HOLOGRAPHIC ENTANGLEMENT ENTROPY

Veronika Hubeny
Durham University

Workshop on Quantum Fields and Strings

Sep.17, 2014

Entanglement

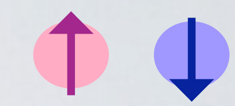
- Most non-classical manifestation of quantum mechanics
  - "Best possible knowledge of a whole does not include best possible knowledge of its parts — and this is what keeps coming back to haunt us'' [Schrodinger '35]
- New quantum resource for tasks which cannot be performed using classical resources [Bennet '98]
- Plays a central role in wide-ranging fields
  - quantum information (e.g. cryptography, teleportation, …)
  - quantum many body systems
  - quantum field theory
- Hints at profound connections to geometry…

OUTLINE

- Entanglement Entropy
- Holographic principle & AdS/CFT
- Holographic Entanglement Entropy

Entanglement in 2 qubit system

Consider a system of 2 spins, labeled A and B



- Simple product state: $|\,\psi\,\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B \equiv |\uparrow\downarrow\rangle$
- More complicated product state:

$$|\,\psi\,\rangle = \frac{|\downarrow\rangle_A + |\uparrow\rangle_A}{\sqrt{2}} \otimes \frac{|\downarrow\rangle_B + |\uparrow\rangle_B}{\sqrt{2}} = \frac{1}{2}\left(|\downarrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)$$

- Generic state (with arbitrary c_{ij} s.t. $\sum c_{ij}^2 = 1$)

$$|\,\psi\,\rangle = c_{00}\,|\downarrow\downarrow\rangle + c_{01}\,|\downarrow\uparrow\rangle + c_{10}\,|\uparrow\downarrow\rangle + c_{11}\,|\uparrow\uparrow\rangle$$

 is entangled when it is not a product state.

- A Bell (EPR) pair, such as $|\,\psi\,\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right)$

 is maximally entangled.

Entanglement in 2 qubit system

Now suppose we can only measure A.

What does that tell us about B?

- For the maximally entangled state, measuring A determines B :

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)$$

- For a non-entangled state, measuring A gives no knowledge of B

$$|\psi\rangle = \frac{1}{2}(|\downarrow\downarrow\rangle+|\downarrow\uparrow\rangle+|\uparrow\downarrow\rangle+|\uparrow\uparrow\rangle)$$

Entanglement Entropy (EE)

The amount of entanglement is characterized by Entanglement Entropy S_A. Since we can only measure A, integrate out B:

- reduced density matrix $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \log \rho_A$

- For the maximally entangled state $|\,\psi\,\rangle = \frac{1}{\sqrt{2}}(\,|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad S_A = \log 2$$

- For the non-entangled state $|\,\psi\,\rangle = \frac{1}{2}(\,|\downarrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\uparrow\uparrow\rangle)$

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad S_A = 0$$

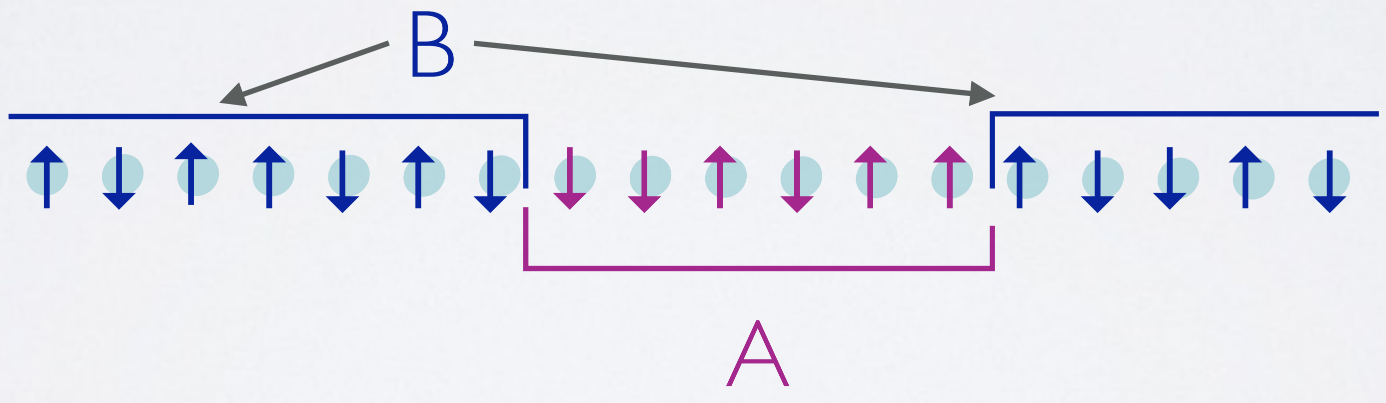
EE more generally

More generally: divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

e.g.:

- spin chain



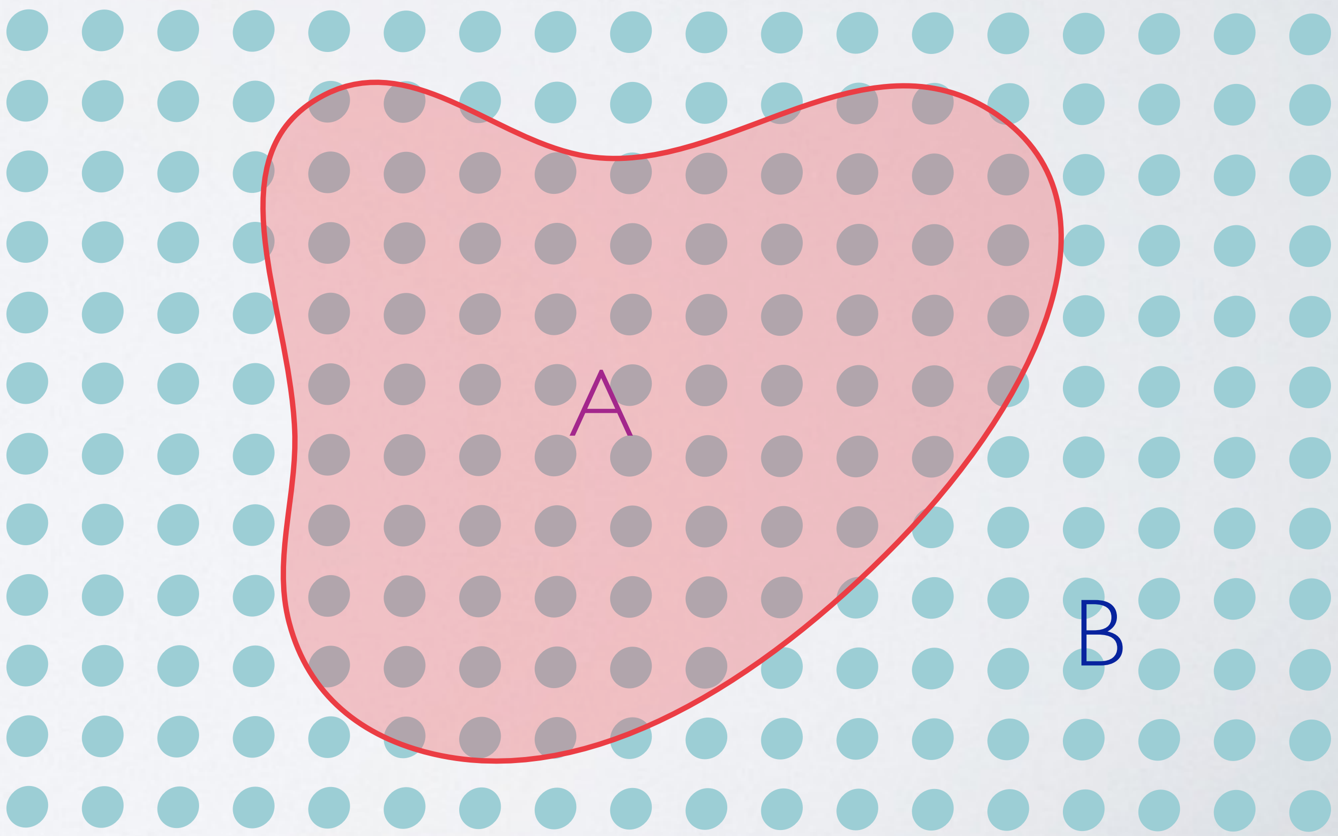
EE more generally

More generally: divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

e.g.:

- spin chain
- many-body quantum system, e.g. on a lattice



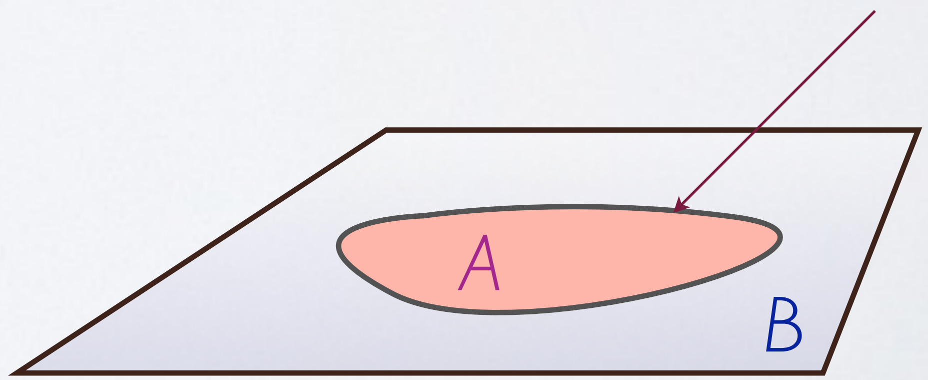
EE more generally

More generally: divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

e.g.:

- spin chain
- many-body quantum system, e.g. on a lattice
- QFT: A and B can be spatial regions, separated by a smooth entangling surface



EE more generally

More generally: divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

e.g.:

- spin chain
- many-body quantum system, e.g. on a lattice
- QFT: A and B can be spatial regions, separated by a smooth entangling surface

In all cases, $S_A = -\mathrm{Tr}\, \rho_A \log \rho_A$, where $\rho_A = \mathrm{Tr}_B \rho$.

Applications of EE

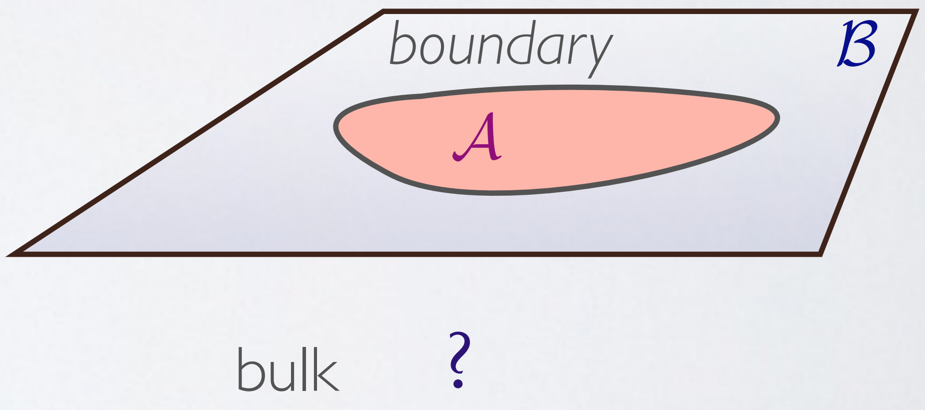
- Quantum Information theory: new quantum resource [Bennett '98 & Masanes '05]
  - quantum cryptography [Ekert, '91]
  - quantum dense coding [Bennett and Wiesner, '92]
  - quantum teleportation [Bennett et al., '93]
- Condensed Matter theory: diagnostic
  - quantum critical points
  - topological phases
  - computational difficulty, e.g. MERA [Vidal '09]
- Quantum Gravity:
  - suggested as origin of black hole entropy [Bombelli,Koul,Lee&Sorkin, '86 Srednicki, Frolov&Novikov, Callan&Wilczek, Susskind ...]
  - origin of macroscopic spacetime [van Raamsdonk et.al., Maldecena&Susskind]

The good news & the bad news

- But EE is hard to deal with…
  - non-local quantity, intricate & sensitive to environment
  - difficult to measure
  - difficult to calculate

… especially in strongly-coupled quantum systems

- AdS/CFT to the rescue?
  - Is there a natural bulk dual of EE?
    (= "Holographic EE")



Yes! - described geometrically…

OUTLINE

- Entanglement Entropy
- Holographic principle & AdS/CFT
- Holographic Entanglement Entropy

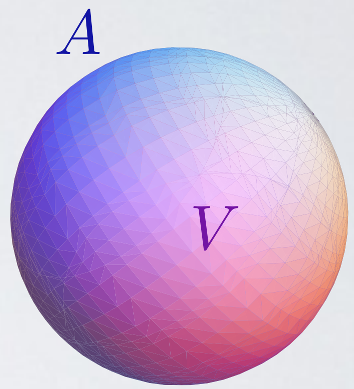
Entropy Bound

- Generalized Second Law: combined matter+BH entropy increases

  ⇒ Bekenstein bound (weakly gravitating systems): $S_{\text{matter}} \leq 2\pi\, E\, R$

  ⇒ Spherical entropy bound (slowly evolving systems): ['t Hooft '93, Susskind]

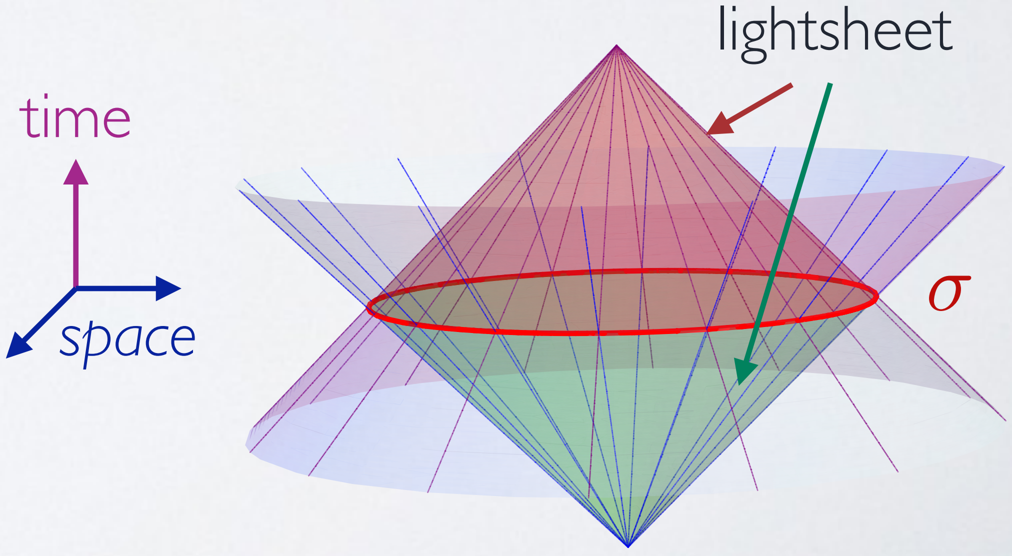
$$S_{\text{matter}} \leq \frac{A}{4} \qquad \Rightarrow \text{ entropy } S \text{ is not extensive: } \quad S \not\propto V$$



- Covariant entropy bound: full spacetime construct [Bousso '99]

  Entropy on any lightsheet L of a surface σ cannot exceed the area of σ:

$$S(L) \leq \frac{A(\sigma)}{4}$$



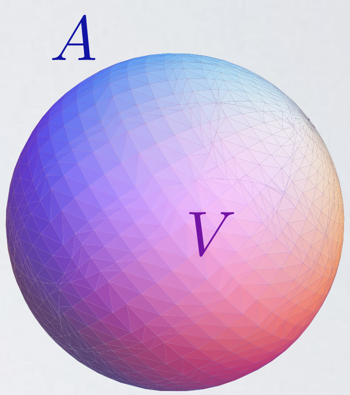
Holographic Principle

- Generalized Second Law: combined matter+BH entropy increases

  ⇒ Bekenstein bound (weakly gravitating systems): $S_{\rm matter} \leq 2\pi\, E\, R$

  ⇒ Spherical entropy bound (slowly evolving systems): ['t Hooft, Susskind]

$$S_{\rm matter} \leq \frac{A}{4} \qquad \Rightarrow \text{ entropy } S \text{ is not extensive: } \quad S \not\propto V$$



- Covariant entropy bound: full spacetime construct [Bousso]

- Holographic Principle: in a theory of gravity, the number of degrees of freedom describing the physics on lightsheet $L(\sigma)$ cannot exceed $A(\sigma)/4$

⇒ physical equivalence between 2 theories living in different # of dimensions!

- Concrete realization: AdS/CFT correspondence

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × K *"on boundary"*

Key aspects:

- ✳ Gravitational theory maps to non-gravitational one!
- ✳ *Holographic*: gauge theory lives in fewer dimensions.
- ✳ Strong/weak coupling duality.

Invaluable tool to:

- ~ Use gravity on AdS to learn about strongly coupled field theory
  (as successfully implemented in e.g. AdS/QCD & AdS/CMT programs)
- ~ Use the gauge theory to define & study quantum gravity in AdS

Pre-requisite: Understand the AdS/CFT 'dictionary'...

Onward from AdS/CFT

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × K

"on boundary"

Applied AdS/CFT:

- study specific system via its dual
- e.g. AdS/QCD, AdS/CMT, …

Fundamentals of AdS/CFT:

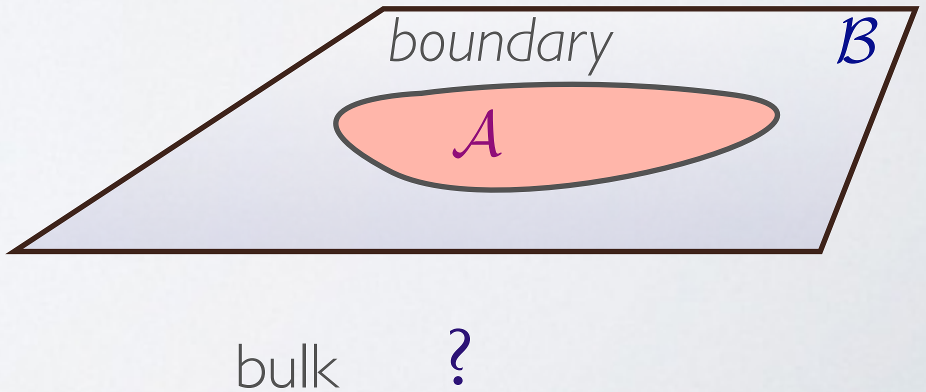
- why/how does the duality work
- map between the 2 sides

Holographic Entanglement Entropy

Quantum Gravity

OUTLINE

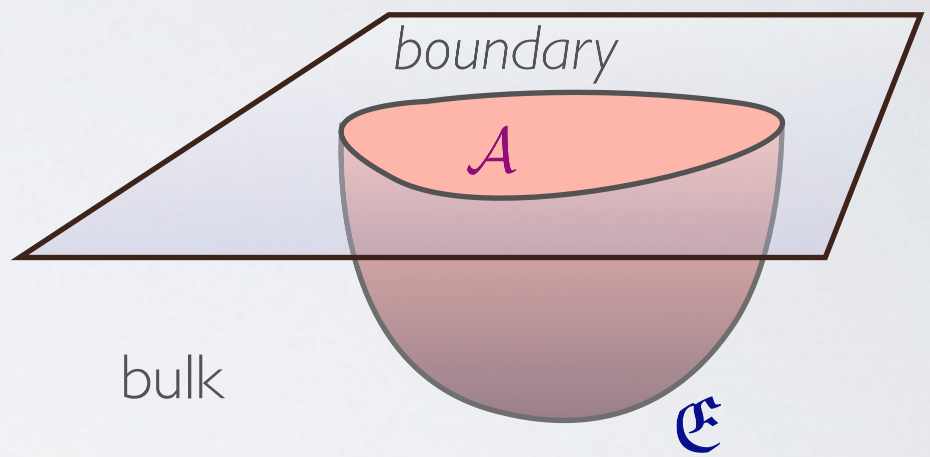
- Entanglement Entropy
- Holographic principle & AdS/CFT
- Holographic Entanglement Entropy
  - bulk dual of EE:
    (= "Holographic EE")



Holographic Entanglement Entropy

Proposal [Ryu & Takayanagi, '06] for *static* configurations:

In the bulk EE $S_{\mathcal{A}}$ is captured by the area of minimal co-dimension 2 bulk surface $\mathfrak{E}$ *(at constant t)* anchored on $\partial\mathcal{A}$.



$$S_{\mathcal{A}} = \min_{\partial\mathfrak{E}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\,G_N}$$

Remarks:

- cf. black hole entropy…
- Minimal surface "hangs" into the bulk due to large distances near bdy.
- Note that both LHS and RHS are in fact infinite…

Area-law divergence of EE

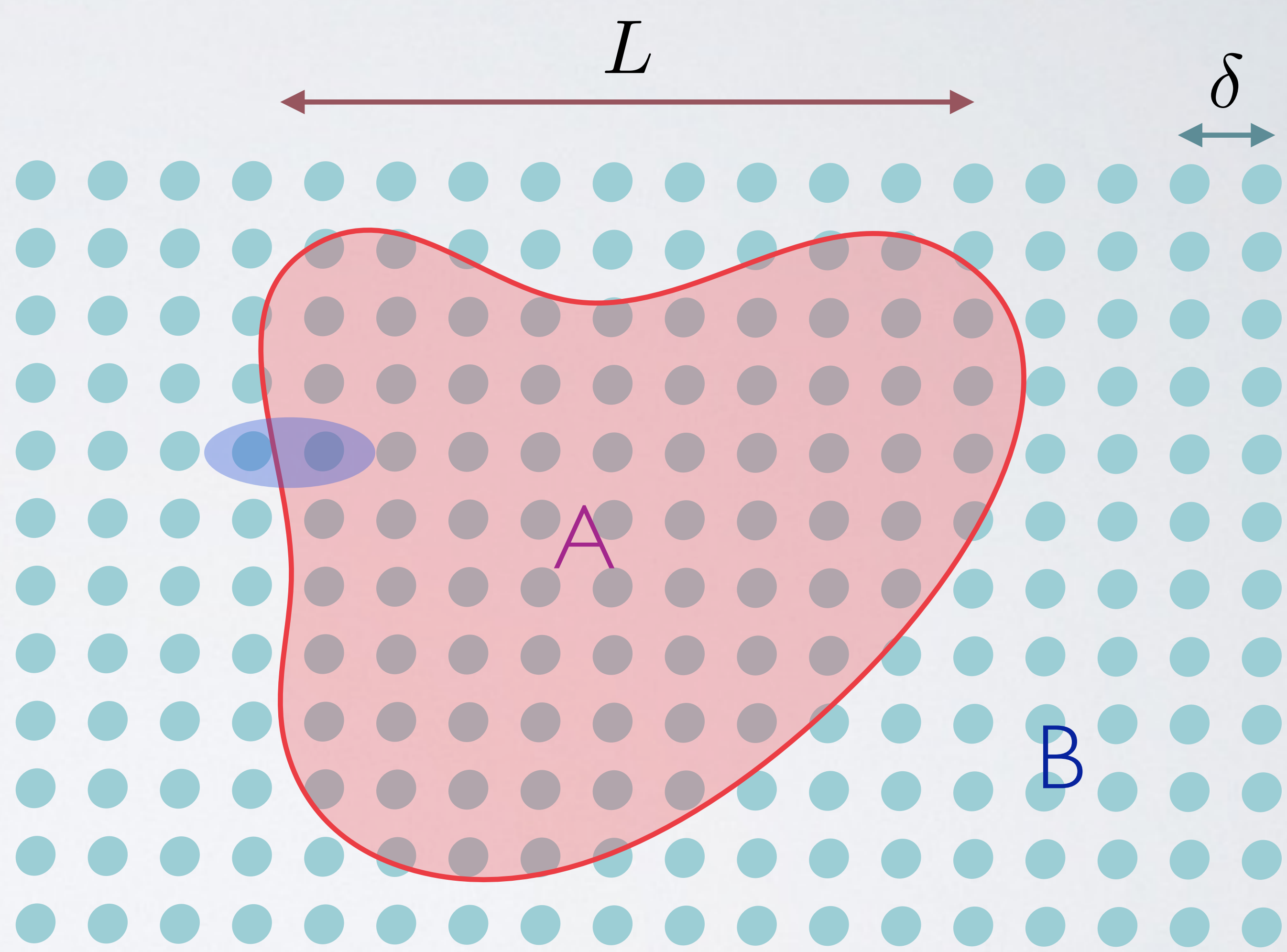
cf. lattice system, with lattice spacing δ and A of size L, in d spacetime dims:

for the full system near the ground state, most of the entanglement is local:

$$S_A \sim \left(\frac{L}{\delta}\right)^{d-2}$$

$\sim$ # of links through ∂A

$\to \infty$ as $\delta \to 0$

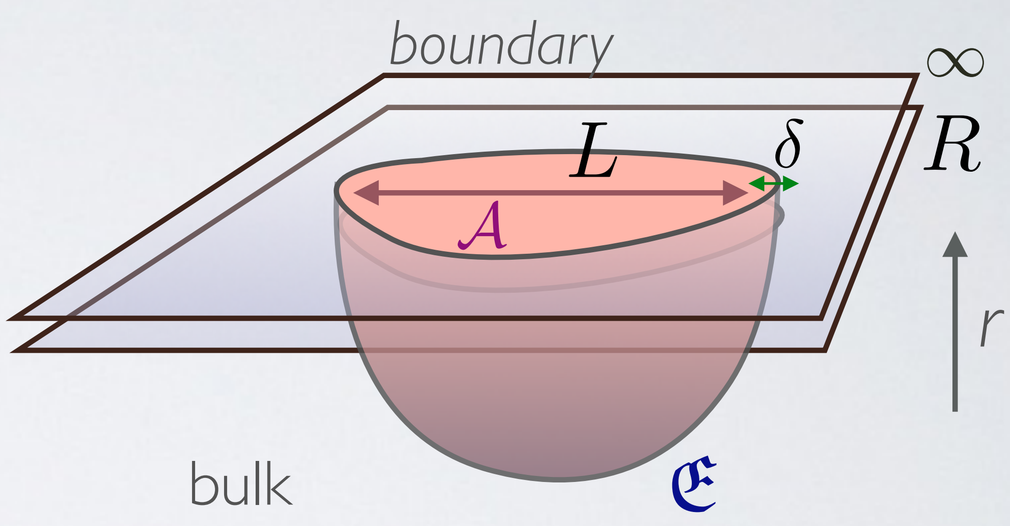


In a QFT, we can regulate the UV divergence by a short distance cutoff δ.

Area-law divergence of HEE

Short-distance cutoff δ in the CFT translates to large-radius cutoff R in AdS

with $\quad \delta = \dfrac{\ell^2}{R}$ (cf. UV/IR duality)



Bulk area reproduces the correct divergence structure:

$$S_{\mathcal{A}} = c_0 \left(\frac{L}{\delta}\right)^{d-2} + c_1 \left(\frac{L}{\delta}\right)^{d-4} + \cdots$$

$$+ \begin{cases} c_{d-2} \log\left(\frac{L}{\delta}\right) + \cdots \quad , & d \ \text{even} \\ c_{d-2} + \cdots \quad , & d \ \text{odd} \end{cases}$$

cutoff-dependent coefficients

universal coefficients

We can regulate EE by e.g. background subtraction.

Evidence for HEE

- ✓ Leading contribution correctly reproduces the area law
- ✓ Recover known results of EE for intervals in 2-d CFT [Calabrese&Cardy] both in vacuum and in thermal state
- ✓ Derivation of holographic EE for spherical entangling surfaces [Cassini,Huerta,&Myers]
- ✓ Attempted proof by [Fursaev] elaborated & refined by [Headrick, Faulkner, Hartman, Maldacena&Lewkowycz]

Further suggestive evidence:

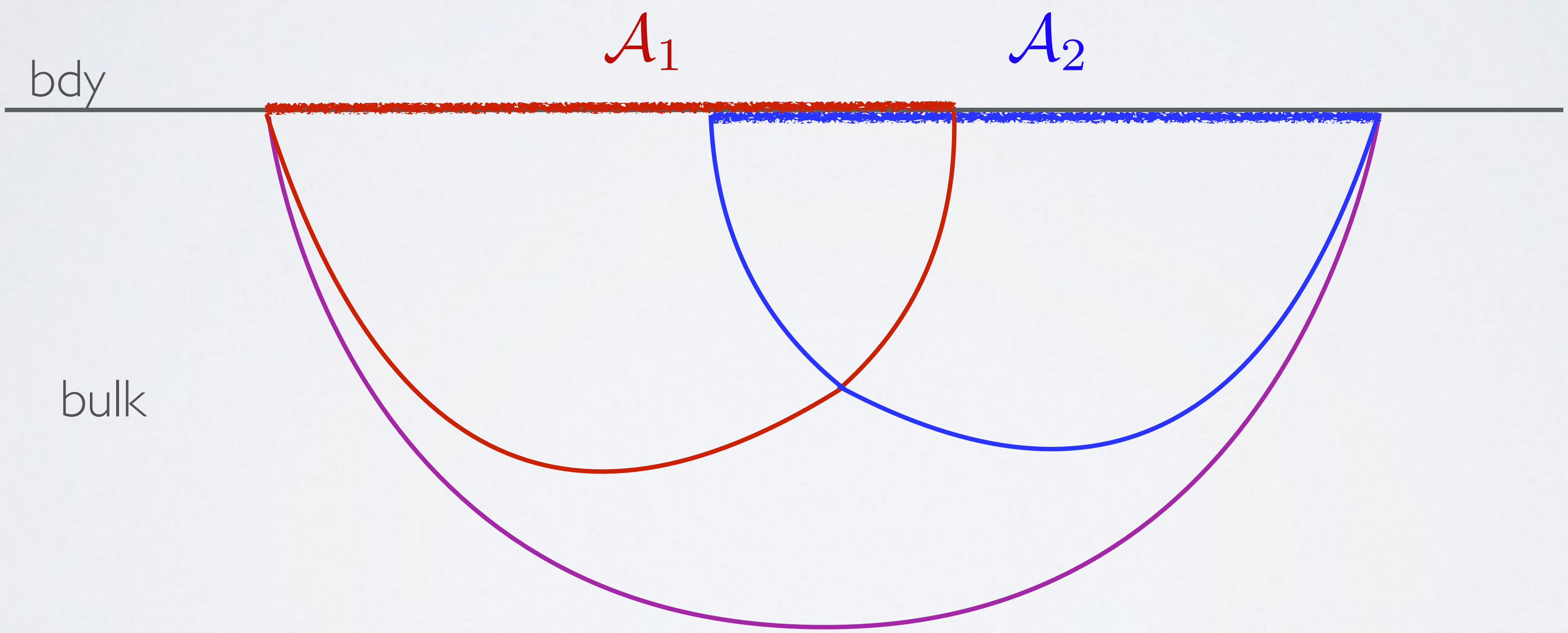
- ✓ Automatically satisfies $S_{\mathcal{A}} = S_{\mathcal{A}^c}$ for pure states
- ✓ Automatically satisfies (strong) subadditivity [Lieb&Ruskai] & Araki-Lieb inequality -- easy to prove on the gravity side, far harder within field theory

Subadditivity

- Subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2}$$

- Manifest in the gravity dual



- Implies positivity of mutual information: $I(\mathcal{A}_1, \mathcal{A}_2) = S_{\mathcal{A}_1} + S_{\mathcal{A}_2} - S_{\mathcal{A}_1 \cup \mathcal{A}_2}$

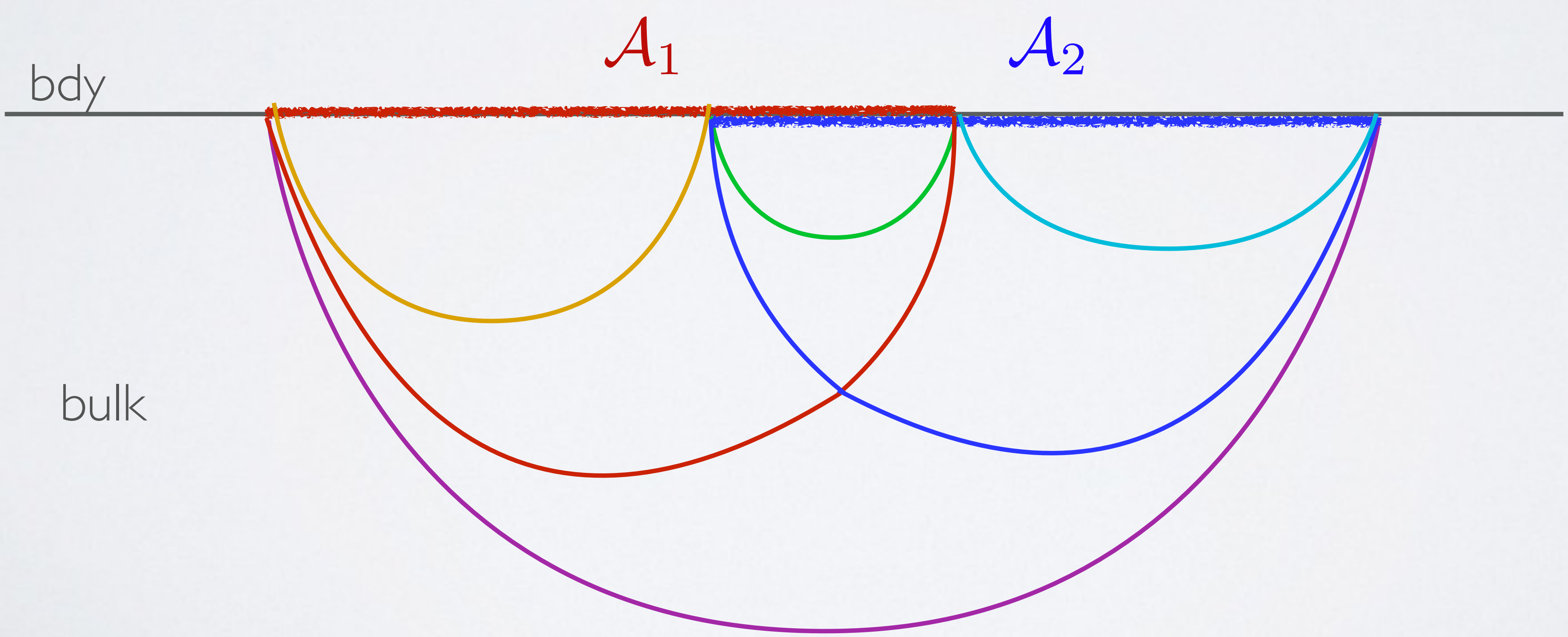
Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

- proof in static configurations [Headrick&Takayanagi]



In time-dependent configurations more involved but true [Headrick et.al., Wall]

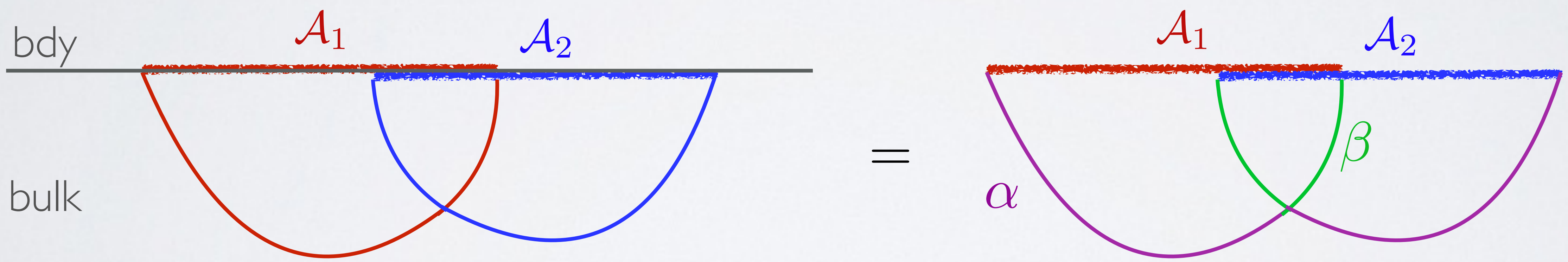
Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \backslash \mathcal{A}_2} + S_{\mathcal{A}_2 \backslash \mathcal{A}_1}$$

- proof in static configurations [Headrick&Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta$$

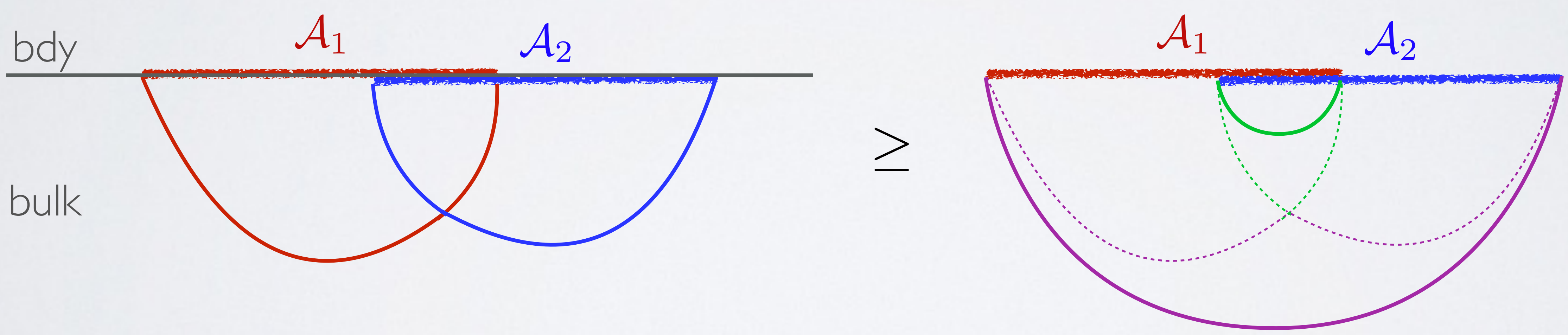
Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

- proof in static configurations [Headrick&Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

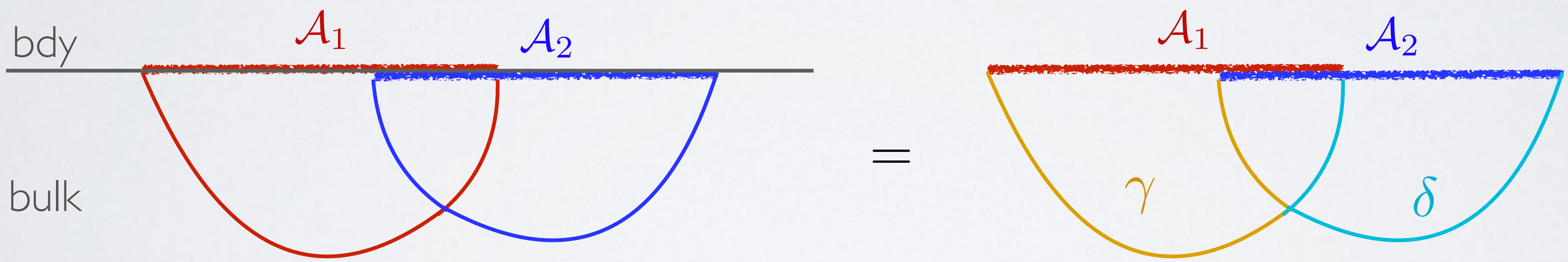
Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

- proof in static configurations [Headrick&Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \gamma + \delta$$

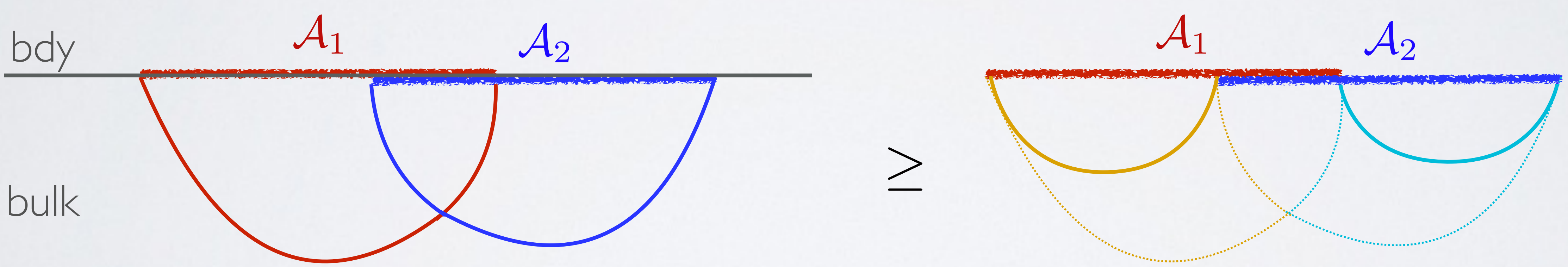
Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

- proof in static configurations [Headrick&Takayanagi]

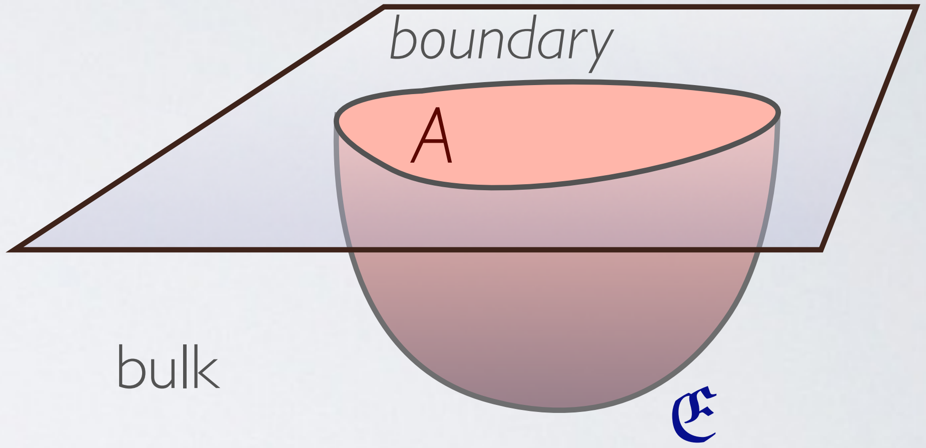


$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \gamma + \delta \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

Covariant Holographic EE

But the RT prescription is not well-defined outside the context of static configurations:

- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…



In *time-dependent* situations, RT prescription must be covariantized:

[VH, Rangamani, Takayanagi '07]

- ✳ minimal surface → extremal surface
- ✳ equivalently, $\mathfrak{E}$ is the surface with zero null expansions; (cf. light sheet construction [Bousso])
- ✳ equivalently, maximin construction: maximize over minimal-area surface on a spacelike slice [Wall]

Summary of HEE proposals:

In all cases, EE is given by Area/4G of a certain surface which is:

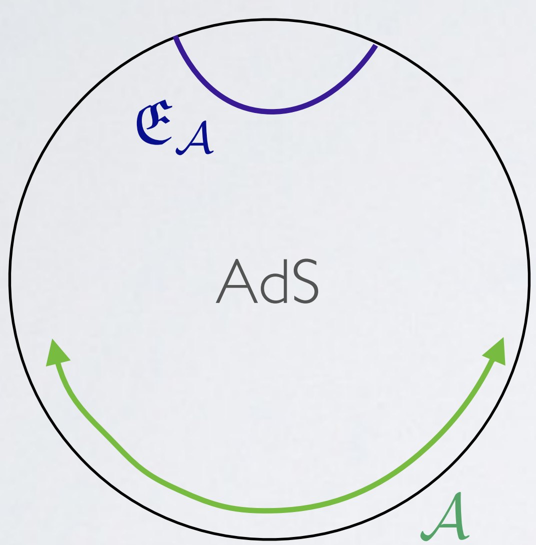
- bulk co-dimension 2 surface
- anchored on the boundary on entangling surface $\partial\mathcal{A}$
- homologous to $\mathcal{A}$ [Headrick, Takayanagi, et.al.]
- in case of multiple surfaces, $S_{\mathcal{A}}$ is given by the one with smallest area.

But the HEE proposals differ in the specification of the surfaces:

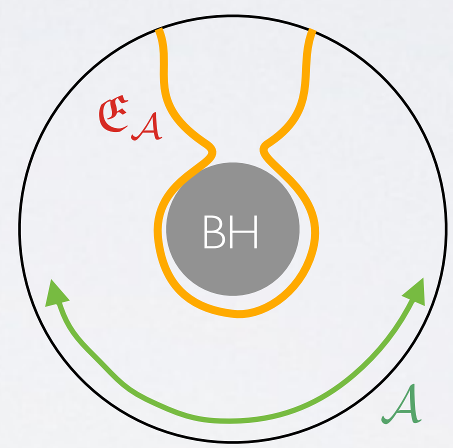
- RT [Ryu & Takayanagi] (static ST only): minimal surface on const. t slice
- HRT [Hubeny, Rangamani, & Takayanagi]: extremal surface in full ST
- maximin [Wall]: minimal surface on bulk achronal slice $\tilde{\Sigma}$, maximized over all $\tilde{\Sigma}$ containing $\mathcal{A}$ (equivalent to extremal surface)

Homology constraint

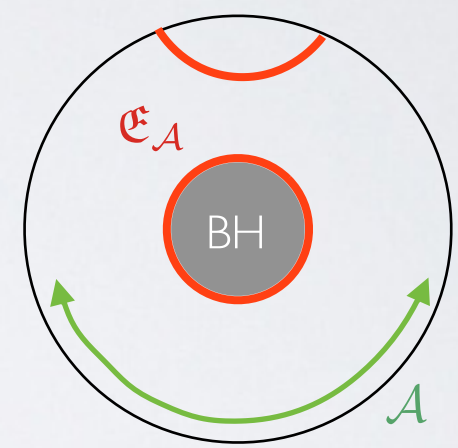
- Usual phrasing: $\mathfrak{E}_{\mathcal{A}}$ is homologous to $\mathcal{A}$ if ∃ a smooth manifold whose only boundaries are $\mathfrak{E}_{\mathcal{A}}$ and $\mathcal{A}$



is OK, and

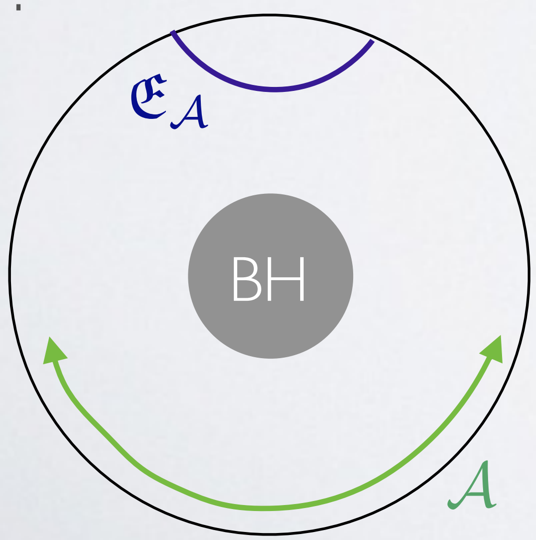


or

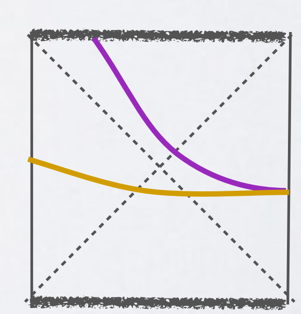


are also OK,

BUT:



is <u>not</u> OK,
since the interpolating manifold
either has another boundary
or hits the singularity:



Entropy inequalities

- consider the full system in the state (density matrix) ρ_Σ
- partition the full space Σ into subsystems $\mathcal{A} \cup \mathcal{A}^c$
- then EE satisfies:

$$\underbrace{|S_\mathcal{A} - S_{\mathcal{A}^c}|}_{\delta S_\mathcal{A}} \leq S_{\rho_\Sigma} \leq S_\mathcal{A} + S_{\mathcal{A}^c}$$

Araki-Lieb   subadditivity

- for a system in a pure state,

$$S_{\rho_\Sigma} = 0 \quad \Rightarrow \quad S_\mathcal{A} = S_{\mathcal{A}^c} \quad \Rightarrow \quad \delta S_\mathcal{A} = 0$$

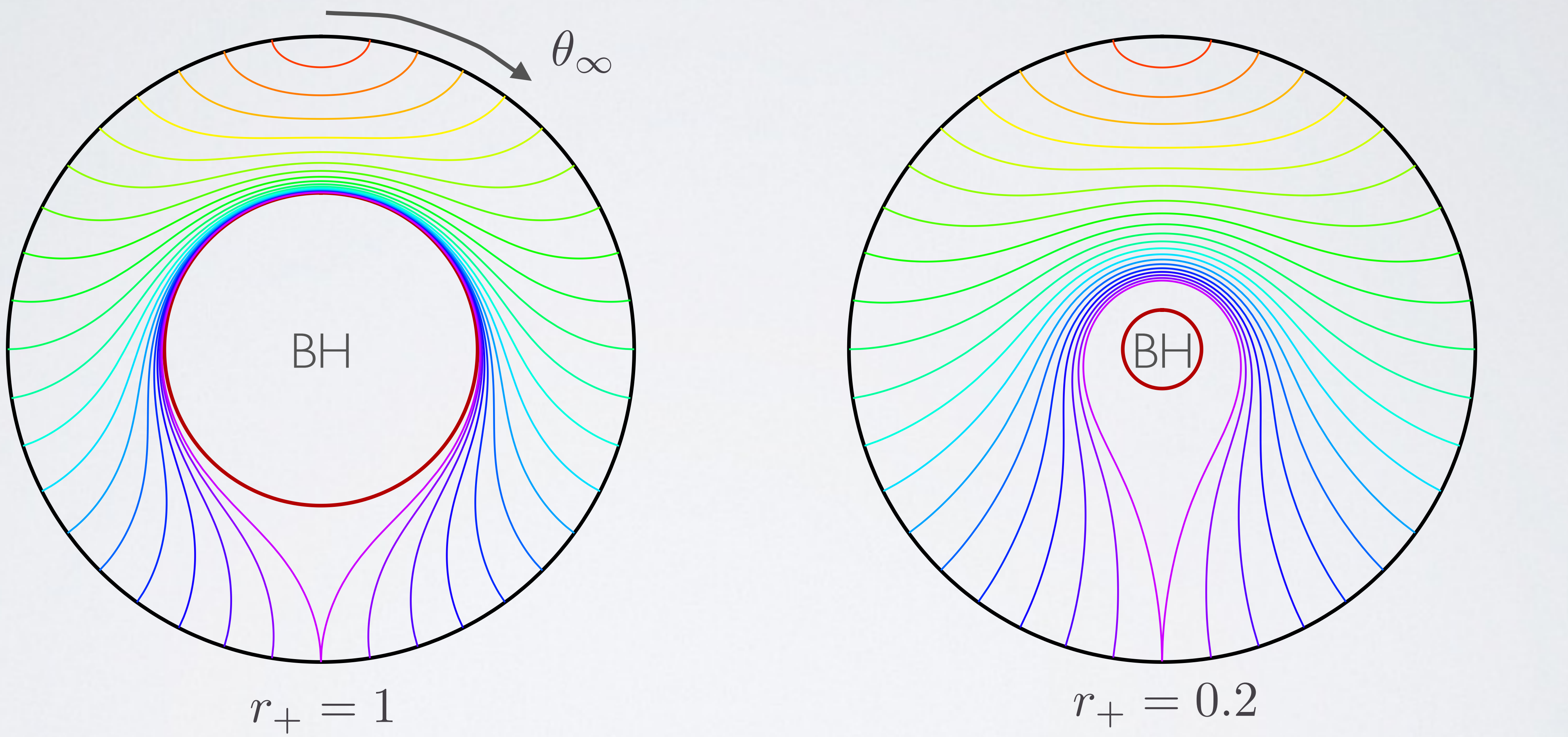
- use $\delta S_\mathcal{A}$ to characterise deviations from purity

Curious properties of EE:

- Entanglement plateaux ($\delta S_{\mathcal{A}}$ saturates to S_{ρ_Σ} for large enough $\mathcal{A}$)
- EE is a 'fine-grained' observable
- EE satisfies very nontrivial causality constraints
- EE has two separate components

Warm-up: EE in 2-d thermal CFT:

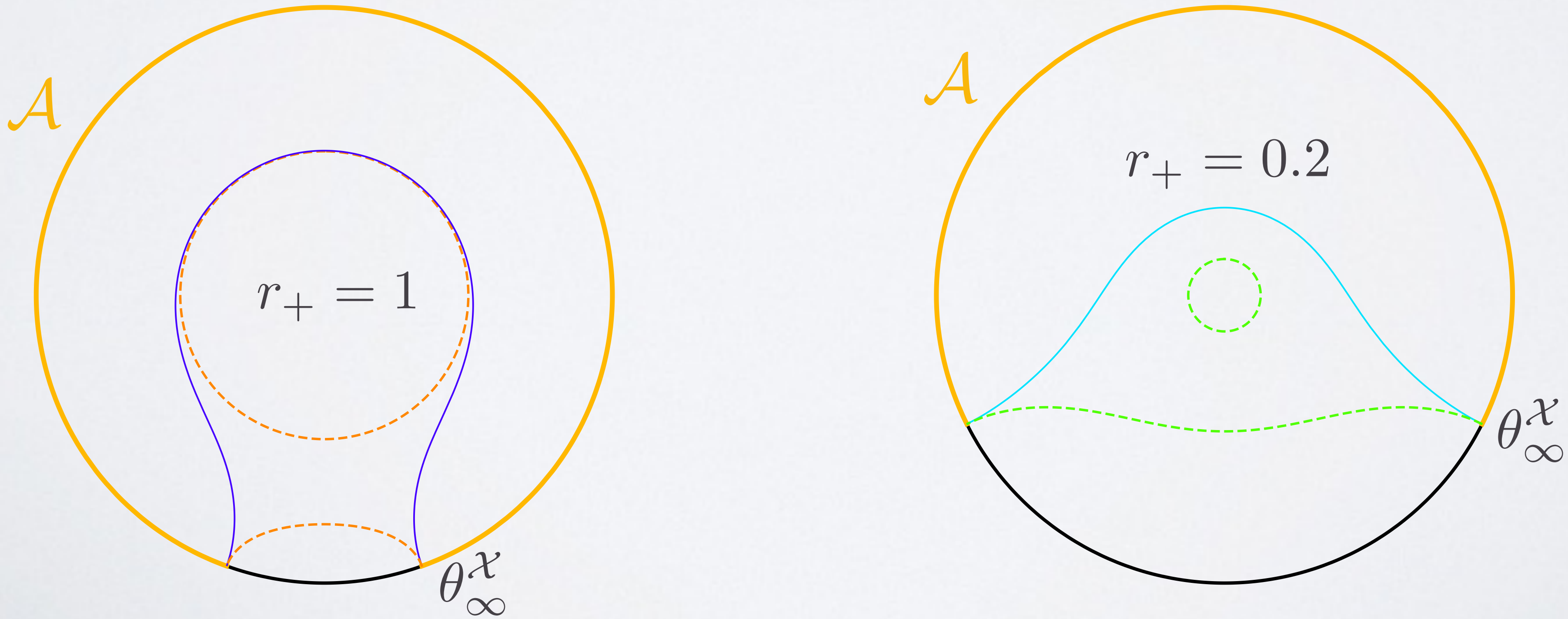
- Consider extremal surfaces (= spacelike geodesics) in BTZ



- Area/4G is given by: $$(S_{\mathcal{A}})_{\text{naive}} = \frac{c}{3}\log\left(\frac{2\,r_\infty}{r_+}\sinh(r_+\,\theta_\infty)\right)$$

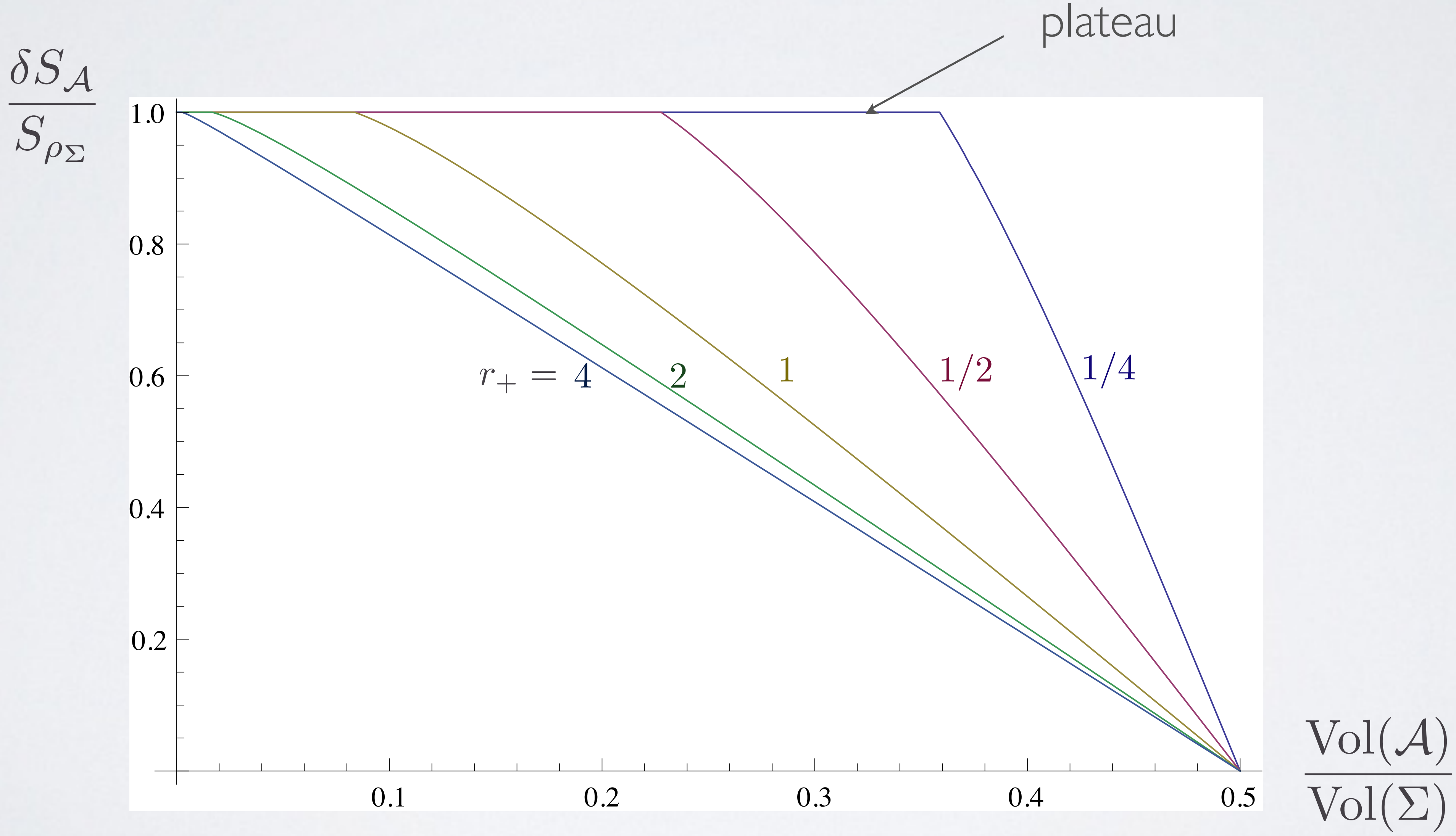
Entanglement plateaux in 2-d CFT:

- Consider global BTZ, and compute $S_{\mathcal{A}}$ for varying $\mathcal{A}$
- Using connected $\mathfrak{E}_{\mathcal{A}}$, $(\delta S_{\mathcal{A}})_{\text{naive}} = \frac{c}{3} \log \left[\sinh(r_+ \, \theta_\infty) \, \text{csch}(r_+ \, (\pi - \theta_\infty) \right]$
- But this would lead to diverging $\delta S_{\mathcal{A}}$ $\Rightarrow$ violates Araki-Lieb
- However, the disconnected $\mathfrak{E}_{\mathcal{A}}$ has smaller area for large enough θ_∞ $\quad \Rightarrow \; S_{\mathcal{A}} = S_{\mathcal{A}^c} + S_{\rho_\Sigma} \;\; \forall \, \theta_\infty \geq \theta_\infty^{\mathcal{X}} = \frac{\coth^{-1}(2\coth(\pi \, r_+) - 1)}{r_+}$
- Where the transition happens depends on BH size:



Entanglement plateaux in higher d:

- In higher dimensions, we can check similar saturation effect takes place: e.g. for Schw-AdS$_5$ [VH, Maxfield, Rangamani, Tonni '13]

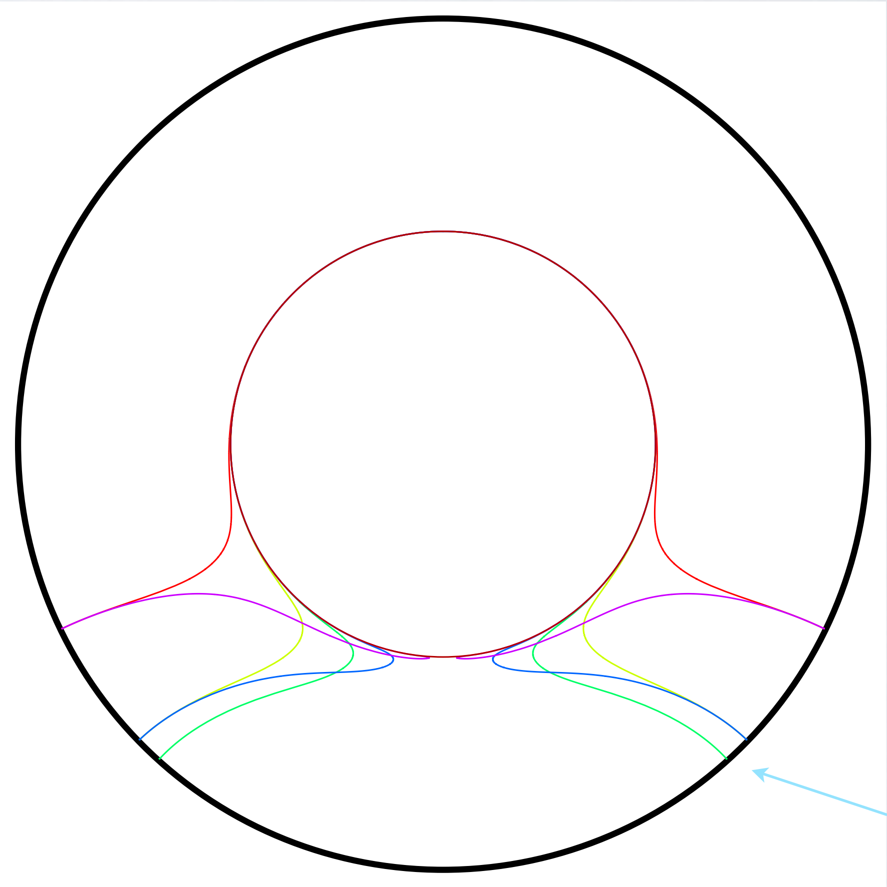


Entanglement plateaux in higher d:

- However, unlike BTZ, new surprise for Schw-AdS$_{d+1}$: connected surface $\mathfrak{E}_{\mathcal{A}}$ actually does not exist for large enough $\mathcal{A}$! What happened to it?

As closest approach to horizon shrinks,

- Size of $\mathcal{A}$ initially increases, but then decreases, and in fact oscillates
- The surface develops thin neck and folds back around the black hole
- There can be arbitrarily many folds, increasingly close to horizon
- This gives an infinite family of minimal surfaces anchored on the same region $\mathcal{A}$



max size

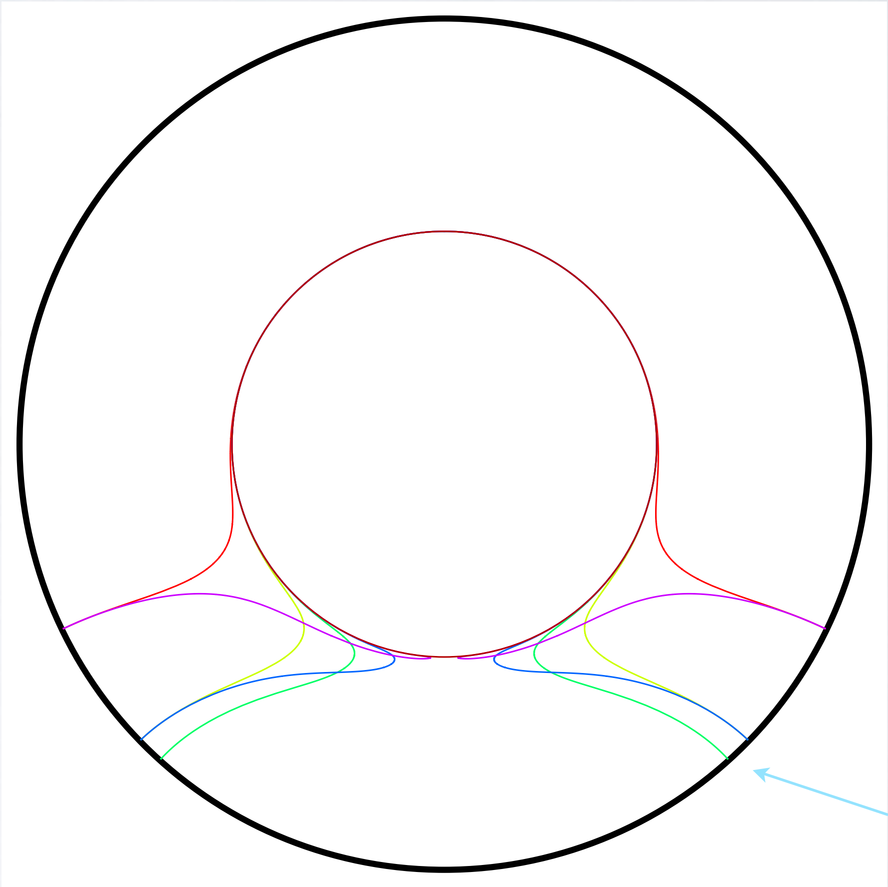
max size

Entanglement plateaux in higher d:

- However, unlike BTZ, new surprise for Schw-AdS$_{d+1}$: connected surface $\mathfrak{E}_{\mathcal{A}}$ actually does not exist for large enough $\mathcal{A}$! What happened to it?

As closest approach to horizon shrinks,

- Size of $\mathcal{A}$ initially increases, but then decreases, and in fact oscillates
- The surface develops thin neck and folds back around the black hole
- There can be arbitrarily many folds, increasingly close to horizon
- This gives an infinite family of minimal surfaces anchored on the same region $\mathcal{A}$



max size

- This nonexistence behaviour is robust to deforming the state, and follows directly from *causal wedge* arguments

Implication for entanglement entropy

- Important implication: whenever $\mathcal{A}$ is large enough for $\Xi_{\mathcal{A}}$ to have two disconnected pieces, there cannot exist a single connected extremal (minimal) surface $\mathfrak{E}_{\mathcal{A}}$ homologous to $\mathcal{A}$!
  - However, the homology constraint is required for EE, i.e. part of $\mathfrak{E}_{\mathcal{A}}$ must reach around the BH.
  - So $\mathfrak{E}_{\mathcal{A}}$ must likewise have two disconnected pieces, one on the horizon and one homologous to $\mathcal{A}^c$ (=complement of $\mathcal{A}$)
- Hence we have the universal formula for the entanglement entropy, whenever $\mathcal{A}$ is large enough: $S_{\mathcal{A}} = S_{\mathcal{A}^c} + S_{\rho_\Sigma}$
  - Automatically saturates the Araki-Lieb inequality

    = *entanglement plateau* [VH, Maxfield, Rangamani, Tonni]
  - So we can extract BH (thermal) entropy from entanglement entropy [cf. Azeyanagi, Nishioka, Takayanagi]

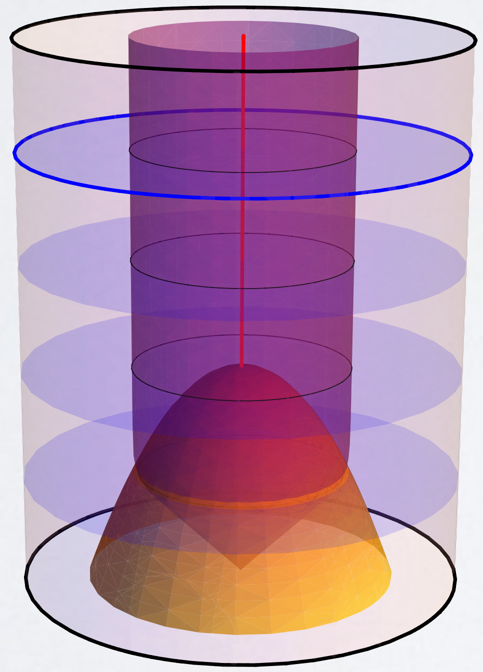
Curious properties of EE:

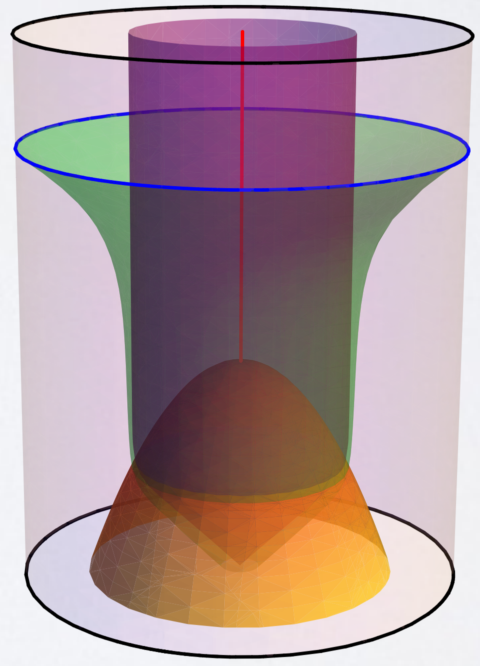
- Entanglement plateaux ($\delta S_{\mathcal{A}}$ saturates to S_{ρ_Σ} for large enough $\mathcal{A}$)
- EE is a ‘fine-grained’ observable
- EE satisfies very nontrivial causality constraints
- EE has two separate components

EE is fine-grained observable!

Example: black hole formed from a collapse

- In contrast to the static (i.e. eternal) black hole, for a collapsed black hole, there is no non-trivial homology constraint on extremal surfaces. [cf. Takayanagi & Ugajin]





- Hence we always have $S_{\mathcal{A}} = S_{\mathcal{A}^c}$ as for a pure state.

EE is fine-grained observable!

- Despite the event horizon formation arbitrarily long in the past, EE `remembers' the state is pure: $S_{\mathcal{A}} = S_{\mathcal{A}^c} \quad \forall\, \mathcal{A}$
- Hence entanglement entropy is sensitive to very 'fine-grained' information: it can tell whether the black hole is eternal or collapsed, arbitrarily late after the collapse (when all 'coarse-grained' observables have thermalized).
- And this in spite of its classical geometrical nature...
- Other diagnostics of thermal vs. pure state (e.g. periodicity in imaginary time appear much more subtle).
- However this wouldn't necessarily tell apart individual microstates.

Curious properties of EE:

- Entanglement plateaux ($\delta S_{\mathcal{A}}$ saturates to S_{ρ_Σ} for large enough $\mathcal{A}$)
- EE is a 'fine-grained' observable
- EE satisfies very nontrivial causality constraints
- EE has two separate components

Two components to HEE

When Araki-Lieb inequality is saturated (sufficiently large $\mathcal{A}$), there is a natural decomposition of HEE:

- In general time-dependent 'eternal' (=2-sided) BH, one component of $\mathfrak{E}_{\mathcal{A}}$ (the one anchored on the boundary) depends on time, while the other (wrapping the horizon) is time-independent.
- Likewise, there are coarse-grained (since geometrical) and fine-grained (due to the minimalisation and homology constraints) aspects to HEE.
- Natural factorisation into two uncorrelated groups of DoFs in $\mathcal{A}$: those capturing the entanglement with $\mathcal{A}^c$ and those giving the entropy of the full system... [Headrick].

[also cf. Zhang&Wu]

Summary

- The extremal surfaces $\mathfrak{E}_{\mathcal{A}}$:
    - can exist in large multiplicities (exhibit remarkably rich structure)
    - new families can appear at some critical region size
    - need not exist for static black hole in single homologous piece
- The entanglement entropy $S_{\mathcal{A}}$:
    - exhibits entanglement plateaux (required by homology & minimality)
    - distinguishes between pure and thermal states (collapsed versus eternal black holes), arbitrarily long after 'themalization'
    - hence has features of a 'fine-grained' observable (though not for distinguishing microstates)

Future directions

- Complete (fully covariant) prescription and proof of HEE...
  - Correct formulation of the homology constraint
  - Interpretation of fine-grained nature of EE
- Does entanglement reconstruct geometry?
  - Metric extraction from set of EE's of subregions $\{S_{\mathcal{A}}\}$
  - Dual of reduced density metrix $\rho_{\mathcal{A}}$
  - Entanglement renormalization (MERA) vs. AdS geometry
- Role of EE in fundamental nature of spacetime…

Space Ref (Harvard-Smithsonian CfA) soihub.org/itschool