arbitrary (real) values of κ, ξ this is a solvable Lie algebra. Firstly we notice that the coordinates $\left(x^{1}, x^{2}, x^{-}\right)$ make a L^{1} Abelian 3 dimensional subalgebra. Thus 4-dimensional algebra can be classified as $M^{3}_{a=1}$ in the following way:

1. firstly we rescale x^{+} as $\frac{\kappa}{i}x^{+} = \tilde{x}^{0}$ to obtain $\left[\tilde{x}^{0}, x^{1}\right] = x^{1} + 2\kappa\xi x^{-}; \quad \left[\tilde{x}^{0}, x^{2}\right] = x^{2}; \quad \left[\tilde{x}^{0}, x^{-}\right] = x^{-}.$
2. then change the generators as $\tilde{x}^{1} = x^{1} + \beta x^{-}$ and $\tilde{x}^{-} = x^{1} + \gamma x^{-}$ to get $\left[\tilde{x}^{0}, \tilde{x}^{1}\right] = \tilde{x}^{-}$ with $\gamma = (2\kappa\xi + \beta)$ and $\left[\tilde{x}^{0}, x^{2}\right] = x^{2}$; together with $\left[\tilde{x}^{0}, \tilde{x}^{-}\right] = -\tilde{x}^{1} + 2\tilde{x}^{-}$.

Finally, this algebra can be classified regardless the specific values for the parameters $\kappa.\xi$ as

$$M^{3}_{a=1} : \left[x^{0}, x^{1}\right] = x^{3}; \quad \left[x^{0}, x^{2}\right] = x^{2}; \quad \left[x^{0}, x^{3}\right] = -ax^{1} + (1 + a)\, x^{3}; \quad f \tag{7}$$

Summary

- The universal formulas for the deformed Poincaré algebra - **depend** on the choice of the **additional vector field** τ and allow one to consider **three cases of deformations** all being the symmetry of the corresponding noncommutative $\kappa(\tau)$-Minkowski spacetimes.
- Thus non-equivalent deformations are classified by the stability subgroups of the vector τ.
- Extending $\kappa-$deformations one obtains a new class of (covariant) quantum spaces (**deformed $\kappa-$Minkowski spaces**). In some cases specialization is possible.

more in SIGMA 10 (2014), 107 [arXiv:1404.2916]