Introduction to gauge-gravity duality and applications

Andrei Starinets

University of Oxford

Corfu Summer School
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Outline

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III. Gauge-gravity duality (holography) at finite temperature and density
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- transport at strong coupling
- universality of the viscosity-entropy ratio
- particle emission rates
- relation to RHIC-LHC and other experiments

Some references:

Old but useful review: O.Aharony, S.Gubser, J.Maldacena, H.Ooguri, Y.Oz, hep-th/9905111

J.Casalderrey-Solana, H.Liu, D.Mateos, K.Rajagopal, U.Wiedemann, 1101.0618 [hep-th]
Updated version is now a book published by Cambridge U.Press (2014)
“Gauge/String Duality, Hot QCD and Heavy Ion Collisions”


H.Nastase, “Introduction to the AdS/CFT correspondence”, 2015


P.K.Kovtun and A.O.S., “Quasinormal modes and holography”, hep-th/0506184
I. Introduction and motivation (physics of heavy ion collisions)
In the last 15 years or so, holographic (gauge/gravity duality) methods were used to study strongly coupled gauge theories at finite temperature and density.

These studies were motivated by the heavy-ion collision programs at RHIC and LHC (ALICE, ATLAS) and the necessity to understand hot and dense nuclear matter in the regime of intermediate coupling $\alpha_s(T_{RHIC}) \sim O(1)$.

As a result, we now have a better understanding of thermodynamics and especially kinetics (transport) of strongly coupled gauge theories. Non-equilibrium behavior (thermalization) is under intense study.

Caution: these calculations are done for theoretical models such as N=4 SYM and its cousins (including non-conformal theories etc). We do not have a string dual description of QCD.
Energy density estimate

Before:

\[ \sim 0.1 \text{ fm} \]

After:

\[ \sim 0.1 \text{ fm} \]

\[ \sim 1 \text{ fm} \]

\[
\varepsilon \sim \frac{dE_T}{d\eta \cdot \Delta \eta} \frac{1}{\tau_0 c \pi R^2} \sim 5 \text{ GeV/fm}^3
\]
Charged particle multiplicity in pp, pPb and central PbPb collisions

\[ \langle \frac{dN_{\text{ch}}}{d\eta} \rangle \langle N_{\text{part}} \rangle \]

\[ s_{\text{NN}}^{0.15} \]

\[ s_{\text{NN}}^{0.11} \]

\[ s_{\text{NN}}^{0.10} \]
Heavy Ion Collisions

RHIC (2000-current): Au+Au

\[ \sqrt{s_{NN}} = 0.2 \text{ TeV per nucleon} \]

LHC (2010-current): Pb+Pb

\[ \sqrt{s_{NN}} = 2.76 \text{ TeV per nucleon} \]

The resulting system is:

* many-body: \( N_{\text{final}} \sim 10^4 \)

* quantum: \( \lambda_{\text{DeBroglie}} \sim l_{mfp} \)

* at very high energy density \( \epsilon \geq 5 \text{ GeV/fm}^3 \)

* in local TD equilibrium at \( T \sim 10^{12} \text{ K} \)

* strongly interacting: \( \alpha_s(T_{\text{RHIC}}) \sim 1 \)

Note: LHC is only about 30% “hotter” than RHIC…
Heavy ion collision experiments at **RHIC** (2000-current) and **LHC** (2010-current) create hot and dense nuclear matter known as the “quark-gluon plasma”

(note: qualitative difference between p-p and Au-Au collisions)

Evolution of the plasma “fireball” is described by relativistic fluid dynamics (relativistic Navier-Stokes equations)

Need to know

- **thermodynamics** (equation of state)
- **kinetics** (first- and second-order transport coefficients)

in the regime of intermediate coupling strength:

\[ \alpha_s(T_{\text{RHIC}}) \sim O(1) \]

- **initial conditions** (initial energy density profile)
- **thermalization time** (start of hydro evolution)
- **freeze-out conditions** (end of hydro evolution)
QCD phase diagram

- Quark-gluon plasma
- Hadron gas
  - Confined, $\chi$-symmetric
- Color superconductor

Variables:
- $T$: Temperature
- $\mu$: Chemical potential
- $\mu_o$: Few times nuclear matter density
- $\sim 170$ MeV
Energy density vs temperature for various gauge theories

Ideal gas of quarks and gluons

Ideal gas of hadrons

Figure: an artistic impression from Myers and Vazquez, 0804.2423 [hep-th]
Pressure in perturbative QCD
Elliptic flow with Glauber initial conditions

Glauber

![Graph showing elliptic flow with Glauber initial conditions](image)

- **STAR non-flow corrected (est.)**
- **STAR event-plane**

Luzum and Romatschke, 0804.4015 [nuc-th]
Shear viscosity in $\mathcal{N} = 4$ SYM

$$\sim \frac{1}{\lambda^2 \log \frac{1}{\lambda}}$$

perturbative thermal gauge theory

$\frac{1}{4\pi} + \frac{15\zeta(3)}{4\pi} \frac{1}{\lambda^{3/2}} + \cdots$

Correction to $\frac{1}{4\pi}$: Buchel, Liu, A.S., hep-th/0406264
Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]
II. Gauge-string (gauge-gravity) duality: Origins and overview
Duality can be viewed as a change of variables in the path integral

Ordinary integrals can often be simplified by a clever change of (dummy) variables

Similarly, path integrals can be simplified by choosing the “right” set of fields (d.o.f.) (for a given corner of the parameter space)

Example: Kramers-Wannier duality (1941)

\[ Z[J] = \sum_{s_j = \pm 1} \exp \left( \sum_{<ij>} J s_i s_j \right) \]

\[ s_i \to \sigma_i \]

\[ Z[J] = 2^N (\cosh J)^{2N} Z[\tilde{J}] \]

\[ \sinh 2J \sinh 2\tilde{J} = 1 \]

Ising model in d=2

Note: can be viewed as Fourier transform on (Abelian) groups
From brane dynamics to AdS/CFT correspondence

Open strings picture:
dynamics of $N_c$ coincident D3 branes
at low energy is described by

$\mathcal{N} = 4$ supersymmetric
$SU(N_c)$ YM theory in 4 dim

Closed strings picture:
dynamics of $N_c$ coincident D3 branes
at low energy is described by

type IIB superstring theory
on $AdS_5 \times S^5$ background

Maldacena (1997); Gubser, Klebanov, Polyakov (1998); Witten (1998)
\[ \mathcal{N} = 4 \] supersymmetric YM theory

- Field content:

\[
A_\mu \quad \Phi_I \quad \Psi^A_{\alpha} \quad \text{all in the adjoint of } SU(N) \\
I = 1 \ldots 6 \quad A = 1 \ldots 4
\]

- Action:

\[
S = \frac{1}{g_{YM}^2} \int d^4x \quad \text{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_I)^2 - \frac{1}{2} [\Phi_I, \Phi_J]^2 \\
+ i \bar{\Psi} \Gamma^\mu D_\mu \Psi - \bar{\Psi} \Gamma^I [\Phi_I, \Psi] \right\}
\]

(super)conformal field theory = coupling doesn’t run
AdS/CFT correspondence

$\mathcal{N} = 4$ supersymmetric $SU(N_c)$ YM theory in 4 dim $\leftrightarrow$ type IIB superstring theory on $AdS_5 \times S^5$ background

conjectured exact equivalence

$Z_{SYM}[J] = \langle e^{-\int J \mathcal{O} d^4 x} \rangle_{SYM} = Z_{string}[J]$  

Generating functional for correlation functions of gauge-invariant operators $\langle \mathcal{O} \mathcal{O} \cdots \mathcal{O} \rangle$ $\leftrightarrow$ String partition function

In particular

$Z_{SYM}[J] = Z_{string}[J] \simeq e^{-S_{grav}[J]}$

$\lambda \equiv g_{YM}^2 N_c \gg 1$

$N_c \gg 1$

Classical gravity action serves as a generating functional for the gauge theory correlators
AdS/CFT correspondence: the role of J

\[ Z_{\text{SYM}}[J] = \langle e^{-\int J \mathcal{O} d^4 x} \rangle_{\text{SYM}} \sim e^{-S_{\text{grav}}[J]} \]

For a given operator \( \mathcal{O} \), identify the source field \( J \), e.g.

\[ T^{\mu \nu} \iff h_{\mu \nu} \]

\[ e^{-S_{\text{grav}, M}[\phi_{BG} + \delta \phi]} = Z[J = \delta \phi \big|_{\partial M}] \]

\( \delta \phi \) satisfies linearized supergravity e.o.m. with b.c.

\[ \delta \phi \rightarrow \delta \phi_0 \equiv J \]

**The recipe:**

To compute correlators of \( \mathcal{O} \), one needs to solve the bulk supergravity e.o.m. for \( \delta \phi \) and compute the on-shell action as a functional of the b.c.

\[ \delta \phi_0 \equiv J \]

**Warning:** e.o.m. for different bulk fields may be coupled: need self-consistent solution

Then, taking functional derivatives of \( e^{-S_{\text{grav}}[J]} \) gives

\[ \langle \mathcal{O} \mathcal{O} \rangle \]
Holography at finite temperature and density

\[ \langle \mathcal{O} \rangle = \frac{\text{tr} \rho \mathcal{O}}{\text{tr} \rho} \]

\[ \rho = e^{-\beta H + \mu Q} \]

\[ H \rightarrow T^{00} \rightarrow T^{\mu \nu} \rightarrow h_{\mu \nu} \]

\[ Q \rightarrow J^0 \rightarrow J^\mu \rightarrow A_\mu \]

Nonzero expectation values of energy and charge density translate into nontrivial background values of the metric (above extremality) = horizon and electric potential = CHARGED BLACK HOLE (with flat horizon)

\[ ds^2 = -F(u) \, dt^2 + G(u) \left( dx^2 + dy^2 + dz^2 \right) + H(u) \, du^2 \]

\[ T = T_H \quad \text{temperature of the dual gauge theory} \]

\[ A_0 = P(u) \]

\[ \mu = P(\text{boundary}) - P(\text{horizon}) \quad \text{chemical potential of the dual theory} \]
The bulk and the boundary in AdS/CFT correspondence

\[ ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{z^2} + dz^2 \]

UV/IR: the AdS metric is invariant under \( z \to \Lambda z \quad x \to \Lambda x \)

\( z \) plays a role of inverse energy scale in 4D theory

5D bulk (+5 internal dimensions)

\textit{strings} or supergravity fields

gauge fields

4D boundary
III. Gauge-gravity duality at finite temperature and density
10-dim gravity

$\begin{array}{c}
T_{\text{Hawking}} \\
S_{\text{Bekenstein-Hawking}}
\end{array}$

Holographically dual system in thermal equilibrium

$\begin{array}{c}
M, J, Q
\end{array}$

4-dim gauge theory – large N, strong coupling

Gravitational+electromag fluctuations

$g^{(0)}_{\mu\nu} + h_{\mu\nu}$

$A^0_\mu + a_\mu$

"$\square$" $h_{\mu\nu} = 0$ and B.C.

Quasinormal spectrum

Deviation from equilibrium

$\begin{array}{c}
????
\end{array}$

$j_i = -D\partial_i j^0 + \cdots$

$\partial_t j^0 + \partial_i j^i = 0$

$\partial_t j^0 = D\nabla^2 j^0$

$\omega = -iDq^2 + \cdots$
Quasinormal spectra of black holes/branes

Schwarzschild black hole
(asymptotically flat)

AdS-Schwarzschild black brane
Computing real-time correlation functions from gravity

To extract transport coefficients and spectral functions from dual gravity, we need a recipe for computing Minkowski space correlators in AdS/CFT.


Quasinormal spectrum of dual gravity = poles of the retarded correlators in 4d theory [D.T. Son & A.S., 2001]
The role of quasinormal modes

G.T.Horowitz and V.E.Hubeny, hep-th/9909056

D.Birmingham, I.Sachs, S.N.Solodukhin, hep-th/0112055


\[ S^{(2)} \sim \lim_{z=\epsilon \to 0} \int d\omega \, dp \, q \, F(\omega, q) \, \Phi'(z) \Phi(z) + \text{contact terms} \]

\[ \Phi(z) = A \varphi_1(z) + B \varphi_2(z) \]

\[ \Phi(z) = A \, z^\Delta^- (1 + \cdots) + B \, z^\Delta^+ (1 + \cdots) \quad \text{for } z \to 0 \]

I. Computing the retarded correlator: inc.wave b.c. at the horizon, normalized to 1 at the boundary

\[ G^R \sim \frac{B}{A} + \text{contact terms} \]

II. Computing quasinormal spectrum: inc.wave b.c. at the horizon, Dirichlet at the boundary

\[ A(\omega, q) = 0 \]
Example: R-current correlator in $4d\;\mathcal{N} = 4\;\text{SYM}$ in the limit $N_c \rightarrow \infty$, $g_{YM}^2 N_c \rightarrow \infty$

Zero temperature: $< J_i(x) J_i(y) > \sim \frac{N_c^2}{|x - y|^6}$

$G_E(k) = \frac{N_c^2 k_E^2}{32 \pi^2} \ln k_E^2$

$G^{\text{ret}}(k) = \frac{N_c^2 k^2}{32 \pi^2} \left( \ln |k^2| - i \pi \theta (-k^2) \text{sgn} \omega \right)$

$\omega^2 = -k^2 + q^2$

Finite temperature: $G^{\text{ret}}(\omega, q)$

$G^{\text{ret}}(\omega, 0) = \frac{N_c^2 T^2}{8} \left\{ \frac{i \omega}{2 \pi T} + \frac{\omega^2}{4 \pi^2 T^2} \left[ \psi \left( \frac{(1 - i) \omega}{4 \pi T} \right) + \psi \left( -\frac{(1 + i) \omega}{4 \pi T} \right) \right] \right\}$

Poles of $G^{\text{ret}} = \text{quasinormal spectrum of dual gravity background}$

Our understanding of gauge theories is limited…
Conjecture:
specific gauge theory in 4 dim =
specific string theory in 10 dim
In practice: gravity (low energy limit of string theory) in 10 dim = 4-dim gauge theory in a region of a parameter space

Can add fundamental fermions with $N_f \ll N_c$
IV. The hydrodynamic regime


The hydrodynamic regime

Hierarchy of times (e.g. in Bogolyubov’s kinetic theory)

$$\tau_{mft} \ll t \ll T_{exist}$$

0 \quad \tau_{inter} \quad \tau_{mft} \quad \tau_{relax} \quad t

Mechanical description \quad Kinetic theory \quad Hydrodynamic approximation \quad Equilibrium thermodynamics

Hierarchy of scales

$$l_{mfp} \ll l \ll L$$

(L is a macroscopic size of a system)
The hydrodynamic regime (continued)

Degrees of freedom

Hydro regime: \[ \tau_{\text{micro}} \ll \tau \ll t_{\text{global}} \quad l_{\text{micro}} \ll l \ll L_{\text{global}} \]
Hydrodynamics: fundamental d.o.f. = densities of conserved charges

Need to add constitutive relations!

**Example: charge diffusion**

Conservation law

$$\partial_t j^0 + \partial_i j^i = 0$$

Constitutive relation

[Fick’s law (1855)]

$$j_i = -D \partial_i j^0 + O[(\nabla j^0)^2, \nabla^2 j^0]$$

Diffusion equation

$$\partial_t j^0 = D \nabla^2 j^0$$

Dispersion relation

$$\omega = -i D q^2 + \cdots$$

Expansion parameters: \( \omega \ll T, \quad q \ll T \)
Fluid dynamics is an effective theory valid in the long-wavelength, long-time limit

Fundamental degrees of freedom = densities of conserved charges

Equations of motion = conservation laws + constitutive relations^*

Example I

\[ \partial_a J^a = 0 \]
\[ J^i = -D \nabla^i J^0 + \ldots \]
\[ \partial_t J^0 = D \nabla^2 J^0 + \ldots \]

Example II

\[ \partial_a T^{ab} = 0 \]
\[ T^{ab} = \varepsilon u^a u^b + P(\varepsilon) (g^{ab} + u^a u^b) + \Pi^{ab} + \ldots \]

^* Modulo assumptions e.g. analyticity

** E.o.m. universal, transport coefficients depend on underlying microscopic theory
Example: momentum diffusion and sound

Thermodynamic equilibrium: \( \langle T^{00} \rangle = \epsilon, \langle T^{0i} \rangle = 0 \)
\[ T^{ij} = P(\epsilon) \delta^{ij} \]

Near-equilibrium: \( T^{00} = \epsilon + \tilde{T}^{00} \)
\[ T^{ij} = P \delta^{ij} + \left( \frac{\partial P}{\partial \epsilon} \right) \tilde{T}^{00} + \tilde{T}^{ij} \]
\[ \tilde{T}^{ij} = -\frac{1}{\epsilon + P} \left[ \eta \left( \partial_i \tilde{T}^{0j} + \partial_j \tilde{T}^{0i} - \frac{2}{3} \delta^{ij} \partial_k \tilde{T}^{0k} \right) + \zeta \delta^{ij} \partial_k \tilde{T}^{0k} \right] + \cdots \]

Eigenmodes of the system of equations \( \partial_\mu T^{\mu\nu} = 0 \)

Shear mode (transverse fluctuations of \( \tilde{T}^{0i} \)):
\[ \omega = -\frac{i \eta}{\epsilon + P} q^2 \]

Sound mode:
\[ \omega = v_s q - \frac{i}{2 \epsilon + P} \left( \zeta + \frac{4}{3} \eta \right) q^2 \]

For CFT we have \( \zeta = 0 \) and \( \epsilon = 3P \)
\[ v_s = 1/\sqrt{3} \]
Transport problems in “real life”

Thermal conductivity (transport of heat), electrical conductivity (transport of charge), diffusion (transport of charge/matter), viscosity (transport of momentum)
Transport problems in “real life”: viscosity

Viscosity of water: 1 centipoise (cP)

Viscosity of honey: 2000-10000 cP

1 Poise = 1 Pa s = 1 kg/(m s)

Viscosity of the quark-gluon plasma: $10^{14}$ cP (almost glass)
What is viscosity?

Friction in Newton’s equation:
\[
\frac{d(mv_i)}{dt} + \gamma v_i = F_i
\]

Friction in Euler’s equations
\[
\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial}{\partial x^k} \left( P \delta_{ik} + \rho v_i v_k \right) + \frac{\partial}{\partial x^k} \sigma_{ik}^{\text{fric}}
\]

\[
\sigma_{ik}^{\text{fric}} \sim \frac{\partial v_i}{\partial x^k}
\]

\[
\sigma_{ik}^{\text{fric}} \sim \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i}
\]

\[
\sigma_{ik}^{\text{fric}} = \eta \left( \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} - \frac{2}{d} \delta_{ik} \frac{\partial v_l}{\partial x^l} \right) + \zeta \delta_{ik} \frac{\partial v_l}{\partial x^l} + \cdots
\]
Viscosity of gases and liquids

Gases (Maxwell, 1867):

\[ \eta \sim \rho \bar{v} l_{mfp} \sim \frac{m_o \bar{v}}{\sigma} \sim \frac{m_o^{1/2}}{\sigma} \sqrt{T} \]

Viscosity of a gas is

- independent of pressure
- scales as square of temperature
- inversely proportional to cross-section

Liquids (Frenkel, 1926):

\[ \eta \sim A(P, T) \exp \frac{W}{T} \]

- \( W \) is the “activation energy”
- In practice, \( A \) and \( W \) are chosen to fit data
First-order transport (kinetic) coefficients

Shear viscosity \( \eta \)

Bulk viscosity \( \zeta \)

Charge diffusion constant \( D_Q \)

Supercharge diffusion constant \( D_s \)

Thermal conductivity \( \kappa_T \)

Electrical conductivity \( \sigma \)

* Expect Einstein relations such as \( \frac{\sigma}{e^2 s} = D_{U(1)} \) to hold
Linear response theory

\[ H(t) = H_0 + H_1(t) \theta(t - t_0) \]

\( t < t_0 \)

\[ \rho_0 = e^{-\beta H_0} = \sum_n |n \rangle \langle n| e^{-\beta E_n^{(0)}} \]

\[ \langle \mathcal{O} \rangle = \frac{1}{Z_0} \text{tr} \rho_0 \mathcal{O} = \frac{1}{Z_0} \sum_n \langle n| \mathcal{O}| n \rangle e^{-\beta E_n^{(0)}} \]

\( t > t_0 \)

\[ \rho(t) = \sum_n |n(t) \rangle \langle n(t)| e^{-\beta E_n} \]

\[ \langle \mathcal{O} \rangle = \frac{1}{Z_0} \text{tr} \rho \mathcal{O} = \frac{1}{Z_0} \sum_n \langle n(t)| \mathcal{O}| n(t) \rangle e^{-\beta E_n} \]

\[ i \partial_t |n(t) \rangle = H(t) |n(t) \rangle \]
Linear response theory (continued)

\[|n(t)\rangle = e^{-iH_0 t}U(t,t_0)|n\rangle\]

\[U(t,t_0) = 1 - i \int_{t_0}^{t} dt' H_1(t') + \cdots\]

Inserting this into \[\rho(t) = \sum_n |n(t)\rangle \langle n(t)| e^{-\beta E_n}\] we get

\[\langle \mathcal{O}(t) \rangle = \frac{1}{Z_0} tr \rho \mathcal{O} =\]

\[= \langle \mathcal{O} \rangle_0 - i \int_{t_0}^{t} dt' \frac{1}{Z_0} \sum_n \langle n(t_0)|\mathcal{O}(t)H_1(t') - H_1(t')\mathcal{O}(t)|n(t_0)\rangle e^{-\beta E_n} + \cdots\]

\[= \langle \mathcal{O} \rangle_0 - i \int_{t_0}^{t} dt' \langle [\mathcal{O}(t)H_1(t')] \rangle + \cdots\]
Computing transport coefficients from “first principles”

Fluctuation-dissipation theory (Callen, Welton, Green, Kubo)

Kubo formulae allows one to calculate transport coefficients from microscopic models

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt d^3 x e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

In the regime described by a gravity dual the correlator can be computed using the gauge theory/gravity duality
Second-order hydrodynamics

Hydrodynamics is an effective theory, valid for sufficiently small momenta

\[ kl_{mfp} \ll 1 \]

First-order hydro eqs are parabolic. They imply instant propagation of signals.

This is not a conceptual problem since hydrodynamics becomes “acausal” only outside of its validity range but it is very inconvenient for numerical work on Navier-Stokes equations where it leads to instabilities [Hiscock & Lindblom, 1985]

These problems are resolved by considering next order in derivative expansion, i.e. by adding to the hydro constitutive relations all possible second-order terms compatible with symmetries (e.g. conformal symmetry for conformal plasmas)
Second-order conformal hydrodynamics (in d dimensions)

\[ T_{\text{conformal}}^{\mu \nu} = \varepsilon u^\mu u^\nu + \frac{\varepsilon}{d-1} \Delta^{\mu \nu} + \Pi^{\mu \nu} \]

\[ \Pi^{\mu \nu} = -\eta \sigma^{\mu \nu} + \tau \eta \left[ \langle D\sigma^{\mu \nu} \rangle + \frac{1}{d-1} \sigma^{\mu \nu} (\nabla \cdot u) \right] + \kappa \left[ R^{\langle \mu \nu \rangle} - (d-2)u_\alpha R^\alpha_{\langle \mu \nu \rangle \beta} u_\beta \right] + \frac{\lambda_1}{\eta^2} \sigma^\lambda \langle \mu \nu \rangle \lambda - \frac{\lambda_2}{\eta} \sigma^\lambda \langle \mu \Omega^\nu \rangle \lambda + \lambda_3 \Omega^\lambda \langle \mu \Omega^\nu \rangle \lambda \]

\[ \Delta^{\mu \nu} = g^{\mu \nu} + u^\mu u^\nu \quad \sigma^{\mu \nu} = 2\nabla \langle \mu u^\nu \rangle \quad u^\mu u_\mu = -1 \]

\[ D \equiv u^\mu \nabla_\mu \quad \Omega^{\mu \nu} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} \left( \nabla_\alpha u_\beta - \nabla_\beta u_\alpha \right) \]

\[ A^{\langle \mu \nu \rangle} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} \left( A_{\alpha \beta} + A_{\beta \alpha} \right) - \frac{1}{d-1} \Delta^{\mu \nu} \Delta^{\alpha \beta} A_{\alpha \beta} \]
Second-order transport (kinetic) coefficients

(for theories conformal at $T=0$)

Relaxation time $\tau_{\Pi}$

Second order transport coefficient $\lambda_1$

Second order transport coefficient $\lambda_2$

Second order transport coefficient $\lambda_3$

Second order transport coefficient $\kappa$

In non-conformal theories such as QCD, the total number of second-order transport coefficients is quite large.
Derivative expansion in hydrodynamics: first order

Hydrodynamic d.o.f. = densities of conserved charges

\[ \partial_\mu T^{\mu \nu} = 0 \quad \text{(4 equations)} \]
\[ T^{00}, T^{0i} \quad \text{or} \quad \varepsilon, \ u^\mu \]

\[ T^{\mu \nu} = \varepsilon u^\mu u^\nu + P(\varepsilon) \Delta^{\mu \nu} + \Pi^{\mu \nu} \]
\[ \Pi^{\mu \nu} = -\eta(\varepsilon) \sigma^{\mu \nu} - \zeta(\varepsilon) \Delta^{\mu \nu} (\nabla \cdot u) + \cdots \]

\[ \Delta^{\mu \nu} = g^{\mu \nu} + u^\mu u^\nu \quad \sigma^{\mu \nu} = 2 \nabla^{<\mu}u^{\nu>} \]

\[ A^{<\mu \nu>} = \frac{1}{2} \Delta^{\mu \alpha} \Delta^{\nu \beta} (A^\alpha_\beta + A^\beta_\alpha) - \frac{1}{d - 1} \Delta^{\mu \nu} \Delta^{\alpha \beta} A^\alpha_\beta \]
\[ u^\mu u_\mu = -1 \]
Predictions of the second-order conformal hydrodynamics

Sound dispersion: \[ \omega_{1,2} = \pm c_s q - i\Gamma q^2 \pm \frac{\Gamma}{c_s} \left( c_s^2 \tau_\Pi - \frac{\Gamma}{2} \right) q^3 + O(q^4) \]

\[ \Gamma = \frac{d-2}{d-1} \frac{\eta}{\varepsilon + P} \]

Kubo: \[ G^{xy,xy}_R(\omega, q) = P - i\eta \omega + \eta \tau_\Pi \omega^2 - \frac{\kappa}{2} \left[ (d-3)\omega^2 + q^2 \right] \]
In quantum field theory, the dispersion relations such as

$$\omega = \pm v_s q - \frac{i}{2(\epsilon + P)} \left( \frac{4}{3} \eta + \zeta \right) q^2$$

appear as poles of the retarded correlation functions, e.g.

$$\langle T_{00}(k) T_{00}(-k) \rangle \sim \frac{q^2 T^4}{\omega^2 - q^2/3 + i\omega q^2/3\pi T}$$

- in the hydro approximation - \( \omega/T \ll 1, \quad q/T \ll 1 \)
Supersymmetric sound mode (“phonino”) in $4d \; \mathcal{N} = 4 \; \text{SYM}$

Conserved charge $\implies$ Hydrodynamic mode (infinitely slowly relaxing fluctuation of the charge density) $\implies$ Hydro pole in the retarded correlator of the charge density

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu}_{\text{equib}} + \delta T^{\mu\nu}$$

$$\langle T_{\mu\nu}(-k) T_{\rho\sigma}(k) \rangle$$

Sound wave pole:

$$\omega = \pm v_s q - i \frac{2}{3sT} \left( \eta + \frac{3}{4} \zeta \right) q^2 + \cdots$$

$$v_s = \sqrt{\frac{\partial P}{\partial \epsilon}}$$

$$\partial_\mu S^\mu_{\alpha} = 0$$

$$S^\mu_{\alpha} + \delta S^\mu_{\alpha}$$

$$\langle \bar{S}^\mu_{\alpha}(-k) S^\nu_{\beta}(k) \rangle$$

Supersound wave pole:

$$\omega = \pm v_{ss} q - i D_s q^2 + \cdots$$

$$v_{ss} = \frac{P}{\epsilon}$$

Lebedev & Smilga, 1988 (see also Kovtun & Yaffe, 2003)
Sound and supersymmetric sound in $4d \ \mathcal{N} = 4 \ \text{SYM}$

In $4d$ CFT

$\epsilon = 3 \ P$  \hspace{1cm} $\implies$

$\zeta = 0$

$v_s = \sqrt{\frac{\partial P}{\partial \epsilon}} = \frac{1}{\sqrt{3}}$

$v_{SS} = \frac{P}{\epsilon} = \frac{1}{3}$

Sound mode:

$\omega = \pm \frac{q}{\sqrt{3}} - i \frac{2 \eta}{3sT} q^2 + \cdots$

Supersound mode:

$\omega = \pm \frac{q}{3} - iD_s q^2 + \cdots$

Quasinormal modes in dual gravity

Graviton:

$\omega = \pm \frac{q}{\sqrt{3}} - i \frac{1}{6\pi T} q^2 + \cdots$  \hspace{1cm} $\implies$  \hspace{1cm} $\frac{\eta}{s} = \frac{1}{4\pi}$

Gravitino:

$\omega = \pm \frac{q}{3} - i \frac{2\sqrt{2}}{9\pi T} q^2 + \cdots$  \hspace{1cm} $\implies$  \hspace{1cm} $D_s = \frac{2\sqrt{2}}{9\pi T}$
Spectral function and quasiparticles

\[ \chi_{\mu\nu,\alpha\beta}(k) = \int d^4x \, e^{-ikx} \langle [T_{\mu\nu}(x)T_{\alpha\beta}(0)] \rangle = -2 \text{Im} \, G^{\text{R}}_{\mu\nu,\alpha\beta}(\omega, q) \]
V. Some applications
First-order transport coefficients in $N = 4$ SYM

in the limit $N_c \rightarrow \infty$, $g_{YM}^2 N_c \rightarrow \infty$

Shear viscosity
$$\eta = \frac{\pi}{8} N_c^2 T^3 \left[ 1 + O \left( \frac{1}{(g^2 N_c)^{3/2}}, \frac{1}{N_c^2} \right) \right]$$

Bulk viscosity
$$\zeta = 0$$

for non-conformal theories see Buchel et al; G.D.Moore et al Gubser et al.

Charge diffusion constant
$$D_R = \frac{1}{2\pi T} + \ldots$$

Supercharge diffusion constant
$$D_s = \frac{2\sqrt{2}}{9\pi T}$$

Thermal conductivity
$$\frac{\kappa T}{\eta T} = 8\pi^2 + \ldots$$

Electrical conductivity
$$\sigma = e^2 \frac{N_c^2 T}{16 \pi} + \ldots$$

(G.Policastro, 2008)
Universality of shear viscosity in the regime described by gravity duals

\[ ds^2 = f(w) \left( dx^2 + dy^2 \right) + g_{\mu\nu}(w) dw^\mu dw^\nu \]

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, dx \, e^{i\omega t} \left\langle \left[ T_{xy}(t, x), T_{xy}(0, 0) \right] \right\rangle \]

\[ \sigma_{abs} = -\frac{16\pi G}{\omega} \text{Im} \, G^R(\omega) \]

\[ = \frac{8\pi G}{\omega} \int dt \, dx \, e^{i\omega t} \left\langle \left[ T_{xy}(t, x), T_{xy}(0, 0) \right] \right\rangle \]

Graviton’s component \( h^x_y \) obeys equation for a minimally coupled massless scalar. But then \( \sigma_{abs}(0) = A_H \).

Since the entropy (density) is \( s = A_H / 4G \) we get

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]
First and second order transport coefficients of \textit{conformal} holographic fluids \textit{to leading order} in supergravity approximation

\[
\begin{align*}
\eta &= s/4\pi, \\
\tau_\Pi &= \frac{d}{4\pi T} \left(1 + \frac{1}{d} \left[\gamma_E + \psi \left(\frac{2}{d}\right)\right]\right), \\
\kappa &= \frac{d}{d - 2} \frac{\eta}{2\pi T}, \\
\lambda_1 &= \frac{d\eta}{8\pi T}, \\
\lambda_2 &= \left[\gamma_E + \psi \left(\frac{2}{d}\right)\right] \frac{\eta}{2\pi T}, \\
\lambda_3 &= 0
\end{align*}
\]

Bhattacharyya et al, 2008

Note: \[2\eta \tau_\Pi - 4\lambda_1 - \lambda_2 = 0\]
Classification of fluctuations and universality

\[ ds^2 = \frac{r^2}{R^2} (-f(r)dt^2 + dx^2 + dy^2 + dz^2) + \frac{R^2}{r^2f}dR^2 \]

\[ \delta g_{\mu\nu} \sim e^{-i\omega t + iqz} h_{\mu\nu}(r) \]

O(2) symmetry in x-y plane

Shear channel: \( h_{tx} \quad h_{zx} \quad h_{ty} \quad h_{zy} \)

Sound channel: \( h_{tt} \quad h_{tz} \quad h_{zz} \quad h_{xx} + h_{yy} \)

Scalar channel: \( h_{xy} \quad h_{xx} - h_{yy} \)

Other fluctuations (e.g. \( \delta \varphi_1, \ldots, \delta \varphi_n \)) may affect sound channel

But not the shear channel

universality of \( \eta/s \)
Two-point correlation function of stress-energy tensor

Field theory

Zero temperature:
$$\langle T_{\mu\nu} T_{\alpha\beta} \rangle_{T=0} = \Pi_{\mu\nu,\alpha\beta} F(k^2) + Q_{\mu\nu,\alpha\beta} G(k^2)$$

Finite temperature:
$$\langle T_{\mu\nu} T_{\alpha\beta} \rangle_{T} = S^{(1)}_{\mu\nu,\alpha\beta} G_1(\omega, q) + S^{(2)}_{\mu\nu,\alpha\beta} G_2(\omega, q)$$
$$+ S^{(3)}_{\mu\nu,\alpha\beta} G_3(\omega, q) + S^{(4)}_{\mu\nu,\alpha\beta} G_4 + S^{(5)}_{\mu\nu,\alpha\beta} G_5$$

Dual gravity

- Five gauge-invariant combinations $Z_1, Z_2, Z_3, Z_4, Z_5$ of $h_{\mu\nu}$ and other fields determine $G_1, G_2, G_3, G_4, G_5$
- $Z_1, Z_2, Z_3, Z_4, Z_5$ obey a system of coupled ODEs
- Their (quasinormal) spectrum determines singularities of the correlator
Example: stress-energy tensor correlator in $4d$ $\mathcal{N} = 4$ SYM in the limit $N_c \to \infty$, $g_{YM}^2 N_c \to \infty$.

Zero temperature, Euclid: $G_E(k) = \frac{N_c^2 k_F^4}{32 \pi^2} \ln k_F^2$

Finite temperature, Mink:

$$< T_{tt}(-\omega, -q), T_{tt}(\omega, q) >_{\text{ret}} = \frac{3 N_c^2 \pi^2 T^4 q^2}{2(\omega^2 - q^2/3 + i\omega q^2/3 \pi T)} + \ldots$$

(in the limit $\omega/T \ll 1$, $q/T \ll 1$)

The pole (or the lowest quasinormal freq.)

$$\omega = \pm \frac{1}{\sqrt{3}} q - \frac{i}{6 \pi T} q^2 + \frac{3 - 2 \ln 2}{24 \pi^2 \sqrt{3} T^2} q^3 + \ldots$$

Compare with hydro:

$$\omega = \pm v_s q - \frac{i}{2 \pi T} \left( \zeta + \frac{4}{3} \eta \right) q^2 + \ldots$$

In CFT: $v_s = \frac{1}{\sqrt{3}}$, $\zeta = 0$

Also, $s = \pi^2 N_c^2 T^3 / 2$ (Gubser, Klebanov, Peet, 1996)
Example 2 (continued): stress-energy tensor correlator in 4d $\mathcal{N} = 4$ SYM in the limit $N_c \to \infty$, $g_{YM}^2 N_c \to \infty$

Zero temperature, Euclid:

$$G_E(k) = \frac{N_c^2 k_E^4}{32 \pi^2} \ln k_E^2$$

Finite temperature, Mink:

$$\langle T_{tx}(-\omega, -q), T_{tx}(\omega, q) \rangle_{\text{ret}} \sim \frac{N_c^2 T^4 \omega^2}{\omega - iq^2/4\pi T} + \ldots$$

(in the limit $\omega/T \ll 1$, $q/T \ll 1$)

The pole (or the lowest quasinormal freq.)

$$\omega = -\frac{i}{4\pi T} q^2 - \frac{i(1 - \ln 2)}{32\pi^3 T^3} q^4 + \ldots$$

Compare with hydro:

$$\omega = -\frac{i \eta}{sT} q^2 + \ldots$$

$$s = \pi^2 N_c^2 T^3 / 2 \quad \Rightarrow \quad \eta = \pi N_c^2 T^3 / 8$$
Sound dispersion in $4d \ N = 4$ SYM

$\text{Re}(\omega)/q$

$\text{Im}(\omega)/q$
Analytic structure of the correlators

\[ g^2 N = 0 \]

\[ g^2 N = \infty \]

Strong coupling: A.S., hep-th/0207133
Weak coupling: S. Hartnoll and P. Kumar, hep-th/0508092
Shear viscosity in $\mathcal{N} = 4$ SYM

perturbative thermal gauge theory

\[
\sim \frac{1}{\lambda^2 \log \frac{1}{\lambda}}
\]

\[
\frac{1}{4\pi} + \frac{15\zeta(3)}{4\pi} \frac{1}{\lambda^{3/2}} + \ldots
\]

Correction to $1/4\pi$: Buchel, Liu, A.S., hep-th/0406264
Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]
Electrical conductivity in $\mathcal{N} = 4$ SYM

Weak coupling: $\lambda \ll 1$

$$\sigma = 1.28349 \frac{e^2 (N_c^2 - 1) T}{\lambda^2 \left[ \ln \lambda^{-1/2} + O(1) \right]}$$

Strong coupling: $\lambda \gg 1$

$$\sigma = \frac{e^2 N_c^2 T}{16 \pi} + O\left(\frac{1}{\lambda^{3/2}}\right)$$

* Charge susceptibility can be computed independently:
$$\Xi = \frac{N_c^2 T^2}{8}$$

D.T.Son, A.S., hep-th/0601157

Einstein relation holds:
$$\frac{\sigma}{e^2 \Xi} = D_{U(1)} = \frac{1}{2\pi T}$$
Universality of $\frac{\eta}{s}$

Theorem:

*For a thermal gauge theory, the ratio of shear viscosity to entropy density is equal to $\frac{1}{4\pi}$ in the regime described by a dual gravity theory.*

(e.g. at $g_{YM}^2 N_c = \infty$, $N_c = \infty$ in $\mathcal{N} = 4$ SYM)

Remarks:

- Extended to non-zero chemical potential:
  
  Benincasa, Buchel, Naryshkin, hep-th/0610145

- Extended to models with fundamental fermions in the limit $\frac{N_f}{N_c} \ll 1$
  
  Mateos, Myers, Thomson, hep-th/0610184

- *String/Gravity dual to QCD is currently unknown*
Universality of shear viscosity in the regime described by gravity duals

\[ ds^2 = f(w) \left( dx^2 + dy^2 \right) + g_{\mu\nu}(w) dw^\mu dw^\nu \]

\[ \eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt \, dx e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

\[ \sigma_{abs} = -\frac{16\pi G}{\omega} \text{Im} \, G^R(\omega) \]

\[ = \frac{8\pi G}{\omega} \int dt \, dx e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle \]

Graviton’s component \( h^x_{y} \) obeys equation for a minimally coupled massless scalar. But then \( \sigma_{abs}(0) = A_H \).

Since the entropy (density) is \( s = A_H/4G \) we get

\[ \eta = \frac{A_H}{4G} \]

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]
Three roads to universality of $\eta/s$

- **The absorption argument**
  D. Son, P. Kovtun, A.S., hep-th/0405231

- **Direct computation of the correlator in Kubo formula from AdS/CFT**
  A. Buchel, hep-th/0408095

- **“Membrane paradigm” general formula for diffusion coefficient**
  + interpretation as lowest quasinormal frequency = pole of the shear mode correlator
  + Buchel-Liu theorem
A viscosity bound conjecture

\[ \frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \approx 6.08 \cdot 10^{-13} K \cdot s \]

Minimum of \( \frac{\eta}{s} \) in units of \( \frac{\hbar}{4\pi k_B} \)

- Xe \quad 84
- Kr \quad 57
- CO_2 \quad 32
- H_2O \quad 25
- C_2H_5OH \quad 22
- Ne \quad 17
- He \quad 8.8

$H_2O$

$(\eta/s)_{\text{min}} \sim 25$ in units of $\frac{\hbar}{4\pi k_B}$

Chernai, Kapusta, McLerran, nucl-th/0604032
\( (\eta/s)_{\text{min}} \sim 8.8 \) in units of \( \frac{\hbar}{4\pi k_B} \)

Chernai, Kapusta, McLerran, nucl-th/0604032
QCD

\[ \eta/s \]

\[ T(\text{MeV}) \]

Chernai, Kapusta, McLerran, nucl-th/0604032
A hand-waving argument

\[ \eta \sim \rho v l \sim \rho v^2 \tau \sim n m v^2 \tau \sim n \epsilon \tau \]

\[ s \sim n \]

Thus

\[ \frac{\eta}{s} \sim \epsilon \tau \geq \hbar \]

Gravity duals fix the coefficient:

\[ \frac{\eta}{s} \geq \hbar / 4\pi \]
Shear viscosity - (volume) entropy density ratio from gauge-string duality

In ALL theories (in the limit where dual gravity valid): \[ \frac{1}{4\pi} + \text{corrections} \]

In particular, in N=4 SYM: \[ \frac{1}{4\pi} + \frac{15\zeta(3)}{4\pi} \frac{1}{\lambda^{3/2}} + \ldots \]

Other higher-derivative gravity actions

\[ S = \int d^D x \sqrt{-g} \left( R - 2\Lambda + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots \right) \]

Y.Kats and P.Petrov: 0712.0743 [hep-th]


R.Myers,M.Paulos, A.Sinha: 0903.2834 [hep-th] (and ref. therein – many other papers)

\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - 8c_1 + \ldots \right) \]

\[ \frac{\eta}{s} = \frac{1}{4\pi} \left( 1 - \frac{1}{2N} \right) \]

for superconformal Sp(N) gauge theory in d=4

Viscosity “measurements” at RHIC

Viscosity is ONE of the parameters used in the hydro models describing the azimuthal anisotropy of particle distribution

\[
\frac{d^2 N^i}{dp_T d\phi} = N_0^i \left[ 1 + 2v_2^i(p_T) \cos 2\phi + \cdots \right]
\]

-elliptic flow for particle species “i”

Elliptic flow reproduced for

\[0 < \eta/s \leq 0.3\]
e.g. Baier, Romatschke, nucl-th/0610108

Perturbative QCD:

\[\eta/s (T_{RHIC}) \approx 1.6 \sim 1.8\]

Chernai, Kapusta, McLerran, nucl-th/0604032

SYM: \[\eta/s \approx 0.09 \sim 0.28\]
Elliptic flow with color glass condensate initial conditions

CGC

v_2 (percent) vs. p_T [GeV]

- ○ STAR non-flow corrected (est).
- ● STAR event-plane

\( \eta/s = 10^{-4} \)
\( \eta/s = 0.08 \)
\( \eta/s = 0.16 \)
\( \eta/s = 0.24 \)

Luzum and Romatschke, 0804.4015 [nuc-th]
Elliptic flow with Glauber initial conditions

Glauber

\[ v_2 (\text{percent}) \]

\[ p_T [\text{GeV}] \]

- \( \eta/s = 10^{-4} \)
- \( \eta/s = 0.08 \)
- \( \eta/s = 0.16 \)

Luzum and Romatschke, 0804.4015 [nuc-th]
Viscosity-entropy ratio of a trapped Fermi gas

\[ \eta/s \sim 4.2 \quad \text{in units of} \quad \frac{\hbar}{4\pi k_B} \]

T.Schafer, cond-mat/0701251

(based on experimental results by Duke U. group, J.E.Thomas et al., 2005-06)
Viscosity/entropy ratio in QCD: current status

Theories with gravity duals in the regime where the dual gravity description is valid

Kovtun, Son & A.S; Buchel; Buchel & Liu, A.S

\[ \frac{\eta}{s} = \frac{1}{4\pi} \approx 0.08 \]
(universal limit)

QCD: RHIC elliptic flow analysis suggests

\[ 0 < \frac{\eta}{s} < 0.2 \]

QCD: (Indirect) LQCD simulations

H.Meyer, 0805.4567 [hep-th]

\[ 0.08 < \frac{\eta}{s} < 0.16 \]

\[ 1.2 T_c < T < 1.7 T_c \]

Trapped strongly correlated cold alkali atoms

T.Schafer, 0808.0734 [nucl-th]

\[ \left( \frac{\eta}{s} \right)_{\text{min}} \approx 0.5 \]

\[ \left( \frac{\eta}{s} \right)_{\text{min}} \approx 0.7 \]

Liquid Helium-3
Shear viscosity at non-zero chemical potential

\[ \mathcal{N} = 4 \text{ SYM} \]
\[ q_i \in U(1)^3 \subset SO(6)_R \]
\[ Z = \text{tr} e^{-\beta H + \mu_i q_i} \]

(see e.g. Yaffe, Yamada, hep-th/0602074)

Reissner-Nordstrom-AdS black hole with three R charges

(Behrnd, Cvetic, Sabra, 1998)

We still have
\[ \frac{\eta}{s} = \frac{1}{4\pi} \]

\[ \eta = \pi N^2 T^3 \frac{m^2(1 - \sqrt{1 - 4m^2 - m^2})^2}{(1 - \sqrt{1 - 4m^2})^3} \]
\[ m \equiv \mu / 2\pi T \]
Spectral function and quasiparticles in finite-temperature “AdS + IR cutoff” model

\[ \chi(\omega) \sim N_c^2 \sum_{n=0}^{\infty} \omega_n^2 \rho(\omega_n) \delta(\omega - \omega_n) \]

\[ \chi(\omega) = \frac{N_c^2}{16\pi} \frac{\omega^2 \sinh(\omega/2T)}{\cosh(\omega/2T) - \cos(\omega/2T)} \]

\[ \mathcal{N} = 4 \text{ SYM} \]
Photon and dilepton emission from supersymmetric Yang-Mills plasma


![Radiated Power Density Planck Law](image1)

\[ S(\lambda) = \frac{2\pi c^2}{\lambda^5} \frac{1}{\lambda c} e^{-\frac{\lambda c}{\hbar k T}} \]

![Irradiance (W/m²·μm)](image2)

- Outside atmosphere
- At Earth's surface (m=2°)

![Sun's spectrum](image3)
Photon emission from SYM plasma

Photons interacting with matter: \( e J_{\mu}^{EM} A^\mu \)

To leading order in \( e \)

\[
\frac{d\Gamma_{\gamma}}{(2\pi)^3} \frac{e^2}{2|k|} \eta^{\mu\nu} C^{\mu\nu}_<(k^0 = |k|) \]

\[
C^{\mu\nu}_< = \int d^4X e^{-iKX} \langle J_{\mu}^{EM}(0) J_{\nu}^{EM}(X) \rangle
\]

Mimic \( J_{\mu}^{EM} \) by gauging global R-symmetry \( U(1) \subset SU(4) \)

\[
\mathcal{L} = \mathcal{L}_{\mathcal{N}=4 SYM} + e J_{\mu}^3 A^\mu - \frac{1}{4} F_{\mu\nu}^2
\]

Need only to compute correlators of the R-currents \( J_{\mu}^3 \)
Photoproduction rate in SYM

(Normalized) photon production rate in SYM for various values of ‘t Hooft coupling

\[
\frac{d\Gamma_\gamma}{dk \alpha_{em} N_c^2 T^3} = n_B(k) \left( \frac{k}{4\pi T} \right)^2 \ _2F_1 \left( 1 - \frac{(1 + i)k}{4\pi T}, 1 + \frac{(1 - i)k}{4\pi T}; 1 - \frac{ik}{2\pi T}; -1 \right)^{-2}
\]
Transport coefficients are non-trivial functions of the parameters of the underlying microscopic theory, even in the simplest case of conformal liquids

\[
\frac{\eta}{s} = F \left( N_c, N_f, \frac{m}{T}, \frac{\Lambda}{T}, \ldots \right)
\]

Some information can be obtained from kinetic theory at weak coupling and from gauge-string duality at strong coupling (for theories with string or gravity duals). In the latter case, it is natural to look for “universal” (independent of the specific string construction) results.

In the limit of infinite \( N \) (gauge group rank) and infinite coupling (i.e. in the supergravity approximation from dual string theory point of view)

\[
\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \\
\frac{\sigma}{\chi} = \frac{1}{4\pi T} \frac{d}{d - 2}
\]

[for conformal relativistic fluids]: Kovtun, Ritz, 2008

Policastro, Kovtun, Son, AOS, 2001-2008
Buchel, J.Liu, 2003; Buchel, 2004
Iqbal, H.Liu, 2008

\[2\eta \tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0 \] [for conformal relativistic fluids]: Haack, Yarom, 2008
More speculative statements inspired by holography

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

Kovtun, Son, AOS, 2004 (violated by some models with Higher derivative gravity, Kats and Petrov, 2007; Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008)

$$\frac{\zeta}{\eta} \geq 2 \left( \frac{1}{d - 1} - c_s^2 \right)$$

Buchel, 2008 (counterexample: Buchel, 2012)
See also Kanitscheider and Skenderis, 2009

$$c_s^2 \leq c_{s,conf}^2 = \frac{1}{d - 1}$$

Hohler and Stephanov, 2009; Cherman, Cohen, Nellore, 2009

$$\frac{\sigma}{\chi} \geq \frac{\hbar v^2}{4\pi T} \frac{d}{d - 2}$$

Kovtun and Ritz, 2008

$$D \gtrsim \frac{\hbar v_F}{k_B T}$$

Hartnoll, 2014

See also “Fluctuation bounds…” by Kovtun 1407.0690 [hep-th]
Energy and Momentum Density

\[ \frac{x|\Delta E(x)}{T^3 \sqrt{\lambda}} \]

\[ \frac{|x|\Delta S(x)}{T^3 \sqrt{\lambda}} \]

(Chesler and Yaffe)
Now consider strongly interacting systems at finite density and LOW temperature.
AdS/CFT and condensed matter physics

S. Hartnoll
“Lectures on holographic methods for condensed matter physics”,
0903.3246 [hep-th]

C. Herzog
“Lectures on holographic superfluidity and superconductivity”,
0904.1975 [hep-th]

M. Rangamani
“Gravity and hydrodynamics: Lectures on the fluid-gravity correspondence”,
0905.4352 [hep-th]

S. Sachdev
“Condensed matter and AdS/CFT”,
1002.2947 [hep-th]
AdS/CFT and condensed matter physics

* Studies of systems at low temperature, finite density

* Quantum critical points, holographic superconductors

* Top-down and bottom up approaches

* Searching for a Fermi surface

\[ G_F^{-1}(\omega, q) = -i\omega + v_F(q - q_F) - c_1\omega^\theta \]

Faulkner, Liu, McGreevy, Vegh, 0907.2694 [hep-th]
Probing quantum liquids with holography

<table>
<thead>
<tr>
<th>Quantum liquid in ( p+1 ) dim</th>
<th>Low-energy elementary excitations</th>
<th>Specific heat at low ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantum Bose liquid</td>
<td>phonons</td>
<td>( \sim T^p )</td>
</tr>
<tr>
<td>Quantum Fermi liquid (Landau FLT)</td>
<td>fermionic quasiparticles + bosonic branch (zero sound)</td>
<td>( \sim T )</td>
</tr>
</tbody>
</table>

Departures from normal Fermi liquid occur in

- 3+1 and 2+1 –dimensional systems with strongly correlated electrons
- In 1+1 –dimensional systems for any strength of interaction (Luttinger liquid)

One can apply holography to study strongly coupled Fermi systems at low \( T \)
L.D. Landau (1908-1968)
The simplest candidate with a known holographic description is

\[ SU(N_c) \ \mathcal{N} = 4 \ \text{SYM} \ \text{coupled to} \ N_f \ \mathcal{N} = 2 \ \text{fundamental hypermultiplets} \]

at finite temperature \( T \) and nonzero chemical potential associated with the “baryon number” density of the charge \( U(1)_B \subset U(N_f) \)

There are two dimensionless parameters:

\[ \frac{n_q^{1/3}}{T}, \quad \frac{M}{T} \]

\( n_q \) is the baryon number density \( M \) is the hypermultiplet mass

The holographic dual description in the limit \( N_c \gg 1, \ g_{YM}^2 N_c \gg 1, \ N_f \) finite is given by the D3/D7 system, with D3 branes replaced by the AdS-Schwarzschild geometry and D7 branes embedded in it as probes.

Karch & Katz, hep-th/0205236
AdS-Schwarzschild black hole (brane) background

\[ ds^2 = \frac{r^2}{R^2} \left[ - \left( 1 - \frac{r_H^4}{r^4} \right) dt^2 + d\vec{x}^2 \right] + \left( 1 - \frac{r_H^4}{r^4} \right)^{-1} \frac{R^2}{r^2} dr^2 \]

D7 probe branes

\[ S_{DBI} = -N_f T_{D7} \int d^8 \xi \sqrt{-\det(g_{ab} + 2\pi \alpha' F_{ab})} \]

The worldvolume U(1) field \( A_\mu \) couples to the flavor current \( J^\mu \) at the boundary.

Nontrivial background value of \( A_0 \) corresponds to nontrivial expectation value of \( J^0 \).

We would like to compute

- the specific heat at low \( (Tn_q^{-1/3} \ll 1) \) temperature
- the charge density correlator \( G^R \sim \langle J^0(k) J^0(-k) \rangle \)
The specific heat (in $p+1$ dimensions):

$$c_V = \mathcal{N}_q p \left( \frac{4\pi}{p+1} \right)^{2p+1} \frac{T^{2p}}{n_q} \left[ 1 + O(Tn_q^{-\frac{1}{p}}) \right]$$

(note the difference with Fermi $c_V \sim T$ and Bose $c_V \sim T^p$ systems)

The (retarded) charge density correlator $G^R \sim \langle J^0(k) J^0(-k) \rangle$

has a pole corresponding to a propagating mode (zero sound)
- even at zero temperature

$$\omega = \pm \frac{q}{\sqrt{p}} - \frac{i \Gamma\left(\frac{1}{2}\right) q^2}{n_q^p \Gamma\left(\frac{1}{2} - \frac{1}{2p}\right) \Gamma\left(\frac{1}{2p}\right)} + O(q^3)$$

(note that this is NOT a superfluid phonon whose attenuation scales as $q^{p+1}$)

New type of quantum liquid?
Other avenues of (related) research

Bulk viscosity for non-conformal theories (Buchel, Gubser,…)

Non-relativistic gravity duals (Son, McGreevy,…)

Gravity duals of theories with SSB (Kovtun, Herzog,…)

Bulk from the boundary (Janik,…)

Navier-Stokes equations and their generalization from gravity (Minwalla,…)

Quarks moving through plasma (Chesler, Yaffe, Gubser,…).
Epilogue

- On the level of theoretical models, there exists a connection between near-equilibrium regime of certain strongly coupled thermal field theories and fluctuations of black holes.

- This connection allows us to compute transport coefficients for these theories.

- At the moment, this method is the only theoretical tool available to study the near-equilibrium regime of strongly coupled thermal field theories.

- The result for the shear viscosity turns out to be universal for all such theories in the limit of infinitely strong coupling.

- Influences other fields (heavy ion physics, condmat).