Power counting and scaling for tensor models

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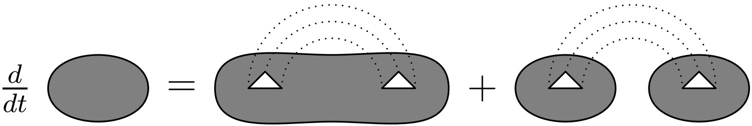
Polchinski's equation for tensor models and tensorial group field theories

Formulation of an exact renormalization group equation in terms of bubble couplings (boundary triangulation)

- Evolution of bubble couplings in tensor models
  $\Rightarrow$ Melonic bubble dominance at large N
- Dimensional analysis for abelian group field theories
  $\Rightarrow$ Classification of renormalizable theories



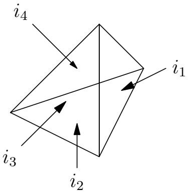
Renormalization group equation for boundary couplings

Tensor modelz

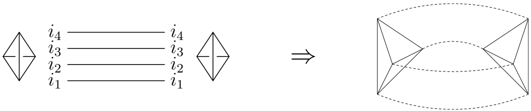
Tensor models = generalizations of matrix models for random geometry in dimension D

$$\int d\overline{T} dT \, \exp\{-\overline{T}\cdot C^{-1}\cdot T + S(\overline{T},T)\} = \sum_{\substack{\text{Feynman graph } \mathcal{G} \Leftrightarrow \\ \text{dimension } D \text{ triangulation}}} \frac{\mathcal{A}_{\mathcal{G}}}{\sigma_{\mathcal{G}}}$$

- $T_{i_1,\ldots,i_D}$ and $\overline{T}_{i_1,\ldots,i_D}$ propagating $(D-1)$-simplex



- Covariance C: identifications of $(D-2)$-simplices



- Action: $S(\overline{T},T)$ basic building blocks (boundary triangulation)

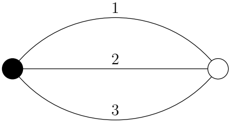
Expansion over bubble couplings ⇔ boundary triangulation

$$S(\Phi,\overline{\Phi}) = \sum_{\substack{\mathcal{B}\\ \text{bubble}}} \frac{1}{\sigma_{\mathcal{B}}} \sum_{\{i_e,\bar{i}_e\}} u_{\mathcal{B}}(\{i_e,\bar{i}_e\}) \prod_{\substack{v\\ \text{white vertex}}} \Phi_{I_{\mathcal{B}}(v)} \prod_{\substack{\bar{v}\\ \text{black vertex}}} \overline{\Phi}_{\bar{I}_{\mathcal{B}}(\bar{v})}$$

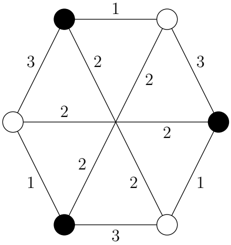
- $\mathcal{B}$ bipartite graph (white vertex T, black vertex $\overline{T}$) with vertices of valence D (dual to a triangulation of dimension $D-1$)
- Proper coloring of the edges by 1, .. ,N
- $I_{\mathcal{B}}(v) = D$-tuple of indices $\{i_e\}$ pertaining to the lines of colour $1,\ldots,D$ attached to the white vertex v ($\bar{I}_{\mathcal{B}}$ for black vertex $\bar{v}$)
- Summation over $\{i_e,\bar{i}_e\}$ from 1 to N for each edge
- $\sigma_{\mathcal{B}}$ = order of the symmetry group (including color permutation)
- $\lambda_{\mathcal{B}}(\{i_e,\bar{i}_e\})$ bubble coupling
- $\mathcal{B}$ not necessarily connected ("multitrace operator" $\prod_i [\mathrm{Tr}(\Phi)^{n_i}]^{k_i}$))

Special case: Invariant models

Invariance under $\mathrm{U}(N)^D$: $T \to U^{\otimes N} T$, $\overline{T} \to \overline{U}^{\otimes N}\overline{T}$
$\Rightarrow \lambda_{\mathcal{B}}\{i_e,\bar{i}_e\} = \lambda_B \prod_e \delta_{i_e,\bar{i}_e}$ with $\lambda_{\mathcal{B}}$ scalar



$$\frac{1}{3}\lambda \sum_{i_a} \overline{T}_{i_1,i_2,i_3} T_{i_1 i_2 i_3}$$



$$\frac{1}{6}\lambda \sum_{i_a, j_b, k_c} \overline{T}_{i_1 i_2 i_3} \overline{T}_{j_1 j_2 j_3} \overline{T}_{k_1 k_2 k_3} T_{i_1 k_2 j_3} T_{j_1 i_2 k_3} T_{k_1 j_2 i_3}$$

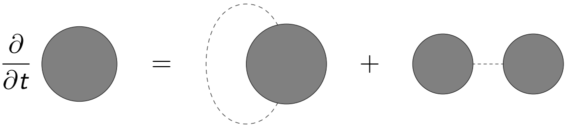
Wilsonian effective action obtained by a partial integration with covariance $C_{t,t_0} = \int_{t_0}^{t} ds\, K_s$ (dsK_s = integration over infinitesimal shell)

$$S_{t,t_0}[\Phi,\overline{\Phi}] = \log \int \frac{d\overline{\Psi} d\Psi}{\mathcal{N}_{t,t_0}} \exp\left\{ -\overline{\Psi} \cdot C_{t,t_0}^{-1} \cdot \Psi + S_{t_0}[\Phi+\Psi, \overline{\Phi}+\overline{\Psi}] \right\},$$

$\mathcal{N}_{t,t_0}$ = normalization factor

Polchinski's exact renormalisation group equation for tensors

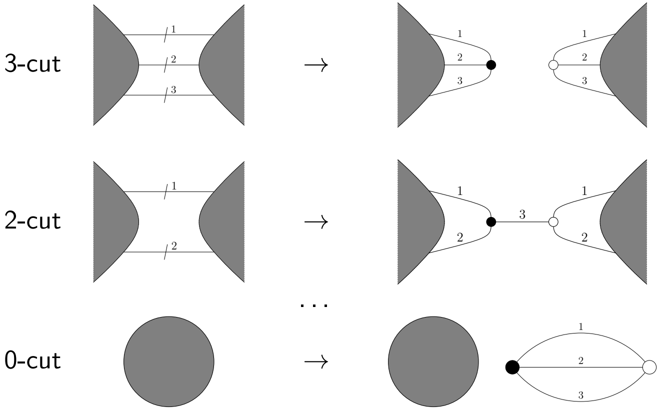
$$\frac{\partial S}{\partial t} = \sum_{I,\bar{I}} K_{I,\bar{I}} \left(\frac{\partial^2 S}{\partial \overline{\Phi}_{\bar{I}} \partial \Phi_I} + \frac{\partial S}{\partial \overline{\Phi}_{\bar{I}}} \frac{\partial S}{\partial \Phi_I} \right)$$



- $\frac{\partial S}{\partial \Phi_I}$ removes a white vertex ($\frac{\partial S}{\partial \overline{\Phi}_{\bar{I}}}$ for black)
- K attaches resulting half-edges respecting colors

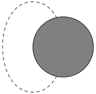
Definition of a c-cut of $\{c \in 0, \ldots, D\}$ colors

- cut in 2 halves c edges of different colors
- attach the resulting half-edges to a new pair $v, \overline{v}$
- complete with $D - c$ new edges (colors not in the cut)



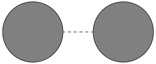
cut= inverse of $\sum_{I,\bar{I}} K_{I,\bar{I}} \left(\frac{\partial^2}{\partial T_I \partial \overline{T}_{\bar{I}}} + \frac{\partial}{\partial \overline{T}_{\bar{I}}} \frac{\partial}{\partial T_I} \right)$

$\Rightarrow$ system of non linear differential equations for $\lambda_{\mathcal{B}}$



$$\frac{\partial \lambda_{\mathcal{B}}}{\partial t} \{i_e, \bar{i}_e\} = \sum_{k=0}^{D} \sum_{k\text{-cut } c} \overbrace{\sum_{\{j_l, \bar{j}_l\}} K\{j_l, \bar{j}_l\} \lambda_{\mathcal{B}_c} \left\{ \{i_e, \bar{i}_e\}_{e \in \mathcal{B}}, \{j_l, \bar{j}_l\} \right\}}$$

$$+ \sum_{\substack{D\text{-cut } c \\ \kappa(\mathcal{B}_c) > \kappa(\mathcal{B}_c)}} \sum_{\substack{\mathcal{B}', \mathcal{B}'' \\ \mathcal{B}_c = \mathcal{B}' \cup \mathcal{B}''}} \sum_{\{j_l, \bar{j}_l\}} \Bigg(\underbrace{K\{j_l, \bar{j}_l\} \lambda_{\mathcal{B}'} \left\{ \{i_e\}_{e \in \mathcal{B}'}, \{\bar{i}_e\}_{e \in \mathcal{B}' - c}, \{\bar{j}_l\} \right\} \lambda_{\mathcal{B}''} \left\{ \{\bar{i}_e\}_{e \in \mathcal{B}''}, \{i_e\}_{e \in \mathcal{B}'' - c}, \{j_l\} \right\}} \Bigg)$$



Invariant tensor model with covariance $C_{I,\bar{I}} = z\int_0^t ds\, \delta_{i_1,\bar{i}_1}\cdots\delta_{i_D,\bar{i}_D}$

$$\frac{\partial\lambda_{\mathcal{B}}}{\partial t} = \sum_{k=0}^{D}\sum_{k\text{-cut } c} z\,N^{D-k}\lambda_{\mathcal{B}_c} + \sum_{\substack{D\text{-cut } c\\ \kappa(\mathcal{B}_c)>\kappa(\mathcal{B}_c)}}\sum_{\substack{\mathcal{B}',\mathcal{B}''\\ \mathcal{B}_c=\mathcal{B}'\cup\mathcal{B}'',v\in\mathcal{B}',\bar{v}\in\mathcal{B}''}} z\lambda_{\mathcal{B}'}\lambda_{\mathcal{B}''}$$

Proof: Each edge not in c contributes $\sum_i = N$

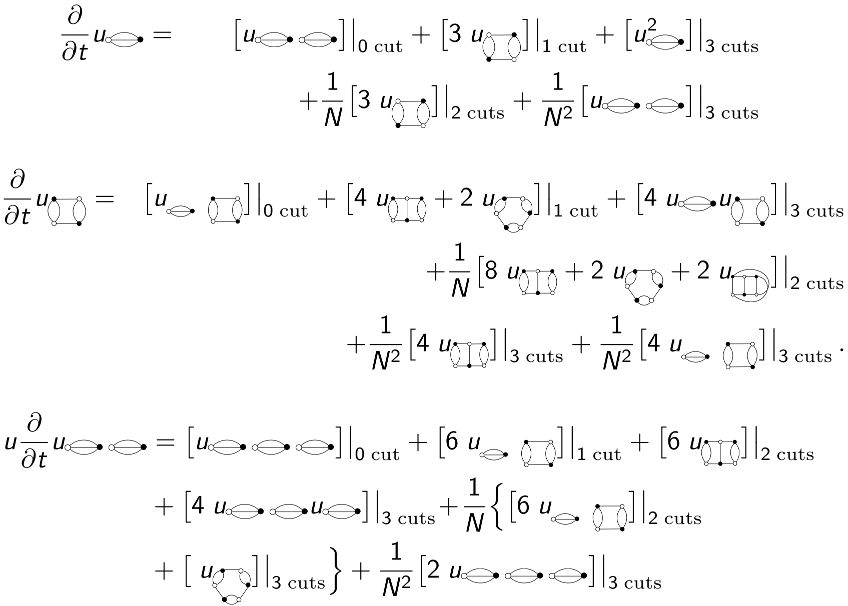
Differential equations for rescaled couplings

Rescaling the covariance $z = \frac{1}{N^{D-1}}$ and bubble couplings $\lambda_{\mathcal{B}} = N^{D-\kappa_{\mathcal{B}}}u_{\mathcal{B}}$ ("dimensionless" quantites), with $\kappa_{\mathcal{B}} = \#\{\text{connected components of } \mathcal{B}\}$

$$\frac{\partial u_{\mathcal{B}}}{\partial t} = \sum_{k=0}^{D}\sum_{k\text{-cut } c} \frac{u_{\mathcal{B}_c}}{N^{k-\kappa_{\mathcal{B},c}}} + \sum_{\substack{D\text{-cut } c\\ \kappa(\mathcal{B}_c)>\kappa(\mathcal{B}_c)}}\sum_{\substack{\mathcal{B}',\mathcal{B}''\\ \mathcal{B}_c=\mathcal{B}'\cup\mathcal{B}'',v\in\mathcal{B}',\bar{v}\in\mathcal{B}''}} u_{\mathcal{B}'}u_{\mathcal{B}''}$$

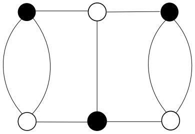
where $\kappa_{\mathcal{B},c} = \#\{\text{connected components of } \mathcal{B} \text{ containing edges in } c\}$

- Proof: $\kappa_{\mathcal{B}_c} = \kappa_{\mathcal{B}} - \kappa_{\mathcal{B},c} + 1$
- $\kappa_{\mathcal{B},c} \leq k \Rightarrow$ only 0-cuts and k-cuts ($k>0$) with one edge in each connected component contribute when $N\to\infty$, others are $O(\frac{1}{N})$

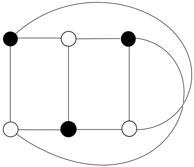

∂/∂t u = [u] |0 cut + [3 u] |1 cut + [u²] |3 cuts
+ 1/N [3 u] |2 cuts + 1/N² [u] |3 cuts
∂/∂t u = [u] |0 cut + [4 u + 2 u] |1 cut + [4 u u] |3 cuts
+ 1/N [8 u + 2 u + 2 u] |2 cuts
+ 1/N² [4 u] |3 cuts + 1/N² [4 u] |3 cuts .
u ∂/∂t u = [u] |0 cut + [6 u] |1 cut + [6 u] |2 cuts
+ [4 u u] |3 cuts + 1/N { [6 u] |2 cuts
+ [u] |3 cuts } + 1/N² [2 u] |3 cuts

Melonic bubble

$\mathcal{B}$ a bubble is melonic if for every white vertex v there is a black vertex $\bar{v}$ such that removing v and $\bar{v}$ and reattaching the lines of identical colours increases the number of connected components by $D-1$.



Melonic bubble



Non melonic bubble

Large N-universality of melonic couplings

At large N, melonic couplings $u_{\mathcal{B}}(t)$ only depend on melonic initial conditions $u_{\mathcal{B}}(t_0)$ (coupling of non melonic bubbles are irrelevant)

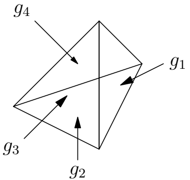
- Non melonic initial conditions yield $\frac{1}{N}$ corrections
- $\frac{\partial u_{\mathcal{B}}(t)}{\partial u_{\mathcal{B}_0}(t_0)}$ can be computed in a Gaußian theory for $\mathcal{B}, \mathcal{B}_0$ melonic (Gurau's Gaußian universality)

What is group field theory?

Group field theory = quantum field theory over D copies of a group SU(2), SO(4), SL$(2,\mathbb{C})$ whose perturbative expansion yields a sum over triangulations weighted by spin foam amplitudes

$$\int [\mathcal{D}\Phi\mathcal{D}\overline{\Phi}] \exp S(\Phi,\overline{\Phi}) = \sum_{\substack{\mathcal{G}\Leftrightarrow \\ \text{triangulation}}} \frac{\mathcal{A}_{\mathcal{G}}}{\mathrm{Sym}_{\mathcal{G}}}$$

- Group field $\Phi(g_1,\ldots,g_D)$: tetrahedron



- Same formalism as for random tensors with

$$T_{i_1,\ldots,i_D} \to \Phi(g_1,\ldots,g_D) \qquad \overline{T}_{\bar{i}_1,\ldots,\bar{i}_D} \to \overline{\Phi}(\bar{g}_1,\ldots,\bar{g}_D).$$

Effective action

Effective action expanded over bubble couplings

$$S(\Phi,\overline{\Phi}) = \sum_{\mathcal{B}} \frac{1}{\sigma_{\mathcal{B}}} \int \prod_e dg_e d\overline{g}_e \, \lambda_{\mathcal{B}}(\{g_e \overline{g}_e^{-1}\}) \prod_v \Phi(G_{\mathcal{B}}(v)) \prod_{\overline{v}} \overline{\Phi}(\overline{G}_{\mathcal{B}}(\overline{v}))$$

where $G_{\mathcal{B}}(v)$ and $\overline{G}_{\mathcal{B}}(\overline{v})$ indicate the D-tuplets of group elements (ordered by their colours) incident to v and $\overline{v}$ in $\mathcal{B}$.

- Covariance and coupling only depend on the products $g_e \overline{g}_e^{-1}$
- Closure constraint (tetrahedron) $\Rightarrow$ gauge invariance (covariance and couplings)

$$C\left\{g_e\overline{g}_e^{-1}\right\} = C\left\{h g_e \overline{g}_e^{-1} \overline{h}^{-1}\right\} \quad \lambda_{\mathcal{B}}\left\{g_e\overline{g}_e^{-1}\right\} = \lambda_{\mathcal{B}}\left\{h_{v_{\mathcal{B}}(e)} g_e \overline{g}_e^{-1} \overline{h}^{-1}_{\overline{v}_{\mathcal{B}}(e)}\right\}$$

- Effective action expanded over bubble couplings with closure constraint

Covariance

$$C_{\Lambda,\Lambda_0}\left(\{g_e\overline{g}_e^{-1}\}\right) = \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\alpha K_\alpha\left(\{g_e\overline{g}_e^{-1}\}\right)$$

with heat kernel on the group manifold

$$K_\alpha\left(\{g_e\overline{g}_e^{-1}\}\right) = \int dh d\overline{h} \prod_{1\leq i\leq D} H_\alpha(hg_i\overline{g}_i^{-1}\overline{h}^{-1}).$$

Λ ($\propto e^t$ for tensor models) = "ultraviolet" cut-off

Polchinski's equation for group field theories

$$\Lambda\frac{\partial S}{\partial\Lambda} = -\frac{2}{\Lambda^2}\int\prod_{1\leq i\leq D} dg_i d\overline{g}_i K_{\frac{1}{\Lambda^2}}\{g_i\overline{g}_i^{-1}\}\left(\frac{\delta S}{\delta\overline{\Phi}(\overline{G})}\frac{\delta S}{\delta\Phi(G)} + \frac{\delta^2 S}{\delta\overline{\Phi}(\overline{G})\delta\Phi(G)}\right)$$

- Same structure as tensor models with $i\to g$ and $\sum_i \to \int dg$
- New ingredients: non trivial propagator and closure constraint

- Replace group by $(\mathrm{U}(1))^d$ (technical simplification) with size L
- Discrete momenta $\Rightarrow$ closure constraint $\delta_{\sum p,0}$
- Heat kernel covariance $\Rightarrow$ Gaußian integrals (sums)

$$C_{\Lambda,\Lambda_0}\left\{\theta_i-\bar{\theta}_i\right\}=\int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\alpha \sum_{\{p_i\}\in\frac{\mathbb{Z}^{dD}}{L}} \exp-\left\{\alpha\sum_i p_i^2+\mathrm{i}\sum p_i(\theta_i-\bar{\theta}_i)\right\}\delta_{\sum p_i,0}$$

Flow equation for bubble couplings in abelian models

$$\Lambda\frac{\partial\lambda_{\mathcal{B}}}{\partial\Lambda}\{p_e\}=$$

$$-\frac{2}{\Lambda^2}\sum_{\substack{0\leq k\leq D\\ k\text{-cut } c}}\sum_{\{p_l\}_{l\notin c}}\delta_{(\sum_{e\in c}p_e),0}\, e^{-\frac{\sum_{l\notin c}p_l^2+\sum_{e\in c}p_e^2}{\Lambda^2}}\lambda_{B_c}\left\{\{p_e\}_{e\in\mathcal{B}},\{p_l\}_{l\notin c}\right\}$$

$$-\frac{2}{\Lambda^2}\sum_{\substack{D\text{-cut } c\\ \kappa(\mathcal{B}_c)>\kappa(\mathcal{B}_c)}}\sum_{\substack{\mathcal{B}',\mathcal{B}''\\ \mathcal{B}_c=\mathcal{B}'\cup\mathcal{B}''}} e^{-\frac{\sum_{e\in c}p_e^2}{\Lambda^2}}\lambda_{\mathcal{B}'}\{p_e\}_{e\in\mathcal{B}'}\lambda_{\mathcal{B}''}\{p_e\}_{e\in\mathcal{B}''}$$

Problem: find the right scaling dimension $\lambda_{\mathcal{B}} \sim \Lambda^{\delta_{\mathcal{B}}}$ when $\Lambda \to \infty$

- Each field has $D-1$ momenta (closure constraint) with d components each

$$\#\left\{\text{number of modes with } \sum p^2 < \Lambda^2\right\} \sim (L\Lambda)^{d(D-1)}$$

$\Rightarrow$ free field effective action scales as $(L\Lambda)^{d(D-1)}$ ("cosmological constant") $\Rightarrow \delta_S = d(D-1)$

- Kinetic term involves $d(D-1)$ momentum integrations and

$$\delta_S = d(D-1)\ (\text{momentum integration}) + 2(\text{Laplacian}) + 2\delta_\Phi$$

$\Rightarrow \delta_\Phi = -1.$

- Each bubble $\mathcal{B}$ with $e_{\mathcal{B}}$ edges, $v_{\mathcal{B}}$ vertices and $\kappa_{\mathcal{B}}$ connected components involves $e_{\mathcal{B}} - v_{\mathcal{B}} + \kappa_{\mathcal{B}}$ momentum summations

$$\delta_S = \delta_{\mathcal{B}} + d\left(e_{\mathcal{B}} - v_{\mathcal{B}} + \kappa_{\mathcal{B}}\right) + v_{\mathcal{B}}\delta_\Phi$$

Scaling dimension of a bubble coupling when $\Lambda \to \infty$ with L fixed

$$\delta_{\mathcal{B}} = d(D-1) - d\kappa_{\mathcal{B}} - \left[d(D-2) - 2\right]\frac{v_{\mathcal{B}}}{2}$$

- Introduce rescaled ("dimensionless") couplings $u_{\mathcal{B}}$ and momenta q

$$\lambda_{\mathcal{B}}(\{p_e\}\,,\Lambda) = \Lambda^{\delta_{\mathcal{B}}} u_{\mathcal{B}}(\{q_e\}\,,\Lambda) \qquad \text{with } q_e = \frac{p_e}{\Lambda}.$$

- Assume L large enough so tha the continuum approximation holds

$$\sum_{p \in \frac{\mathbb{Z}^{(D-k)d}}{L}} \to L^{(D-k)d} \int dp \qquad \text{and} \qquad \delta_{\sum p_i,0} \to \frac{1}{L^d}\delta(\sum p_i).$$

Flow equation for rescaled variables

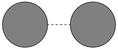
$$\Lambda \frac{\partial u_{\mathcal{B}}}{\partial \Lambda}\{q_e\} = -\delta_{\mathcal{B}}\, u_{\mathcal{B}}\{q_e\} + \sum_e q_e \frac{\partial u_{\mathcal{B}}}{\partial q_e}\{q_e\} \quad \text{(dimensional analysis)}$$

$$-2\sum_{k=0}^{D} \sum_{k\text{-cut } c} \frac{L^{d(D-k-1)}}{\Lambda^{d(k-\kappa_{\mathcal{B},c})}} \int \prod_{l\notin c} dq_l\, \delta\Big(\sum_{e\in c} q_e\Big)\, e^{-\left(\sum_{\substack{l\notin c\\ e\in c}} q_{l,e}^2\right)} u_{B_c}\{q_{e,l}\}_{e\in\mathcal{B},l\notin c}$$

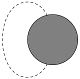
$$-\,2 \sum_{\substack{D\text{-cut } c\\ \kappa(\mathcal{B}_c)>\kappa(\mathcal{B})}} \sum_{\substack{\mathcal{B}',\mathcal{B}''\\ \mathcal{B}_c=\mathcal{B}'\cup\mathcal{B}''}} e^{-\sum_{e\in c} q_e^2}\, u_{\mathcal{B}'}\{q_e\}_{e\in\mathcal{B}'}\, u_{\mathcal{B}''}\{q_e\}_{e\in\mathcal{B}''}$$

$\Rightarrow$ only non positive powers of the large cut-off Λ

Proof based on comparison of scaling dimensions on each side

- Tree term 

$$\begin{aligned}\delta_{\mathcal{B}''}+\delta_{\mathcal{B}'} &= 2d(D-1)-d(\kappa_{\mathcal{B}'}+\kappa_{\mathcal{B}''})-\big[d(D-2)-2\big]\frac{v_{\mathcal{B}'}+v_{\mathcal{B}''}}{2}\\ &= 2d(D-1)-d(\kappa_{\mathcal{B}}+1)-\big[d(D-2)-2\big]\frac{v_{\mathcal{B}}+2}{2}\\ &= \delta_{\mathcal{B}}+2\end{aligned}$$

- Loop term 

$$\begin{aligned}\delta_{\mathcal{B}_c} &= d(D-1)-d\kappa_{\mathcal{B}_c}-\big[d(D-2)-2\big]\frac{v_{\mathcal{B}_c}}{2}\\ &= d(D-1)-d(\kappa_{\mathcal{B}}-\kappa_{\mathcal{B},c}+1)-\big[d(D-2)-2\big]\frac{v_{\mathcal{B}}+2}{2}\\ &= \delta_{\mathcal{B}}+d\kappa_{\mathcal{B},c}-d(D-1)+2\end{aligned}$$

$p=\Lambda q$ in integration $\rightarrow \Lambda^{d(D-k)}$ and constraint $\rightarrow \Lambda^{-d}$

Renormalization based on Polchinski's equation

Renormalizable couplings have positive scaling dimension

$$\delta_{\mathcal{B}} = d(D-2) - d(\kappa_{\mathcal{B}}-1) - [d(D-2)-2]\frac{v_{\mathcal{B}}}{2} \geq 0 \Rightarrow v_{\mathcal{B}} \leq 2 + \frac{4}{d(D-2)-2}$$

List of renormalizable theories (finite number of couplings):

- <u>Case $d(D-2)=3$</u>

  $d=3, D=3$ and $D=5, d=1$

  2 vertices: $\overline{\Phi}\Phi$ ($\delta=2$ mass), $\partial^2\overline{\Phi}\Phi$ ($\delta=0$, kinetic)

  4 vertices: $(\overline{\Phi}\Phi)^2$ ($\delta=2$)

  $(\overline{\Phi}\Phi)(\overline{\Phi}\Phi)$ ($\delta=1$, non connected, only for $d=1$)

  6 vertices: $(\overline{\Phi}\Phi)^3$ ($\delta=0$, melonic and non melonic)

- <u>Case $d(D-2)=4$</u>

  ($d=4, D=3$, $D=2, d=4$ and $D=6, d=1$)

  2 vertices: $\overline{\Phi}\Phi$ ($\delta=2$ mass), $\partial^2\overline{\Phi}\Phi$ ($\delta=0$, kinetic)

  4 vertices: $(\overline{\Phi}\Phi)^2$ ($\delta=0$)

We recover results by Ben Geloun, Carrozza Rivasseau and Oriti based on multiscale analysis

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Outlook

- Rigorous bounds (perturbative and non perturbative)
- Systematic analysis of $\frac{1}{N}$ corrections
- Fixed points and truncations using Wetterich type equation
- Non abelian models
- Models involving a simplicity constraint ($4d$ quantum gravity)