Extended Supersymmetry and Four Dimensional Geometry

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Background: Work with

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Superspace

$d = 2\ ,\ \ N = (2,2)$

Algebra:

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\substack{+\\=}}$$

Constrained superfields:

$$\begin{aligned} \bar{\mathbb{D}}_\pm \phi^a &= 0\ , \\ \bar{\mathbb{D}}_+ \chi^{a'} = \mathbb{D}_- \chi^{a'} &= 0\ , \\ \bar{\mathbb{D}}_+ \mathbb{X}^\ell &= 0\ , \\ \bar{\mathbb{D}}_- \mathbb{X}^r &= 0\ . \end{aligned}$$

Notation: $\quad c := a, \bar{a}\ , \quad t := a', \bar{a}'\ , \quad L := \ell, \bar{\ell}\ , \quad R := r, \bar{r}\ .$

Gates-Hull-Roček=Bihermitean=Generalized Kähler.

$$(M, g, J_{(\pm)}, H)$$

$$\boxed{\begin{array}{c} J^2_{(\pm)} = -\mathbf{1}\ , \quad J^t_{(\pm)} g J_{(\pm)} = g\ , \quad \nabla^{(\pm)} J_{(\pm)} = 0 \\ \Gamma^{(\pm)} = \Gamma^0 \pm \tfrac{1}{2} g^{-1} H\ , \quad H = dB\ . \end{array}}$$

$$E := g + B$$

Superspace Action

The $(2,2)$ superspace formulation of GKG uses the generalized Kähler Potential.

$$S = \int \mathbb{D}_+\bar{\mathbb{D}}_+\mathbb{D}_-\bar{\mathbb{D}}_- K(\phi^c, \chi^t, \mathbb{X}^L, \mathbb{X}^R)$$

$K \to K(\mathbb{X}^L, \mathbb{X}^R)$

Reduction to $(1,1)$ superspace

$$S = \int D_+ D_- \left(D_+ X E D_- X \right) \ .$$

The reduction goes via

$$\mathbb{D}_\pm =: D_+ - iQ_\pm \ , \ \ Q_+\mathbb{X}^R| =: \psi^R_+, \ \ Q_-\mathbb{X}^L| =: \psi^L_-$$

Both the auxiliary spinors ψ^L_- and ψ^R_+ have been eliminated and $E := g + B$.

Semichiral model with $4D$ target space

So, we consider the action,

$$S = \int d^2xd^2\theta d^2\bar{\theta}K(\mathbb{X}^L, \mathbb{X}^R)$$

where $L = (\ell, \bar{\ell})$ and $R = (r, \bar{r})$, the SUSY algebra is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\stackrel{+}{=}}$$

and the semichiral fields satisfy

$$\bar{\mathbb{D}}_+\mathbb{X}^\ell = 0\ , \quad \bar{\mathbb{D}}_-\mathbb{X}^r = 0\ .$$

$4D$ Geometry

Complex structures

$$(J^{(\pm)})^2 = -\mathbf{1}$$

$$J^{(\pm)\,t} g\, J^{(\pm)} = g$$

In $4D$

$$\{J^{(+)}, J^{(-)}\} = 2c\mathbf{1}\,,$$

where $c = c(\mathbb{X}^L, \mathbb{X}^R)$. This allows us to construct an $SU(2)$ worth of almost (pseudo-) complex structures $J^{(1)}, J^{(2)}, J^{(3)}$,

$$\begin{aligned} J^{(1)} &:= \frac{1}{\sqrt{1-c^2}}\left(J^{(-)} + cJ^{(+)}\right)\,, \\ J^{(2)} &:= \frac{1}{2\sqrt{1-c^2}}[J^{(+)}, J^{(-)}]\,, \\ J^{(3)} &:= J^{(+)}\,. \end{aligned}$$

For $|c| < 1$ the geometry is almost hyperkähler, while for $|c| > 1$ the geometry is almost pseudo-hyperkähler.

Additional SUSY

Ansatz for the additional supersymmetry,

M. Goteman, U.L., and M. Roček, I.Ryb 2011; M. Goteman, U. L. and M. Roček, 2012

$$
\begin{aligned}
\delta\mathbb{X}^{\ell} &= \bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}f(\mathbb{X}^{L},\mathbb{X}^{R}) + g(\mathbb{X}^{\ell})\bar{\epsilon}^{-}\bar{\mathbb{D}}_{-}\mathbb{X}^{\ell} + h(\mathbb{X}^{\ell})\epsilon^{-}\mathbb{D}_{-}\mathbb{X}^{\ell}\ , \\
\delta\mathbb{X}^{\bar{\ell}} &= \epsilon^{+}\mathbb{D}_{+}\bar{f}(\mathbb{X}^{L},\mathbb{X}^{R}) + \bar{g}(\mathbb{X}^{\bar{\ell}})\epsilon^{-}\mathbb{D}_{-}\mathbb{X}^{\bar{\ell}} + \bar{h}(\mathbb{X}^{\bar{\ell}})\bar{\epsilon}^{-}\bar{\mathbb{D}}_{-}\mathbb{X}^{\bar{\ell}}\ , \\
\delta\mathbb{X}^{r} &= \bar{\epsilon}^{-}\bar{\mathbb{D}}_{-}\tilde{f}(\mathbb{X}^{L},\mathbb{X}^{R}) + \tilde{g}(\mathbb{X}^{r})\bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\mathbb{X}^{r} + \tilde{h}(\mathbb{X}^{r})\epsilon^{+}\mathbb{D}_{+}\mathbb{X}^{r}\ , \\
\delta\mathbb{X}^{\bar{r}} &= \epsilon^{-}\mathbb{D}_{-}\bar{\tilde{f}}(\mathbb{X}^{L},\mathbb{X}^{R}) + \bar{\tilde{g}}(\mathbb{X}^{\bar{r}})\epsilon^{+}\mathbb{D}_{+}\mathbb{X}^{\bar{r}} + \bar{\tilde{h}}(\mathbb{X}^{\bar{r}})\bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\mathbb{X}^{\bar{r}}\ .
\end{aligned}
$$

$$[\delta_1^{(+)}, \delta_2^{(+)}]\mathbb{X}^\ell \stackrel{?}{=} i\epsilon^+_{[2}\bar{\epsilon}^+_{1]}\partial\mathbb{X}^\ell$$

$$[\delta_1, \delta_2]\mathbb{X}^\ell =$$
$$-\epsilon^+_{[2}\bar{\epsilon}^+_{1]}\left(|f_{\bar{\ell}}|^2 i\partial_{+\!\!+}\mathbb{X}^\ell + (f_{\bar{\ell}}\bar{f}_r + f_r\tilde{h})\bar{\mathbb{D}}_+\mathbb{D}_+\mathbb{X}^r + \dots\right)$$
$$+\bar{\epsilon}^-_{[2}\epsilon^-_{1]}(-gh)i\partial_{=}\mathbb{X}^\ell + \dots\ ,$$

- $|f_{\bar{\ell}}| > 0 \Rightarrow$ on-shell algebra
- $gh = -1 \Rightarrow$ Only Hyperkähler geometries
- $\delta\mathbb{X}^\ell = \bar{\mathbb{D}}_+\mathbb{D}_-(\epsilon F^\ell(\mathbb{X}^L, \mathbb{X}^R))$?
- Central charge transformations. Vanish on-shell.

Counterexample?

A $SU(2) \otimes U(1)$ WZW model may be described as the target space of a $(2,2)$ SUSY sigma model in many ways. See e.g., A. Sevrin, W. Staessens and D. Terryn 2011

In terms of chiral fields $\bar{\mathbb{D}}_{\pm}\hat{\phi} = 0$ and twisted chiral fields $\bar{\mathbb{D}}_{+}\hat{\chi} = \mathbb{D}_{-}\hat{\chi} = 0$ we have

$$K = -ln\hat{\chi}ln\hat{\bar{\chi}} + \int^{\frac{\hat{\phi}\hat{\bar{\phi}}}{\hat{\chi}\hat{\bar{\chi}}}} dq \frac{ln(1+q)}{q}$$

$$K_{\hat{\phi}\hat{\bar{\phi}}} + K_{\hat{\chi}\hat{\bar{\chi}}} = 0 \ , \Rightarrow (4,4)$$

It has a nontrivial $H = dB$.

Changing coordinates $\phi = \mathit{ln}\hat{\phi}, \quad \chi = \mathit{ln}\hat{\chi}$ implies

$$K \to K = -\chi\bar{\chi} + \int^{\phi+\bar{\phi}-\chi-\bar{\chi}} dq \; \mathit{ln}(1 + e^{q}) \; ,$$

and makes it amenable to dualization of the translation symmetry

$$\phi \to \phi + \lambda, \; \chi \to \chi + \lambda \; .$$

The dual potential is semichiral and reads
I.Ryb, U.L, M. Roček, R. von Unge and M.Zabzine 2009

$$-\frac{1}{2\alpha^2}X_\phi^2 + \frac{1}{\alpha}X_\phi X_\chi - \int^X dq\; ln(e^q - 1)\;,$$

where the semichiral fields enter in the combinations

$$\begin{aligned} X_\phi &= \tfrac{i}{2}(\mathbb{X}^\ell - \bar{\mathbb{X}}^\ell - \mathbb{X}^r + \bar{\mathbb{X}}^r) \\ X_\chi &= \tfrac{i}{2}(-\mathbb{X}^\ell + \bar{\mathbb{X}}^\ell - \mathbb{X}^r + \bar{\mathbb{X}}^r) \\ X &= \tfrac{1}{2}(\mathbb{X}^\ell + \bar{\mathbb{X}}^\ell - \mathbb{X}^r - \bar{\mathbb{X}}^r) \end{aligned}$$

.

This is the semichiral model which we expect to carry $(4,4)$.
Note that it is NOT Hyperkähler.

How is the $(4,4)$ realized?

- Analyze at the $(2,2)$ level?
- Reduce to $(1,1)$!

In this case the form of the dualization shows that the $(1,1)$ descriptions are related by coordinate transformations. So, find the coordinate transformation from the $(1,1)$ reduction of the chiral-twisted chiral model to the reduction of the semichiral one. Use this to find the expressions for the extra complex structures in the latter. Then check if it can be lifted to the $(2,2)$ semichiral formulation. In a picture:

$$
\begin{tikzcd}
\boxed{K(\phi,\chi)} \arrow[d] & & \boxed{K(\mathbb{X}^L,\mathbb{X}^R)} \\
\boxed{\begin{array}{c} g_{a\bar{a}}\partial\phi^a\partial\bar{\phi}^a + \\ \dots \end{array}} \arrow[rr, dashed] & & \boxed{\begin{array}{c} g_{\ell\bar{\ell}}\partial X^\ell\partial\bar{X}^\ell + \\ \dots \end{array}} \arrow[u]
\end{tikzcd}
$$

The reduction of $K(\phi, \bar{\phi}, \chi, \bar{\chi})$ identifies the right-handed $SU(2)$ of complex structures

$$I^{(A)} I^{(B)} = -\delta^{AB} + \epsilon^{ABC} I^{(C)}$$

as generated by

$$I^{(1)} = \begin{pmatrix} 0 & 0 & 0 & e^{\bar{\chi}-\phi} \\ 0 & 0 & e^{\chi-\bar{\phi}} & 0 \\ 0 & -e^{\bar{\phi}-\chi} & 0 & 0 \\ -e^{\phi-\bar{\chi}} & 0 & 0 & 0 \end{pmatrix}$$

$$I^{(2)} = \begin{pmatrix} 0 & 0 & 0 & ie^{\bar{\chi}-\phi} \\ 0 & 0 & -ie^{\chi-\bar{\phi}} & 0 \\ 0 & -ie^{\bar{\phi}-\chi} & 0 & 0 \\ ie^{\phi-\bar{\chi}} & 0 & 0 & 0 \end{pmatrix} ,$$

with $I^{(3)} = J^{(+)} = J$, the canonical complex structure.

- From the $(1,1)$ version of the dualization, we find the coordinate transformation $(\phi,\chi) \to (\mathbb{X}^L, \mathbb{X}^R)$ for three combinations of the coordinates.
- The full transformations are then determined by the requirement that it takes $J = J_{(+)} = I^{(3)} \to J^{(+)}$ (Check: the transformation of g).
- The coordinate transformation of $I^{(1)}_{(+)}$ and $I^{(2)}_{(+)}$ give us the additional complex structures in $(\phi,\chi) \to (\mathbb{X}^L, \mathbb{X}^R)$ space.
- We may then consider the question of lifting to $(2,2)$ Can they be realised as transformations of a manifiest $(2,2)$ semichiral model?

In terms of a direct product of 2*x*2 matrices they are:

$$I^{(A)} = \mathtt{j} \otimes \mathbb{A}^{(A)} = \mathtt{j} \otimes (a^{(A)}\sigma_1 - b^{(A)}\sigma_2)$$

$$\mathtt{j}^2 = -N\ , \qquad (\mathbb{A}^{(A)})^2 = N^{-1}\ ,$$

where $A = 1, 2$.

Need to discuss the full symmetries at the $(1, 1)$ level.

Extra SUSY for semis in $(1,1)$

$(2,2) \to (1,1) \Rightarrow$

$$\mathbb{X}^L \to (X^L, \psi_-^L)$$
$$\mathbb{X}^R \to (X^R, \psi_+^R)$$

The Lagrangian is

$$\mathcal{L} = D_+ X^A E_{AB}(X) D_- X^C + \Psi_+^R K_{RL} \Psi_-^L := \mathcal{L}_1 + \mathcal{L}_2 \ ,$$

where we define

$$\Psi_+^R := \psi_+^R - D_+ X^A J^R_{(+)A}$$
$$\Psi_-^L := \psi_-^L - J^R_{(-)A} D_- X^A \ .$$

On shell $\Psi_\pm = 0$ and $\mathcal{L}_1(X^L, X^R)$ is the $(1,1)$ Lagrangian.

Assume that we have found the additional transformations of the $(1,1)$ coordinates generated by the $SU(2)$ set of complex structures $J^{(A)}_{(+)}$ that leave the $\mathcal{L}_1$ part of the action invariant. We would now like to extend them to symmetries of the full action and subsequently check if the full set can come from transformations of the $(2,2)$ semichiral fields.
We know one invariance of the full action, that generated by $J^{(+)}$. Guided by how that invariance works we analyse the conditions for an invariance of $\mathcal{L}_1$ to extend to the full action.

The most general ansatz for the X transformation is

$$\delta X^A = \epsilon^+ \left[I^A_{(+)B} D_+ X^B + M^A_R \Psi^R_+ \right] \qquad A = L, R \ .$$

We find the transformations for the Ψs via their definitions and the fact that

$$\boxed{\psi^L_- := Q_- \mathbb{X}^L| \ , \qquad \psi^R_+ := Q_+ \mathbb{X}^R|}$$

which is the $(1,1)$ manifestation of that $\mathbb{X}^A| = X^A$ and ψ^A sit in the same semichiral multiplet. The algebra closes on-shell.

One finds that the conditions for invariance of the action are related to the commutator of the extra complex structure I and $J_{(-)}$

$$
\begin{aligned}
&(1) \quad M_{L[R,\dot{R}]} - M_{[R\dot{R}],L} = 0 \\
&(2) \quad M^{L}{}_{\dot{R}} - K_{\dot{R}\dot{L}} M^{\dot{L}}{}_{R} K^{RL} = \tfrac{1}{2} K_{\dot{R}\dot{L}} [I_{(+)}, J_{(-)}]^{\dot{L}}{}_{A} G^{AL} \\
&(3) \quad M^{R}{}_{\dot{R}} = \tfrac{1}{2} K_{\dot{R}L} [I_{(+)}, J_{(-)}]^{L}{}_{A} G^{AR} \\
&(4) \quad -K_{(\dot{R}|L|} [M, J_{(-)}]^{L}{}_{R)} = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&(5) \quad K_{[\dot{R}|L|} \mathcal{M}(M, J_{(-)})^{L}_{R]A} D_{-} X^{A} = -\tfrac{1}{2} D_{-}(K_{[\dot{R}|L|} [M, J_{(-)}]^{L}{}_{R]}) \\
&(6) \quad \ldots
\end{aligned}
$$

Test?

Hyperkähler

$$\mathcal{I} := J_{(+)} \ , \quad \mathcal{J} := \frac{1}{\sqrt{1-c^2}}(J_{(-)} + cJ_{(+)}) \ , \quad \mathcal{K} := \frac{1}{2\sqrt{1-c^2}}[J_{(+)}, J_{(-)}] \ .$$

$$\mathcal{I}: \qquad M^R{}_{\dot R} = \delta^R{}_{\dot R} \ , \quad M_{[\dot R R]} = 0 \iff K_L M^L{}_{\dot R} = \mu_{,\dot R}$$

$$\mathcal{J}: \qquad M^R{}_{\dot R} = \frac{c\ \delta^R{}_{\dot R}}{\sqrt{1-c^2}} \ , \quad K_L M^L{}_{\dot R} = \mu_{,\dot R}$$

$$\mathcal{K}: \qquad M^R{}_{\dot R} K_{LR} = -\frac{1}{\sqrt{1-c^2}} K_{\dot R L} J^L_{(-)L} = -\frac{1}{\sqrt{1-c^2}} JK_{\dot R L}$$

$$M_{[\dot R R]} = -\frac{1}{\sqrt{1-c^2}} K_{\dot R L} J^L_{(-)R} = -\frac{1}{\sqrt{1-c^2}} C_{\dot R R}$$

$$\Leftarrow M^L{}_{\dot R} = -\frac{1}{\sqrt{1-c^2}} K^{LR} JK_{R\dot R}$$

Each case satisfies the requirements when c constant .

The complex structures from $SU(3) \otimes U(1)$

$I^{(1)}$ *and* $I^{(2)}$ *satisfy criteria* $(1) - (4)$

- *They fail to satisfy* (5)

Conclusions

- Problems with $(4,4)$ away from Hyperkähler semichiral model with $4D$ target space.
- $(2,2)$ in $(1,1)$ in new formulation adapted to on-shell
- $(1,1)$ characterisation of additional supersymmetries of the $(2,2)$ model.
- Hyperkähler examples shown to satisfy these.
- The $S^3 \otimes S^1$ dualized to semichiral model
- Does not fulfil the conditions for having a $(2,2)$ origin.

Maybe we studied the problem from the wryng point of view, as exemplified by my son in the following picture:

THANK YOU FOR YOUR ATTENTION!