Large N phase transitions in Supersymmetric gauge theories with massive matter



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Based on:

J. Russo and K. Zarembo, arxiv:1309.1004, 1312.1214, 1302.6968

A. Barranco and J. Russo, arxiv:1401.3672

J. Russo, G. Silva and M.T., arxiv:1407.4794

Plan

Study of exact results in gauge theories both at large N and finite N

"Exact" = all order in coupling, including both perturbative and non-perturbative contributions

- 1. Four-dimensional \mathcal{N} = 2 SQCD and \mathcal{N} = 2* SYM at large N
- 2. \mathcal{N} = 2 *U*(*N*) CS-matter theory at large *N*
- 3. \mathcal{N} = 2 *U*(*N*) CS-matter theory with finite *N*

Localization

I. Exact partition function for \mathcal{N} = 2 supersymmetric YM theories on S⁴, with arbitrary matter content. [Pestun, 0712.2824]

II. Exact partition function for \mathcal{N} = 2 supersymmetric CS –matter theories on S^3 [Kapustin, Willett and Yaakov, 0909.4559]

 $\langle \Phi \rangle = diag(a_1, \dots, a_N)$ Partition function localizes to a matrix integral over Coulomb moduli

VEV of scalar of vector multiplet

Z = Z(q)Exact g dependence

This a complicated integral which we still need to compute in order to understand the underlying physics.

$$\mathcal{N}$$
= 2 SYM theories in four dimensions:

 $\left| Z(g) = \int d^{N-1}a \prod_{i < i} (a_i - a_j)^2 e^{-S_{cl}(a)} z_{1-loop}(a) \right| \left| z_{inst}(a;g^2) \right|^2$ $S_{cl} = \frac{1}{4g^2} \int_{a^4} d^4 x \sqrt{g} \, \mathrm{R} \, \mathrm{tr} \Phi^2 = \frac{8\pi^2}{g^2} \sum_i a_i^2$

\mathcal{N} = 4 Super Yang-Mills theory on S⁴

•Instantons do not contribute.

•1-loop corrections cancel

Gaussian matrix model:
$$Z = \int d^{N-1}a \prod_{i < j} (a_i - a_j)^2 e^{-\frac{8\pi^2}{g^2}\sum_i a_i^2}$$

At large *N* the integral is dominated by a saddle-point.

Introducing the eigenvalue density:



semicircle (Wigner's law)

\mathcal{N} = 4 SYM : Wilson loop

[Erickson, Semenoff and Zarembo, 0003055] [Drukker, Gross, 0010274]

$$W(C) = \int_{-\mu}^{\mu} dx \ \rho(x) \ e^{2\pi x} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \xrightarrow{\lambda \to \infty} \sqrt{\frac{2}{\pi}} \ \lambda^{-3/4} \ e^{\sqrt{\lambda}}$$
Reproduced by holography

Expanding the Bessel function at small λ one gets the perturbative series that must reproduce the Feynman diagram calculations

$$W(C) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \quad \xrightarrow{\lambda <<1} \quad 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \dots$$

There is a smooth dependence with λ all the way from 0 to infinity.

No phase transition between the perturbative $\lambda \ll 1$ regime and the strong coupling $\lambda \gg 1$ regime (described by AdS/CFT duality).

What about \mathcal{N} = 2 SYM theories?

Is the interpolation between weak and strong coupling still smooth? Could there be a quantum phase transition at some value of λ ?

2. Supersymmetric Chern-Simons with massive matter

Do Chern-Simons-matter theories undergo quantum phase transitions like in the analogous four-dimensional case?

Consider the \mathcal{N} = 2 supersymmetric U(N) Chern-Simons theory with level k, coupled to a matter content given by $N_{\rm f}$ fundamentals and $N_{\rm f}$ antifundamentals chiral multiplets of mass m.

The partition function localizes to

[Kapustin, Willett, Yaakov, 1003.5694]

$$Z_{N_f}^{U(N)} = \int d^N \mu \quad e^{-\frac{1}{2g}\sum_i \mu_i^2} \frac{\prod_{i < j} (4\sinh^2 \frac{1}{2}(\mu_i - \mu_j))}{\prod_i [4\cosh \frac{1}{2}(\mu_i - m)\cosh \frac{1}{2}(\mu_i + m)]^{N_f}} \quad , \quad g = \frac{2\pi i}{k}$$

Can this integral be computed explicitly? It depends on four parameters (g, m, N, N_{f})

Consider the infinite *N* (planar) limit. Then the partition function can be determined by a saddle-point calculation.

Veneziano limit

$$N \to \infty$$
, $t \equiv gN = \text{fixed}$, $\zeta \equiv \frac{N_f}{N} = \text{fixed}$

The saddle-point equations are then

$$\int dv \,\rho(v) \coth \frac{1}{2}(\mu - v) = \frac{\mu}{t} + \frac{\zeta}{2} \tanh \frac{1}{2}(\mu + m) + \frac{\zeta}{2} \tanh \frac{1}{2}(\mu - m)$$

N7

This is an exactly solvable model

[Barranco, J.R, 1401.3672]

Large N solution in the decompactification limit

$$\int dv \ \rho(v) \coth \frac{1}{2}(\mu - v) = \frac{\mu}{t} + \frac{\zeta}{2} \tanh \frac{1}{2}(\mu + m) + \frac{\zeta}{2} \tanh \frac{1}{2}(\mu - m)$$

$$\bigwedge$$
Repulsion attraction to origin attraction to $\mu = +-m$ (con

attraction to μ = +- m (combined effect: attraction to μ = 0)

Restore R dependence and take decompactification limit

$$m \to mR \to \infty, \qquad \mu \to \mu R \to \infty$$

If *t* is fixed, then this limit just decouples matter multiplets, which get an infinite mass. The theory reduces to pure \mathcal{N} = 2 Chern-Simons theory.

An inspection of the saddle-point equation shows that the limit R = infinity is regular if at the same time t goes to infinity with

$$\lambda \equiv \frac{t}{mR} = \text{fixed}$$

Then the dependence on R completely cancels out and one obtains

$$\int d\nu \ \rho(\nu) \operatorname{sign}(\mu - \nu) = \frac{\mu}{\lambda m} + \frac{\zeta}{2} \operatorname{sign}(\mu + m) + \frac{\zeta}{2} \operatorname{sign}(\mu - m)$$

Potential

$$V = \frac{\mu^2}{2\lambda m} + \frac{N_f}{2N} (|\mu + m| + |\mu - m|)$$



Consider the integral equation:

$$\int_{-A}^{A} dv \ \rho(v) \operatorname{sign}(\mu - v) = \frac{\mu}{\lambda m} + \frac{\zeta}{2} \operatorname{sign}(\mu + m) + \frac{\zeta}{2} \operatorname{sign}(\mu - m)$$

As the coupling λ is increased from 0, the system goes through different phases I, II, III, where: I. A < mII. A = mIII. A > m

Phase I ($\lambda < 1$): arises when A < m, implying that $|\mu| < m$. Then the sign functions cancel out.

$$\rho(\mu) = \frac{1}{2m\lambda} , \qquad \mu \in [-m\lambda, m\lambda]$$

The distribution expands as λ is increased, until the endpoints hit + /- *m*.

Phase II
$$(1 < \lambda < 1/(1-\zeta))$$
: $A = m$

$$\rho(\mu) = \frac{1}{2m\lambda} + \frac{\lambda - 1}{2\lambda} (\delta(\mu + m) + \delta(\mu - m)) , \qquad \mu \in [-m,m]$$

Phase III $(\lambda > 1/(1-\zeta))$: arises when A > m.

$$\rho(\mu) = \frac{1}{2m\lambda} + \frac{\zeta}{2} \left(\delta(\mu + m) + \delta(\mu - m) \right) \quad , \qquad \mu \in [-m\lambda(1 - \zeta), m\lambda(1 - \zeta)]$$

We assumed: $0 < \zeta < 1$ $(0 < N_f < N)$

When $\zeta \ge 1$ $(N_f \ge N)$ only phase I, II remain

Eigenvalue density





This implies a discontinuity in the **third** derivative at both critical points, $\lambda = 1$ and $\lambda = (1 - \zeta)^{-1}$.

$$\partial^{3}(F_{I} - F_{II})\Big|_{\lambda=1} = -mR$$
, $\partial^{3}(F_{II} - F_{III})\Big|_{\lambda=1/(1-\zeta)} = mR(1-\zeta)^{5}$

Therefore, both phase transitions are third order.

3. $\mathcal{N} = 2 U(N)$ Chern-Simons-matter with finite N

[J. Russo, G. Silva and M.T., arxiv:1407.4794]

Consider

$$Z_{N_{f}}^{U(N)} = \int d^{N} \mu \quad e^{-\frac{1}{2g}\sum_{i}\mu_{i}^{2}} \frac{\prod_{i< j} (4\sinh^{2}(\mu_{i} - \mu_{j}))}{\prod_{i} [4\cosh(\mu_{i} - m)\cosh(\mu_{i} + m)]^{N_{f}}} \quad , \quad g = \frac{2\pi i}{k}$$

The previous saddle-point method only computes Z in the special corner of the parameter space

$$(g, m, N, N_f)$$

involving the planar N=infinity limit

Finite *N*: One can use the method of orthogonal polynomials. Then computing *Z* reduces to the problem of computing the integral:

$$I = \int_{-\infty}^{\infty} \frac{e^{at^2 + bt}}{e^{ct} + d} dt$$

Since 19th century, this integral appeared in many interesting contexts. In particular, studies of Riemann zeta functions by Siegel and Mock theta functions by Ramanujan. In 1933, it was studied in great detail in a classical paper by Mordell, who gave the result for all possible values of parameters a, b, c, d.

U(N) partition function from orthogonal polynomials

Using the method of orthogonal polynomials, the partition function can be written as follows:

$$Z_{N_f}^{U(N)} = N! e^{-\frac{gN}{2}(N^2 - N_f^2)} \det(f_i, f_j)$$

$$(f_i, f_j) = e^{g(N-N_f)(i+j+1-\frac{1}{2}(N-N_f))} \int_{-\infty}^{\infty} dx \frac{e^{x(i+j+1+N_f-N) - \frac{1}{2g}x^2}}{(1+e^{x+m})^{N_f} (1+e^{x-m})^{N_f}}$$

Case N_f = 1
$$(f_i, f_j) = e^{\frac{g}{2}(N^2 - 1) + \ell_g(N - 1)} \frac{I(\ell, m) - I(\ell, -m)}{2\sinh m}$$

$$\ell = i + j + 1 - N, \qquad \ell = 1 - N, \dots, N - 1$$

$$I(\ell,m) = e^{m} \int_{-\infty}^{\infty} dx \frac{e^{(\ell+1)x - \frac{1}{2g}x^{2}}}{1 + e^{x+m}}$$

The integral $I(\ell,m)$ is a particular case of the Mordell integral. It can in general be evaluated in terms of expressions involving infinite sums. However, in two cases, it reduces to a finite sum of terms.

Luckily, one of these cases is when $k = 2\pi i/g$ is an integer, which in CS theory is required by gauge invariance.

Using Mordell's formula, we find

$$I(\ell,m) = \frac{2\pi e^{-i\pi(\ell+\frac{k}{4})} e^{-m(\ell+\frac{k}{2})+\frac{ikm^2}{4\pi}}}{e^{-km}-1} \left(-\sqrt{\frac{i}{k}} \sum_{n=0}^{k-1} e^{\frac{i\pi}{k}(n-\ell-\frac{k}{2}+\frac{ikm}{2\pi})^2} + i\right)$$

Examples:

$$Z_{k}^{U(1)} = \frac{2\pi e^{-m + \frac{ik(m-i\pi)^{2}}{4\pi}}}{(1 - e^{-2m})(e^{km} - 1)} \left(\sqrt{\frac{i}{k}} \sum_{n=0}^{k-1} \left(e^{\frac{i\pi}{k}(n - \frac{k}{2} - \frac{ikm}{2\pi})^{2}} + e^{\frac{i\pi}{k}(n - \frac{k}{2} + \frac{ikm}{2\pi})^{2}} \right) - 2i \right)$$

The formula contains perturbative as well as non-perturbative terms Perturbative: e.g. $exp(gn^2/2)$ Non-perturbative: e.g. $exp(2\pi im/g)$

Massless case

$$Z_{k}^{U(1)}(m=0) = \frac{1}{2}e^{-\frac{i\pi k}{4}} + \frac{\pi}{k^{\frac{3}{2}}}e^{\frac{i\pi}{4}} \left(\sum_{n=0}^{k-1}(-1)^{n}e^{\frac{i\pi}{k}n^{2}}(n-\frac{k}{2})^{2}\right)$$

Finite dimensional Hilbert space?

Higher rank

$$Z_{k=2}^{U(3)} = \frac{24\sqrt{2}\pi^{3}e^{\frac{i\pi}{4}}e^{m+\frac{im^{2}}{\pi}}}{(e^{2m}-1)^{2}} \left(e^{2m}+2ie^{m}+1-2\sqrt{2}e^{\frac{i\pi}{4}}e^{m-\frac{im^{2}}{2\pi}}\right)$$

Large g coupling and phase transitions

The partition function is proportional to the determinant of J_{ij} defined by the basic integral

$$J_{ij} = \int_{-\infty}^{\infty} dx \, \frac{e^{g(x\ell - \frac{1}{2}x^2)}}{\left(4\cosh\frac{1}{2}(gx + m)\cosh\frac{1}{2}(gx - m)\right)^{N_f}} \quad , \qquad \ell = i + j + 1 - N$$

When g is large, the main contribution of the integral comes from the saddle-point at $x = \ell$ Define m = g p, and consider large g with fixed p. This is equivalent to the decompactification limit considered before, but now we take it at finite N (even low N, e.g. N = 2)

This implies

$$\begin{aligned} 2\cosh \frac{1}{2}(gx \pm m) &\to \exp(\frac{1}{2}g \mid x \pm p \mid) \\ J_{ij} &\to \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}g(x^2 - 2x\ell + N_f \mid x + p \mid + N_f \mid x - p \mid)} \\ &= \int_{-\infty}^{p} dx \; e^{-\frac{1}{2}g(x^2 - 2x(\ell + N_f))} + e^{-N_f gp} \int_{-p}^{p} dx \; e^{-\frac{1}{2}g(x^2 - 2x\ell)} + \int_{p}^{\infty} dx \; e^{-\frac{1}{2}g(x^2 - 2x(\ell - N_f))} \end{aligned}$$

Which term is dominant depends on whether the saddle-point lies within the interval (-p,p) or outside. This leads to three different cases , which are in one-to-one correspondence to the three phases encountered in the planar limit).

I) g(N-1) < m. In this case we simply have

$$Z_{N_f}^{U(N)} = N! (2\pi g)^{N/2} e^{-NN_f m} e^{\frac{1}{6}gN(N^2 - 1)}$$
$$= e^{-NN_f m} Z_{CS}(\mathbf{S}^3)$$

Large N: the free energy becomes

$$F_{N_f}^{U(N)} \equiv -\frac{1}{N^2} \ln Z_{N_f}^{U(N)} \rightarrow m(\zeta - \frac{1}{6}\lambda) \quad , \qquad \zeta = \frac{N_f}{N} \quad , \quad \lambda = gN/m$$

II) $g(N-1-N_f) < m \le g(N-1)$. In this case Z is given by a finite product. At large N

$$F_{N_f}^{U(N)} \rightarrow \frac{m}{6\lambda^2} (3(2\zeta - 1)\lambda^2 + 3\lambda - 1)$$

III) $m \leq g(N-1-N_f)$

$$F_{N_f}^{U(N)} \rightarrow \frac{m}{6\lambda}((\zeta - 1)^3\lambda^2 + 3\zeta^2\lambda + 3\zeta)$$

We thus reproduce the free energies found in the planar limit in phases 1,2 and 3 starting with exact finite N expressions.

Unitary matrix model formulation and large N

An interesting limit is the large N limit at fixed Chern-Simons level k.

A convenient approach is to formulate the unitary version of the matrix model, where the eigenvalues lie on S¹. In pure CS theory one can show that this approach gives equivalent results [Romo, Tierz, 1103.2421]

The unitary matrix model can be viewed as a deformation of the contour of integration to imaginary eigenvalues. The partition function now has trigonometric functions

$$\widetilde{Z}_{N_{f}}^{U(N)} = \int d^{N} \mu \quad e^{-\frac{1}{2g}\sum_{i}\mu_{i}^{2}} \frac{\prod_{i < j} (4\sin^{2}\frac{1}{2}(\mu_{i} - \mu_{j}))}{\prod_{i} [4\cos\frac{1}{2}(\mu_{i} + im)\cos\frac{1}{2}(\mu_{i} - im)]^{N_{f}}}$$

This can be written as an integration on the compact interval

$$\begin{split} \widetilde{Z}_{N_{f}}^{U(N)} &= \int_{[0,2\pi]^{N}} d^{N} \mu \quad \prod_{i=1}^{N} \left(\sum_{n_{i}=-\infty}^{\infty} e^{-\frac{1}{2g^{2}}(\mu_{i}+2\pi n_{i})^{2}} \right) \frac{\prod_{i$$

where we used Poisson resummation formula. This form permits to compute the large N limit by writing Z as a Toeplitz determinant and use the Szegő theorem.

Toeplitz determinants and Szegő theorem

Let

Toeplitz matrix

$$f(z) = \sum_{k \in \mathbb{Z}} f_k z^k , \qquad T_N(f) = (f_{i-j})_{i,j=1,...,N} \qquad \begin{pmatrix} a & b & c & d \\ f & a & b & c \\ g & f & a & b \\ h & g & f & a \end{pmatrix}$$
mula

Heine-Szegő formula

$$Z_{N}[f] \equiv \det T_{N}(f) = \int_{(0,2\pi)^{N}} \frac{d^{N}\phi}{(2\pi)^{N}} \prod_{l < k} \left| e^{i\phi_{l}} - e^{i\phi_{k}} \right|^{2} \prod_{j=1}^{N} f(e^{i\phi_{j}})$$

i.e. the Toeplitz determinant is the partition function of a U(N) unitary matrix model Then the strong Szegő limit theorem states that

$$\frac{\det T_N(f)}{G(f)^N} \qquad \xrightarrow{N \to \infty} \quad \exp(\sum_{k=1}^\infty k[\ln f]_k[\ln f]_{-k})$$

where

$$\ln f(z) = \sum_{k=-\infty}^{\infty} [\ln f]_k z^k$$

 $[\ln f]_k$ are the coefficients of the Fourier expansion of $\ln f_k$ and $G(f) = \exp([\log f]_0)$

In our case this gives

$$\widetilde{Z}_{N_{f}}^{U(N)} = \left(\frac{g}{2\pi}\right)^{\frac{N}{2}} \frac{e^{-NN_{f}|m|}}{\left(1 - e^{-2|m|}\right)^{N_{f}^{2}}} \prod_{j=1}^{\infty} \left(1 - q\right)^{N-j} \left(1 - q^{j-\frac{1}{2}} e^{-|m|}\right)^{2N_{f}} , \qquad N >> 1$$

If we further take the limit of g to infinity at fixed m/g, then this reproduces the expression of Z for phase I.

The other phases II and III cannot be recovered because in the unitary model $|m \pm i \mu|$ can never be zero.

Conclusions

Massive supersymmetric gauge theories exhibit large N phase transitions at critical couplings. Transitions occur when the eigenvalue distribution expands and the largest eigenvalue hits the mass. Then extra massless states contribute to the planar free energy.

Examples:

Four dimensional \mathcal{N} = 2^{*} SYM and \mathcal{N} = 2 SQCD.

 $\mathcal{N}=2^*$ theory has a gravity dual. It predicts something special occurring at $\sqrt{\lambda} \approx n\pi$, n >> 1

 $\mathcal{N}=2$ U(N) Chern-Simons with $2N_f$ massive flavors: The planar theory presents three phases when $0 < N_f < N$ and two phases when $N_f \ge N$.

Finite N: The matrix integral defining Z can be computed explicitly using Mordell integrals. **Large N with fixed k:** Z can be computed using Szegő limit theorem.

Mass deformed ABJM Model

Rich structure of phase transitions at imaginary values of the couplings, analogous to $\mathcal{N}=2^*$ SYM [Anderson, Zarembo 1406.3366]. Gravity dual phenomenon in terms of M2 branes?

Phase transitions also in 5d \mathcal{N} = 1 SYM+CS term and adjoint matter [Minahan, Nedelin 1408.2767] •Phase transition at some critical relative value of the two couplings. •Pure CS case with adjoint matter: evidence for infinite number of phase transitions.