Photon Self-Interaction in a Deformed U(1) Gauge Theory⋆

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- Partially based on 1506.02864, see also 0807.4886, 1109.2485, 1111.4951, 1306.1239, 1402.6184 and 1501.00276.

Outline

Seiberg-Witten map deformed $U(1)$ gauge theory

3-photon interaction and 1-loop photon bubble

The 4-photon interaction and 4-photon tadpole

Summing tadpole into 1-loop photon two-point function

Summary

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Summary

IR divergences

- Quadratic and logarithmic IR divergences exist in NC gauge theory, for example $\mathrm{U}_\star(1)$

$$\Pi^{\mu\nu}(p) \sim \frac{e^2}{(4\pi)^2}\left(\frac{10}{3}(g^{\mu\nu}p^2 - p^\mu p^\nu)\right.$$
$$\left.\underbrace{\left(\frac{2}{\epsilon} - \ln\frac{p^2}{\mu^2} + \ln(p^2(\theta p)^2)\right)}_{+\ln(\mu^2(\theta p)^2)} + 32\underbrace{\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}}_{\text{quadratic}}\right)$$

- Due to the popular expansion over $\theta^{\mu\nu}$, situation was unclear after we add SW map.

$$S = -\frac{1}{4}\int F^{\mu\nu}\star F_{\mu\nu} = -\frac{1}{4}\int f^{\mu\nu}f_{\mu\nu} + S^{\theta^2} + S^{\theta^3} + S^{\theta^4} + \ldots\ldots$$
$$= -\frac{1}{4}\int f^{\mu\nu}f_{\mu\nu} - 2\theta^{ij}f^{\mu\nu}\left(\frac{1}{4}f_{ij}f_{\mu\nu} - f_{\mu i}f_{\nu j}\right) + \theta^{ij}\theta^{kl}\left(\ldots\ldots\right)$$
$$+\frac{1}{3}\theta^{ij}\theta^{kl}\theta^{mn}\left(\frac{1}{4}\partial_k\partial_m f_{ij}\partial_l\partial_n f_{\mu\nu} - \partial_k\partial_m f_{\mu i}\partial_l\partial_n f_{\nu j}\right) + \ldots\ldots$$

θ-exact SW-map expansion

- Expansion over a_μ preserves the nonlocality, as used here for $\mathrm{U}_\star(1)$ theory

$$A_\mu = a_\mu - \frac{1}{2}\theta^{ij} a_i \star_2 (\partial_j a_\mu + f_{j\mu}) + \mathcal{O}(a^3)\,,$$
$$F_{\mu\nu} = f_{\mu\nu} + \theta^{ij}\Big(f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu\nu}\Big) + \mathcal{O}(a^3)\,.$$

- New generalized star product $\star_2$ is commutative and non-associative.

$$f \star_2 g = f(x_1)\frac{\sin\frac{\partial_1\theta\partial_2}{2}}{\frac{\partial_1\theta\partial_2}{2}} g(x_2)\Big|_{x_1=x_2=x} = g(x_2)\frac{\sin\frac{\partial_2\theta\partial_1}{2}}{\frac{\partial_2\theta\partial_1}{2}} f(x_1)\Big|_{x_1=x_2=x} = g \star_2 f\,,$$
$$(f \star_2 g)\star_2 h \neq f \star_2 (g \star_2 h), \quad \int (f \star_2 g)\star_2 h = \int f \star_2 (g \star_2 h)\,.$$

- $F_{\mu\nu}$ is fine-tunable

$$F_{\mu\nu}(\kappa) = f_{\mu\nu} + \theta^{ij}\Big(\kappa f_{\mu i} \star_2 f_{\nu j} - a_i \star_2 \partial_j f_{\mu\nu}\Big) + \mathcal{O}(a^3)\,,$$
$$\delta_\lambda F_{\mu\nu}(\kappa) = -\theta^{ij}\partial_i\lambda \star_2 \partial_j f_{\mu\nu} + \mathcal{O}(a^2)\lambda = i[\lambda \stackrel{\star}{,} f_{\mu\nu}] + \mathcal{O}(a^2)\lambda.$$

Outline

SW mapped action

- Action is expanded over formal power of fields then truncated according to the amplitude wanted

$$
\begin{aligned}
S &= -\frac{1}{4}\int f^{\mu\nu}f_{\mu\nu} + S^{a^3} + S^{a^4} + S^{a^5}\ldots\ldots\\
&= -\frac{1}{4}\int f^{\mu\nu}f_{\mu\nu} + 2f^{\mu\nu}F^{(1)}_{\mu\nu} + \left(F^{(1)}_{\mu\nu}F^{(1)\mu\nu} + 2f^{\mu\nu}F^{(2)}_{\mu\nu}\right) + S^{a^5}\ldots\ldots
\end{aligned}
$$

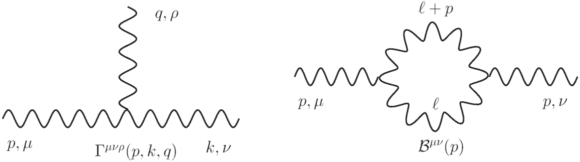
- In $\mathrm{U}(1)$ theory each order is gauge invariant by itself.
- Perturbative quantization (in Feyman gauge) straightforward.

$$
S^{a^3} = -\frac{1}{2}\int f^{\mu\nu}F^{(1)}_{\mu\nu} = \frac{1}{2}\int \theta^{ij}f^{\mu\nu}\left(\frac{1}{4}f_{ij}\star_2 f_{\mu\nu} - \kappa f_{\mu i}\star_2 f_{\nu j}\right)
$$

$$
\Gamma^{\mu\nu\rho}_{\kappa}(p,k,q) = F_{\star_2}(k,q)V^{\mu\nu\rho}_{\kappa}(p,k,q); \quad F_{\star_2}(k,q) = \frac{\sin\frac{k\theta q}{2}}{\frac{k\theta q}{2}},
$$

$$
\begin{aligned}
V^{\mu\nu\rho}_{\kappa}(p,k,q) = \kappa\Big\{ &-(p\theta k)(p-k)^{\rho}g^{\mu\nu} - \theta^{\mu\nu}\left[p^{\rho}(kq) - k^{\rho}(pq)\right]\\
&+(\theta p)^{\nu}\left[g^{\mu\rho}q^2 - q^{\nu}q^{\rho}\right] + (\theta p)^{\rho}\left[g^{\mu\nu}k^2 - k^{\mu}k^{\nu}\right] + \theta^{\mu\sigma}(k+q+\kappa^{-1}p)_{\sigma}\\
&\cdot\left[g^{\nu\rho}(kq) - q^{\nu}k^{\rho}\right]\Big\} + \text{cyclic permutations.}
\end{aligned}
$$

The bubble diagram



$$\mathcal{B}^{\mu\nu}(p) = \frac{1}{2}\int \frac{d^D\ell}{(2\pi)^D}\Gamma_\kappa^{\mu\rho\sigma}(p,\ell,-p-\ell)\frac{-ig_{\rho\rho'}}{\ell^2}\Gamma_\kappa^{\nu\rho'\sigma'}(-p,-\ell,\ell+p)\frac{-ig_{\sigma\sigma'}}{(p+\ell)^2}.$$

$$= \frac{1}{(4\pi)^2}\Big\{\Big[g^{\mu\nu}p^2 - p^\mu p^\nu\Big]B_1^\kappa(p) + (\theta p)^\mu(\theta p)^\nu B_2^\kappa(p)$$

$$+ \Big[g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}\Big]B_3^\kappa(p)$$

$$+ \Big[(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu(\theta\theta p)^\nu\Big]B_4^\kappa(p) + (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}}B_5^\kappa(p)\Big\}.$$

Divergences at $D \to 4$

$$B_1^\kappa(p) \sim \left(\frac{1}{3}(1-3\kappa)^2 + \frac{1}{3}(1+2\kappa)^2\,\frac{p^2(\mathrm{tr}\theta\theta)}{(\theta p)^2} + \frac{2}{3}(1+4\kappa+\kappa^2)\,\frac{p^2(\theta\theta p)^2}{(\theta p)^4}\right) \cdot \underline{\left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right]} - \frac{8}{3}(1-\kappa)^2\,\frac{1}{(\theta p)^6}\left((\mathrm{tr}\theta\theta)(\theta p)^2 + 4(\theta\theta p)^2\right),$$

$$B_2^\kappa(p) \sim \left(\frac{4}{3}(1-\kappa)^2\,\frac{p^4(\theta\theta p)^2}{(\theta p)^6} + \frac{1}{3}(1-2\kappa-5\kappa^2)\,\frac{p^4(\mathrm{tr}\theta\theta)}{(\theta p)^4} + \frac{1}{3}(25-86\kappa + 73\kappa^2)\,\frac{p^2}{(\theta p)^2}\right)\underline{\left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right]} - \frac{8}{3}(1-3\kappa)(3-\kappa)\,\frac{1}{(\theta p)^4} + \frac{16}{3}(1-\kappa)^2\,\frac{1}{(\theta p)^8}\left((\mathrm{tr}\theta\theta)(\theta p)^2 + 6(\theta\theta p)^2\right),$$

$$B_3^\kappa(p) \sim -\frac{1}{6}(1-2\kappa-11\kappa^2)\,\frac{p^2}{(\theta p)^2}\,\underline{\left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right]} - \frac{4}{3(\theta p)^4}(1-10\kappa+17\kappa^2),$$

$$B_4^\kappa(p) \sim -(1+\kappa)^2\,\frac{p^4}{(\theta p)^4}\,\underline{\left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right]} - \frac{16p^2}{3(\theta p)^6}(1-6\kappa+7\kappa^2),$$

$$B_5^\kappa(p) \sim \frac{2}{3}(1+\kappa+4\kappa^2)\,\frac{p^4}{(\theta p)^4}\,\underline{\left[\frac{2}{\epsilon} + \ln(\mu^2(\theta p)^2)\right]} + \frac{32p^2}{3(\theta p)^6}(1-\kappa)(1-2\kappa).$$

Simplify the bubble

- Introducing a nondegenerate noncommutative parameter

$$\theta_2^{\mu\nu} = \frac{1}{\Lambda_{\rm NC}^2}\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \; (\theta\theta)^{\mu\nu} = -\frac{1}{\Lambda_{\rm NC}^4}\delta^{\mu\nu}.$$

- Five tensor structures reduce to two in Euclidean space

$$\mathcal{B}^{\mu\nu}(p)_4\Big|^{\theta_2} = \frac{1}{(4\pi)^2}\Big\{\Big[g^{\mu\nu}p^2 - p^\mu p^\nu\Big] B_I^\kappa(p) + (\theta p)^\mu(\theta p)^\nu B_{II}^\kappa(p)\Big\}$$

$$B_I^\kappa(p) = \left(B_1^\kappa + 2\frac{B_3^\kappa}{\Lambda_{\rm NC}^4} - \frac{B_4^\kappa}{\Lambda_{\rm NC}^8}\right), \quad B_{II}^\kappa(p) = \left(B_2^\kappa - 2\frac{B_5^\kappa}{\Lambda_{\rm NC}^4}\right).$$

- Single divergence cancelation point $\kappa = 1/3$

$$B_I^\kappa(p) \sim (1-3\kappa)\left\{\frac{2(1-3\kappa)}{3}\left(\frac{2}{\epsilon} + \ln\left(\mu^2(\theta p)^2\right)\right) + \frac{8}{3}\frac{(1+\kappa)}{p^2(\theta p)^2}\right\},$$

$$B_{II}^\kappa(p) \sim (1-3\kappa)\left\{p^2\frac{(7-9\kappa)}{(\theta p)^2}\left(\frac{2}{\epsilon} + \ln\left(\mu^2(\theta p)^2\right)\right) - \frac{8}{3}\frac{(7-5\kappa)}{(\theta p)^4}\right\}.$$

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Solving SW map θ-exactly

- Existence of SW map can be shown by solving the differential equations

$$\frac{d}{dt}\Lambda(x) = -\frac{1}{4}\theta^{ij}\left\{A_i \stackrel{\star_t}{,} \partial_j \Lambda\right\},$$

$$\frac{d}{dt}A_\mu(x) = -\frac{1}{4}\theta^{ij}\left\{A_i \stackrel{\star_t}{,} \partial_j A_\mu + F_{j\mu}\right\},$$

$$\frac{d}{dt}F_{\mu\nu}(x) = \frac{1}{4}\theta^{ij}\left[\left\{F_{\mu i} \stackrel{\star_t}{,} F_{\nu j}\right\} - \left\{A_i \stackrel{\star_t}{,} \left(D_j^{\star t} + \partial_j\right) F_{\mu\nu}\right\}\right],$$

- These equations can be solved iteratively as expansion over formal power of fields

$$A_\mu^{(0)} = a_\mu$$

$$A_\mu^{(1)} = \int_0^t dt' - \frac{1}{4}\theta^{ij}\left\{a_i \stackrel{\star_{t'}}{,} \partial_j a_\mu + f_{j\mu}\right\}$$

$$= -\frac{1}{4}\int_0^t dt' \int dp_1 \int dp_2 \theta^{ij}\tilde{a}_i(-ip_{2_j}\tilde{a}_\mu + \tilde{f}_{j\mu})e^{-i(p_1+p_2)x}\cos t'\frac{p_1\theta p_2}{2}$$

$$= -\frac{1}{2}\int dp_1 \int dp_2 \theta^{ij}\tilde{a}_i(-ip_{2_j}\tilde{a}_\mu + \tilde{f}_{j\mu})e^{-i(p_1+p_2)x}\frac{\sin t\frac{p_1\theta p_2}{2}}{\frac{p_1\theta p_2}{2}} = -\frac{1}{2}a_i \star_{2_t} (\partial_j a_\mu + f_{j\mu}).$$

Second order SW map solution

$$\frac{d}{dt}A^{(2)}_\mu(x) = \frac{1}{8}\theta^{ij}\theta^{kl}\Big[\Big\{a_i \overset{\star_t}{,} \Big(\partial_j\big(a_k \star_{2_t} (\partial_l a_\mu + f_{l\mu})\big) - 2(f_{jk} \star_{2_t} f_{\mu l} - a_i \star_{2_t} \partial_l f_{j\mu})\Big)\Big\}$$

$$+ \Big\{\big(a_k \star_{2_t} (\partial_l a_i + f_{li})\big) \overset{\star_t}{,} (\partial_j a_\mu + f_{j\mu})\Big\}\Big].$$

$$\begin{aligned} A_\mu =& a_\mu - \frac{1}{2}\theta^{ij} a_i \star_2 (\partial_j a_\mu + f_{j\mu}) \\ & - \frac{1}{8}\theta^{ij}\theta^{kl}\big[(\partial_i a_\mu + f_{i\mu})a_k(\partial_l a_j + f_{lj}) - a_i\partial_j(a_k(\partial_l a_\mu + f_{l\mu})) \\ & + 2a_i(f_{jk}f_{\mu l} - a_k\partial_l f_{j\mu})\big]_{\star_{3'}}, \end{aligned}$$

$$[fgh]_{\star_{3'}}(x) = \int_0^t dt'\, \{f \overset{\star_{t'}}{,} (g \star_{2_{t'}} h)\}$$

$$= \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)\cdot\left[\frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})-1}{(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})\frac{q\theta k}{2}} - \frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})-1}{(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})\frac{q\theta k}{2}}\right].$$

$$[f(x)g(x)h(x)]_{\star_{3'}} = [f(x)h(x)g(x)]_{\star_{3'}}, \quad \int f_1(x)[f_2(x)g(x)h(x)]_{\star_{3'}} = \int f_2(x)[f_1(x)g(x)h(x)]_{\star_{3'}}.$$

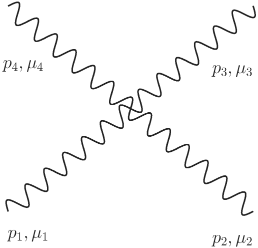
Four photon self-interactions (I)

- The four photon self-interactions can be shown to be explicitly gauge invariant after considerable integral-by-part operations.

$$
\begin{aligned}
S^{a^4}_{\kappa,\kappa_1,\kappa_2,\kappa_3,\kappa_4} = & -\frac{1}{2}\theta^{ij}\theta^{kl}\int \kappa^2 (f_{\mu i}\star_2 f_{\nu j})(f^{\mu}{}_k \star_2 f^{\nu}{}_l) \\
& -\kappa(f_{ij}\star_2 f_{\mu\nu})(f^{\mu}{}_k \star_2 f^{\nu}{}_l) + 2\kappa_1 f^{\mu\nu}[f_{\mu i}f_{\nu k}f_{jl}]_{\star_{3'}} \\
& + 2\kappa_2 f^{\mu\nu}\left(a_i \star_2 \partial_j(f_{\mu k}\star_2 f_{\nu l}) - [f_{\mu k}a_i\partial_j f_{\nu l}]_{\star_{3'}} - [a_i f_{\mu k}\partial_j f_{\nu l}]_{\star_{3'}}\right)) \\
& - \frac{\kappa_3}{4} f^{\mu\nu}\left[f_{\mu\nu}f_{ik}f_{jl}\right]_{\star_{3'}} + \frac{\kappa_4}{8}\left(f^{\mu\nu}\star_2 f_{ij}\right)\left(f_{kl}\star_2 f_{\mu\nu}\right) \\
& + \frac{1}{2}\theta^{pq} f_{\mu\nu}\left[\partial_i f_{jk} f_{lp}\partial_q f_{\mu\nu}\right]_{\mathcal{M}},
\end{aligned}
$$

$$
\begin{aligned}
[fgh]_{\mathcal{M}}(x) = \int e^{-i(p+q+k)x}\tilde{f}(p)\tilde{g}(q)\tilde{h}(k)\Bigg(& \frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})}{2\frac{p\theta q}{2}\frac{q\theta k}{2}(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})} \\
& + \frac{\cos(\frac{p\theta q}{2}-\frac{p\theta k}{2}-\frac{q\theta k}{2})}{2\frac{p\theta q}{2}(\frac{p\theta q}{2}-\frac{q\theta k}{2})(\frac{p\theta q}{2}-\frac{p\theta k}{2}-\frac{q\theta k}{2})} + \frac{\cos(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})}{2\frac{q\theta k}{2}(\frac{p\theta q}{2}-\frac{q\theta k}{2})(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})} \\
& + \frac{2}{(\frac{p\theta q}{2}-\frac{p\theta k}{2}-\frac{q\theta k}{2})(\frac{p\theta q}{2}+\frac{p\theta k}{2}-\frac{q\theta k}{2})(\frac{p\theta q}{2}+\frac{p\theta k}{2}+\frac{q\theta k}{2})}\Bigg).
\end{aligned}
$$

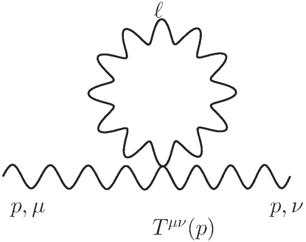
Four photon self-coupling vertices



$\Gamma^{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)$

$$\Gamma^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) = -\frac{i}{4}\Big(\kappa^2\Gamma_{\mathrm{A}}^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) + \kappa\Gamma_{\mathrm{B}}^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)$$
$$+\kappa_1\Gamma_1^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) + \kappa_2\Gamma_2^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) + \kappa_3\Gamma_3^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)$$
$$+\kappa_4\Gamma_4^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) + \Gamma_5^{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4)\Big)$$
$$+\,\text{all } \mathrm{S}_4 \text{ permutations over } \{p_i\} \text{ and } \{\mu_i\} \text{ simutaneously,}$$

Tadpole integral



$$T^{\mu\nu}(p) = \frac{1}{2}\mu^{d-D}\int \frac{d^D\ell}{(2\pi)^D}\frac{-ig_{\rho\sigma}}{\ell^2}\Gamma^{\mu\nu\rho\sigma}_{(\mathrm{I},\mathrm{II})}(p,-p,\ell,-\ell)$$

$$= \tau^{\mu\nu}_{(\mathrm{I},\mathrm{II})}\,\mu^{d-D}\underbrace{\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^2}{\ell^2}}_{=0} + \mathcal{T}^{\mu\nu\rho\sigma}_{(\mathrm{I},\mathrm{II})}\mu^{d-D}\underbrace{\int \frac{d^D\ell}{(2\pi)^D}\frac{\ell_\rho\ell_\sigma}{\ell^2}f^2_{\star_2}(p,\ell)}_{I^{\mu\nu}}.$$

$$
\begin{aligned}
\tau^{\mu\nu} = & - \frac{1}{2D} \Big\{ \Big[g^{\mu\nu} p^2 - p^\mu p^\nu \Big] (\mathrm{tr}\theta\theta)(\kappa_3 - 1) \\
& + \Big[g^{\mu\nu} (\theta p)^2 - (\theta\theta)^{\mu\nu} p^2 + p^{\{\mu} (\theta\theta p)^{\nu\}} \Big] 4(\theta p)^2 (\kappa_1 - \kappa_2) \\
& + (\theta p)^\mu (\theta p)^\nu \Big(4 + (1 - \kappa_3) D(D-1) - 16\kappa + 8\kappa^2 + 8(D-1)(\kappa_1 - \kappa_2) + 4\kappa_4 \Big) \Big\} ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}^{\mu\nu\rho\sigma} = & - \frac{1}{2} \Big\{ \Big(g^{\mu\nu} (\theta p)^\rho (\theta p)^\sigma + \theta^{\mu\rho} p^\nu (\theta p)^\sigma + \theta^{\nu\rho} p^\mu (\theta p)^\sigma + \theta^{\mu\rho} \theta^{\nu\sigma} p^2 \Big) \\
& \qquad \cdot 2 \Big((D-3)\kappa^2 - 2\kappa + \kappa_1 + \kappa_2 \Big) \\
& + \Big(2 g^{\mu\nu} p^\rho (\theta\theta p)^\sigma - g^{\mu\rho} p^\nu (\theta\theta p)^\sigma - g^{\nu\rho} p^\mu (\theta\theta p)^\sigma \\
& - p^\mu (\theta\theta)^{\nu\rho} p^\sigma - p^\nu (\theta\theta)^{\mu\rho} p^\sigma + g^{\mu\rho} \theta^{\nu\sigma} p^2 + g^{\nu\rho} \theta^{\mu\sigma} p^2 \Big) \cdot (2\kappa - \kappa_1 - \kappa_2) \\
& + \Big(g^{\mu\rho} (\theta p)^\nu (\theta p)^\sigma + g^{\nu\rho} (\theta p)^\mu (\theta p)^\sigma + \theta^{\mu\rho} (\theta p)^\nu p^\sigma + \theta^{\nu\rho} (\theta p)^\mu p^\sigma \Big) \\
& \qquad \cdot \Big(-1 - 2\kappa_3 + \kappa_4 + (2 + D)\kappa_1 + (D-2)(\kappa_2 - 2\kappa) + 4\kappa^2 \Big) \\
& + \Big(g^{\mu\nu} g^{\rho\sigma} (\theta p)^2 + (\theta\theta)^{\mu\nu} (p^\rho p^\sigma - p^2 g^{\rho\sigma}) + (p^\mu (\theta\theta p)^\nu + p^\nu (\theta\theta p)^\mu) g^{\rho\sigma} \\
& - g^{\mu\rho} g^{\nu\sigma} (\theta p)^2 - g^{\mu\rho} (\theta\theta p)^\nu p^\sigma - g^{\nu\rho} (\theta\theta p)^\mu p^\sigma \Big) \cdot 2\kappa^2 \\
& + (\theta p)^\mu (\theta p)^\nu g^{\rho\sigma} \cdot 4 \Big(\kappa_1 + \kappa_2 - 2\kappa + (D-1)\kappa_4 \Big) \\
& - \Big[g^{\mu\nu} p^2 - p^\mu p^\nu \Big] (\theta\theta)^{\rho\sigma} (\kappa_4 - 1) \Big\} .
\end{aligned}
$$

Evaluate the NC massless tadpole

- NC tadpole can be simply made into the NC bubble

$$I^{\mu\nu} = \int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{\ell^2} = \int \frac{d^D\ell}{(2\pi)^D}\frac{\ell^\mu\ell^\nu}{\ell^2}\frac{(\ell+p)^2}{(\ell+p)^2} f_{\star_2}(\ell,p)^2.$$

- Out of the same machinery as used for the bubble diagram

$$I^{\mu\nu}_{D\to 4} = \frac{1}{6\pi^2}(\theta p)^{-4}\cdot\left(g^{\mu\nu} - 4\frac{(\theta p)^\mu(\theta p)^\nu}{(\theta p)^2}\right).$$

- Tadpole is solely quadratic IR divergent.
- A few other methods (n-nested zero regulator etc.) yields the same result at the $D \to 4$ limit.

4-photon tadpole at $D \to 4$

$$
\begin{aligned}
T^{\mu\nu}(p) =& \frac{1}{(4\pi)^2}\Bigg\{\Big[g^{\mu\nu}p^2 - p^\mu p^\nu\Big]\,T_1^\kappa(p) + (\theta p)^\mu(\theta p)^\nu\,T_2^\kappa(p) \\
&+ \Big[g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}\Big]\,T_3^\kappa(p) \\
&+ \Big[(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu(\theta\theta p)^\nu\Big]\,T_4^\kappa(p) + (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}}\,T_5^\kappa(p)\Bigg\}, \\
=& \frac{1}{12\pi^2}\Bigg\{\Big[g^{\mu\nu}p^2 - p^\mu p^\nu\Big]\left(\frac{tr\theta\theta}{(\theta p)^4} + 4\frac{(\theta\theta p)^2}{(\theta p)^6}\right)(1-\kappa_4) \\
&+ (\theta p)^\mu(\theta p)^\nu\,\frac{4}{(\theta p)^4}\left(2\kappa^2 - 4\kappa + 6\kappa_1 + 2\kappa_2 - 2\kappa_3 + \kappa_4 - 1\right) \\
&+ \Big[g^{\mu\nu}(\theta p)^2 - (\theta\theta)^{\mu\nu}p^2 + p^{\{\mu}(\theta\theta p)^{\nu\}}\Big]\frac{4}{(\theta p)^4}\left(2\kappa^2 - 2\kappa + \underline{\kappa_1 + \kappa_2}\right) \\
&+ \Big[(\theta\theta)^{\mu\nu}(\theta p)^2 + (\theta\theta p)^\mu(\theta\theta p)^\nu\Big]\frac{8p^2}{(\theta p)^6}\left(\kappa^2 - 2\kappa + \underline{\kappa_1 + \kappa_2}\right) \\
&+ (\theta p)^{\{\mu}(\theta\theta\theta p)^{\nu\}}\,\frac{4p^2}{(\theta p)^6}\left(2\kappa - (\underline{\kappa_1 + \kappa_2})\right)\Bigg\}.
\end{aligned}
$$

Outline

Summing up the quadratic IR divergence

$$\Pi_i(p)_{\mathrm{IR}} = B_i(p)_{\mathrm{IR}} + T_i(p)_{\mathrm{IR}}, \ \forall i = 1, ..., 5.$$

$$\Pi_1(p)_{\mathrm{IR}} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(tr\theta\theta + 4\frac{(\theta\theta p)^2}{(\theta p)^2}\right)\left(2(\kappa-1)^2 - \kappa_4 + 1\right),$$

$$\Pi_2(p)_{\mathrm{IR}} \sim \frac{8}{3}\frac{1}{(\theta p)^4}\left(\kappa^2 + 2\kappa + 12\kappa_1 + 4\kappa_2 - 4\kappa_3 + 2\kappa_4 - 5\right)$$

$$+ \frac{16}{3}\frac{p^2}{(\theta p)^4}\left(\frac{tr\theta\theta}{(\theta p)^2} + 6\frac{(\theta\theta p)^2}{(\theta p)^4}\right)\left(\kappa - 1\right)^2,$$

$$\Pi_3(p)_{\mathrm{IR}} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(9\kappa^2 - 2\kappa - 4\kappa_1 - 4\kappa_2 + 1\right),$$

$$\Pi_4(p)_{\mathrm{IR}} \sim -\frac{16}{3}\frac{p^2}{(\theta p)^6}\left(5\kappa^2 - 2\kappa - 2\kappa_1 - 2\kappa_2 + 1\right),$$

$$\Pi_5(p)_{\mathrm{IR}} \sim \frac{16}{3}\frac{p^2}{(\theta p)^6}\left(4\kappa^2 - 4\kappa - \kappa_1 - \kappa_2 + 2\right).$$

- "Stacking θ" scalar factor in Π_2 vanishes when $\kappa = 1$.

Fine-tuning the IR divergence

- Set $\kappa = 1$

$$\Pi_1(p)_{\mathrm{IR}} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(tr\theta\theta + 4\frac{(\theta\theta p)^2}{(\theta p)^2}\right)\left(1-\kappa_4\right),$$

$$\Pi_2(p)_{\mathrm{IR}} \sim \frac{8}{3}\frac{1}{(\theta p)^4}\left(12\kappa_1 + 4\kappa_2 - 4\kappa_3 + 2\kappa_4 - 2\right),$$

$$\Pi_3(p)_{\mathrm{IR}} \sim -\frac{4}{3}\frac{1}{(\theta p)^4}\left(8 - 4\kappa_1 - 4\kappa_2\right),$$

$$\Pi_4(p)_{\mathrm{IR}} \sim -\frac{16}{3}\frac{p^2}{(\theta p)^6}\left(4 - 2\kappa_1 - 2\kappa_2\right),$$

$$\Pi_5(p)_{\mathrm{IR}} \sim \frac{16}{3}\frac{p^2}{(\theta p)^6}\left(2 - \kappa_1 - \kappa_2\right),$$

- Set $1 = \kappa = \kappa_4 = (\kappa_1 + \kappa_2)/2$.

$$\Pi_2(p)_{\mathrm{IR}} \sim \frac{8}{3}\frac{1}{(\theta p)^4}\left(8\kappa_1 + 8 - 4\kappa_3\right),\ \Pi_{i\neq 2}(p)_{\mathrm{IR}} \sim 0.$$

Outline

Summary

- What we have finished:
  - The quadratic and logarithmic IR divergence behave differently in bubble diagram, only log really mixes with UV.
  - Four photon tadpole is purely quadratic IR divergent.
  - Tadpole-equivalent integrals may be the (technical) origin of the quadratic IR divergence.
  - Canceling quadratic IR divergence via freedoms in 4-photon coupling requires the first order (3-photon) freedom parameter $\kappa = 1$.
- What we still don't know
  - The UV divergence?
  - Why $\kappa = 1$?
  - Stability of the fine-tuning?
  - non-Abelian theory?

Thank you.