Symmetries of curved superspace

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Based on:
SMK, arXiv:1212.6179;
SMK, U. Lindström, M. Roček, I. Sachs
& G. Tartaglino-Mazzucchelli, arXiv:1312.4267;
SMK, J. Novak & G. Tartaglino-Mazzucchelli, arXiv:1406.0727.

Outline

Supersymmetric backgrounds in supergravity

Supersymmetric solutions of supergravity

M. Duff *et al.* (1981,1982)

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P. van Nieuwenhuizen & N. Warner (1984)

M. Duff, B. Nilsson & C. Pope, *Kaluza-Klein Supergravity* (PR, 1986)

Concept of Killing spinors

All supersymmetric solutions of minimal (gauged) supergravity in 5D

J. Gauntlett, J. Gutowski, C. Hull, S. Pakis & H. Reall (2003)

J. Gauntlett & J. Gutowski (2003)

Superspace formalism to determine (super)symmetric backgrounds in off-shell supergravity

I. Buchbinder & SMK, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, IOP, 1995 + 1998

Superspace formalism to construct (super)symmetric backgrounds is universal since

- it is geometric;
- it may be extended to any off-shell supergravity theory formulated in superspace.

Some applications of the superspace formalism:

- Rigid supersymmetric field theories in 5D $\mathcal{N} = 1$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2007)
- Rigid supersymmetric field theories in 4D $\mathcal{N} = 2$ AdS superspace
  SMK & G. Tartaglino-Mazzucchelli (2008)
  D. Butter & SMK (2011)
  D. Butter, SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
- Rigid supersymmetric field theories in 3D (p, q) AdS superspaces
  SMK, & G. Tartaglino-Mazzucchelli (2012)
  SMK, U. Lindström & G. Tartaglino-Mazzucchelli (2012)
  D. Butter, SMK & G. Tartaglino-Mazzucchelli (2012)

Recent developments

Exact results (partition functions, Wilson loops etc.)
in rigid supersymmetric field theories on curved backgrounds
(e.g., S^3, S^4, $S^3 \times S^1$ etc.) using localization techniques

V. Pestun (2007, 2009)
A. Kapustin, B. Willett & I. Yaakov (2010)
D. Jafferis (2010)
. . . .

Necessary technical ingredients:

- Curved space $\mathcal{M}$ has to admit some unbroken rigid supersymmetry (supersymmetric background);
- Rigid supersymmetric field theory on $\mathcal{M}$ should be off-shell.

These developments have inspired much interest in the construction and classification of supersymmetric backgrounds that correspond to off-shell supergravity formulations.

Classification of supersymmetric backgrounds in off-shell supergravity

Component approaches

G. Festuccia and N. Seiberg (2011)
H. Samtleben and D. Tsimpis (2012)
C. Klare, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu, G. Festuccia and N. Seiberg (2012)
D. Cassani, C. Klare, D. Martelli, A. Tomasiello and A. Zaffaroni (2012)
T. Dumitrescu and G. Festuccia (2012)
A. Kehagias and J. Russo (2012)
. . . .

Such results also naturally follow from the superspace formalism developed in the mid 1990s

4D $\mathcal{N} = 1$ SMK (2012)
3D $\mathcal{N} = 2$ SMK, U. Lindström, M. Roček, I. Sachs & G. Tartaglino-Mazzucchelli (2013)
5D $\mathcal{N} = 1$ SMK, J. Novak & G. Tartaglino-Mazzucchelli (2014)

Component approaches vs superspace formalism

- Both component approaches and superspace formalism can be used to derive supersymmetric backgrounds in off-shell supergravity. Practically all classification results have been obtained within the component settings.
- Superspace formalism is more useful in order to determine all (conformal) isometries of a given backgrounds.
- Superspace formalism is more powerful for constructing the most general rigid supersymmetric field theories on a given background.

(Conformal) symmetries of curved spacetime

(Conformal) symmetries of a curved superspace may be defined similarly to those corresponding to a curved spacetime within the Weyl-invariant formulation for gravity (variation on a theme by Hermann Weyl).

S. Deser (1970)
P. Dirac (1973)

Three formulations for gravity in d dimensions:

- Metric formulation;
- Vielbein formulation;
- Weyl-invariant formulation.

I briefly recall the metric and vielbein approaches and then concentrate in more detail of the Weyl-invariant formulation.

Metric and vielbein formulations for gravity

Metric formulation

Gauge field: metric $g_{mn}(x)$

Gauge transformation: $\delta g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m$

$\xi = \xi^m(x)\partial_m$ a vector field generating an infinitesimal diffeomorphism.

Vielbein formulation

Gauge field: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

The metric is a composite field $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$

Gauge transformation: $\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a]$

Gauge parameters: $\xi^a(x) = \xi^m e_m{}^a(x)$ and $K^{ab}(x) = -K^{ba}(x)$

Covariant derivatives (M_{bc} the Lorentz generators)

$$\nabla_a = e_a{}^m \partial_m + \frac{1}{2}\omega_a{}^{bc} M_{bc} , \qquad [\nabla_a, \nabla_b] = \frac{1}{2} R_{ab}{}^{cd} M_{cd}$$

$e_a{}^m$ the inverse vielbein, $e_a{}^m e_m{}^b = \delta_a{}^b$;
$\omega_a{}^{bc}$ the torsion-free Lorentz connection.

Weyl transformations

Weyl transformations

The torsion-free constraint

$$T_{ab}{}^{c}=0 \quad \Longleftrightarrow \quad [\nabla_a,\nabla_b]\equiv T_{ab}{}^{c}\nabla_c+\frac{1}{2}R_{ab}{}^{cd}M_{cd}=\frac{1}{2}R_{ab}{}^{cd}M_{cd}$$

is invariant under Weyl (local scale) transformations

$$\nabla_a \to \nabla'_a = \mathrm{e}^{\sigma}\Big(\nabla_a+(\nabla^b\sigma)M_{ba}\Big)~,$$

with the parameter $\sigma(x)$ being completely arbitrary.

$$e_a{}^m \to \mathrm{e}^{\sigma}e_a{}^m~, \qquad e_m{}^a \to \mathrm{e}^{-\sigma}e_m{}^a~, \qquad g_{mn}\to \mathrm{e}^{-2\sigma}g_{mn}$$

Weyl transformations are gauge symmetries of conformal gravity.
Einstein gravity possesses no Weyl invariance.

Weyl-invariant formulation for Einstein gravity

Weyl-invariant formulation for Einstein gravity

Gauge fields: vielbein $e_m{}^a(x)$, $e := \det(e_m{}^a) \neq 0$

& conformal compensator $\varphi(x)$, $\varphi \neq 0$

Gauge transformations ($\mathcal{K} := \xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}$)

$$\delta \nabla_a = [\xi^b \nabla_b + \frac{1}{2} K^{bc} M_{bc}, \nabla_a] + \sigma \nabla_a + (\nabla^b \sigma) M_{ba} \equiv (\delta_\mathcal{K} + \delta_\sigma) \nabla_a ,$$

$$\delta \varphi = \xi^b \nabla_b \varphi + \frac{1}{2}(d-2)\sigma\varphi \equiv (\delta_\mathcal{K} + \delta_\sigma)\varphi$$

Gauge-invariant gravity action

$$S = \frac{1}{2} \int \mathrm{d}^d x \, e \left(\nabla^a \varphi \nabla_a \varphi + \frac{1}{4} \frac{d-2}{d-1} R \varphi^2 + \lambda \varphi^{2d/(d-2)} \right)$$

Imposing a Weyl gauge condition $\varphi = \frac{2}{\kappa}\sqrt{\frac{d-1}{d-2}} = \mathrm{const}$

reduces the action to

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \, e \, R - \frac{\Lambda}{\kappa^2} \int \mathrm{d}^d x \, e$$

Conformal isometries

Conformal Killing vector fields

A vector field $\xi = \xi^m \partial_m = \xi^a e_a$, with $e_a := e_a{}^m \partial_m$, is conformal Killing if there exist local Lorentz, $K^{bc}[\xi]$, and Weyl, $\sigma[\xi]$, parameters such that

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b \sigma[\xi]) M_{ba} = 0$$

A short calculation gives

$$K^{bc}[\xi] = \frac{1}{2}\big(\nabla^b \xi^c - \nabla^c \xi^b\big) , \qquad \sigma[\xi] = \frac{1}{d} \nabla_b \xi^b$$

Conformal Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 2\eta^{ab} \sigma[\xi]$$

Equivalent spinor form in $d = 4$: $\nabla_{(\alpha}{}^{(\dot\alpha} \xi_{\beta)}{}^{\dot\beta)} = 0$

($\nabla_a \to \nabla_{\alpha\dot\alpha}$ and $\xi_a \to \xi_{\alpha\dot\alpha}$)

Equivalent spinor form in $d = 3$: $\nabla_{(\alpha\beta} \xi_{\gamma\delta)} = 0$

Conformal isometries

- Lie algebra of conformal Killing vector fields
- Conformally related spacetimes (∇_a, φ) and $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b\rho)M_{ba}\Big) , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi$$

have the same conformal Killing vector fields $\xi = \xi^a e_a = \tilde{\xi}^a \tilde{e}_a$.

The parameters $K^{cd}[\tilde{\xi}]$ and $\sigma[\tilde{\xi}]$ are related to $K^{cd}[\xi]$ and $\sigma[\xi]$ as follows:

$$\begin{aligned} \mathcal{K}[\tilde{\xi}] &:= \tilde{\xi}^b \widetilde{\nabla}_b + \frac{1}{2}K^{cd}[\tilde{\xi}]M_{cd} = \mathcal{K}[\xi] , \\ \sigma[\tilde{\xi}] &= \sigma[\xi] - \xi\rho \end{aligned}$$

- Conformal field theories

Isometries

Killing vector fields

Let $\xi = \xi^a e_a$ be a conformal Killing vector,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\nabla_a = \Big[\xi^b \nabla_b + \frac{1}{2}K^{bc}[\xi]M_{bc}, \nabla_a\Big] + \sigma[\xi]\nabla_a + (\nabla^b\sigma[\xi])M_{ba} = 0 \ .$$

It is called Killing if it leaves the compensator invariant,

$$(\delta_{\mathcal{K}} + \delta_\sigma)\varphi = \xi\varphi + \frac{1}{2}(d-2)\sigma[\xi]\varphi = 0 \ .$$

These Killing equations are Weyl invariant in the following sense: Given a conformally related spacetime $(\widetilde{\nabla}_a, \widetilde{\varphi})$

$$\widetilde{\nabla}_a = \mathrm{e}^{\rho}\Big(\nabla_a + (\nabla^b\rho)M_{ba}\Big) \ , \qquad \widetilde{\varphi} = \mathrm{e}^{\frac{1}{2}(d-2)\rho}\varphi \ ,$$

the above Killing equations have the same functional form when rewritten in terms of $(\widetilde{\nabla}_a, \widetilde{\varphi})$, in particular

$$\xi\widetilde{\varphi} + \frac{1}{2}(d-2)\sigma[\tilde{\xi}]\widetilde{\varphi} = 0 \ .$$

Isometries

Because of Weyl invariance, we can work with a conformally related spacetime such that

$$\varphi = 1$$

Then the Killing equations turn into

$$\Big[\xi^b \nabla_b + \frac{1}{2} K^{bc}[\xi] M_{bc}, \nabla_a\Big] = 0 \ , \qquad \sigma[\xi] = 0$$

Standard Killing equation

$$\nabla^a \xi^b + \nabla^b \xi^a = 0$$

- Lie algebra of Killing vector fields
- Rigid symmetric field theories in curved space

(Conformal) symmetries of curved superspace

Weyl-invariant approach to spacetime symmetries has a natural superspace extension in all cases when supergravity is formulated as conformal supergravity coupled to certain conformal compensator(s) Ξ

$z^M = (x^m, \theta^\mu)$ local coordinates of curved superspace

$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha) = E_A + \Omega_A + \Phi_A$ superspace covariant derivatives

$E_A = E_A{}^M(z)\partial_M$ superspace inverse vielbein

$\Omega_A = \frac{1}{2}\Omega_A{}^{bc}(z)M_{bc}$ superspace Lorentz connection

$\Phi = \Phi_A{}^I(z)T_I$ superspace R-symmetry connection

Supergravity gauge transformation

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] , \qquad \delta_{\mathcal{K}}\Xi = \mathcal{K}\Xi , \qquad \mathcal{K} := \xi^B\mathcal{D}_B + \frac{1}{2}K^{bc}M_{bc} + K^I T_I$$

Super-Weyl transformation

$$\delta_\sigma\mathcal{D}_a = \sigma\mathcal{D}_a + \cdots , \qquad \delta_\sigma\mathcal{D}_\alpha = \frac{1}{2}\sigma\mathcal{D}_\alpha + \cdots , \qquad \delta_\sigma\Xi = w_\Xi\sigma\Xi ,$$

with w_Ξ a non-zero super-Weyl weight

Conformal isometries of curved superspace

Let $\xi = \xi^B E_B$ be a real supervector field. It is called conformal Killing if

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 ,$$

for some Lorentz $K^{bc}[\xi]$, R-symmetry $K^I[\xi]$ and super-Weyl $\sigma[\xi]$ parameters.

- All parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$ are uniquely determined in terms of ξ^B.
- The spinor component ξ^β is uniquely determined in terms of ξ^b.
- The vector component obeys an equation that contains all the information about the conformal Killing supervector field.
  Examples:

$$d = 3 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta\gamma)} = 0$$

$$d = 4 \qquad \mathcal{D}^I_{(\alpha}\xi_{\beta)\dot\beta} = 0$$

Isometries of curved superspace

Let $\xi = \xi^B E_B$ be a conformal Killing supervector field,

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathcal{D}_A = 0 \ , \qquad (\star)$$

for uniquely determined parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$.
It is called Killling if the compensators are invariant,

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\Xi = 0 \ . \qquad (\star\star)$$

The Killing equations $(\star)$ and $(\star\star)$ are super-Weyl invariant in the sense that they hold for all conformally related superspace geometries.

Using the compensators Ξ we can always construct a scalar superfield $\phi = \phi(\Xi)$, which (i) is an algebraic function of Ξ; (ii) nowhere vanishing; and (iii) has a nonzero super-Weyl weight w_ϕ, $\delta_\sigma \phi = w_\phi \sigma \phi$.

$$(\delta_{\mathcal{K}[\xi]} + w_\Xi \sigma[\xi])\phi = 0 \ .$$

Super-Weyl invariance may be used to impose the gauge $\phi = 1$, and then

$$\sigma[\xi] = 0 \ .$$

(Conformal) supersymmetries of curved superspace

Of special interest are curved backgrounds which admit at least one (conformal) supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A of the type

$$\xi^a| = 0 \ , \qquad \xi^\alpha| \neq 0$$

and describe a bosonic background with the property that all spinor components of the superspace torsion and curvature tensors

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^I T_I$$

have vanishing bar-projections,

$$\varepsilon(T_{\dots}{}^{\dots}) = 1 \to T_{\dots}{}^{\dots}| = 0 \ , \qquad \varepsilon(R_{\dots}{}^{\dots}) = 1 \to R_{\dots}{}^{\dots}| = 0 \ .$$

These conditions are supersymmetric.
At the component level, all spinor fields may be gauged away.

$d = 3$, $\mathcal{N} = 2$ supergravity

3D $\mathcal{N} = 2$ curved superspace, $\mathcal{M}^{3|4}$, parametrized by local coordinates
$z^M = (x^m, \theta^\mu, \bar{\theta}_\mu)$, $m = 0, 1, 2$ and $\mu = 1, 2$
Superspace structure group $\mathrm{SO}(2,1) \times \mathrm{U}(1)_R$
Superspace covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A + \Omega_A + \mathrm{i}\,\Phi_A \mathcal{J} \ .$$

Algebra of covariant derivatives

$$\begin{aligned} \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar{R}\mathcal{M}_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4R\mathcal{M}_{\alpha\beta} \ , \\ \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}\mathcal{J} \\ &\quad +4\mathrm{i}\mathcal{S}\mathcal{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} \ . \end{aligned}$$

$\mathcal{M}_{ab} = -\mathcal{M}_{ba} \longleftrightarrow \mathcal{M}_{\alpha\beta} = \mathcal{M}_{\beta\alpha}$ Lorentz generators
Dimension-1 torsion superfields: (i) real scalar $\mathcal{S}$; (ii) complex scalar R such that $\mathcal{J}R = -2R$; (iii) real vector $\mathcal{C}_a \longleftrightarrow \mathcal{C}_{\alpha\beta}$.
Bianchi Identities:

$$\bar{\mathcal{D}}_\alpha R = 0 \ , \qquad (\bar{\mathcal{D}}^2 - 4R)\mathcal{S} = 0 \quad \dots$$

Conformal isometries

The conformal Killing supervector fields obey the equation

$$(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A = 0 ,$$

where

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A] , \qquad \mathcal{K} = \xi^C \mathcal{D}_C + \frac{1}{2}K^{cd}\mathcal{M}_{cd} + \mathrm{i}\,\tau\mathcal{J}$$

and

$$\delta_\sigma \mathcal{D}_\alpha = \frac{1}{2}\sigma\mathcal{D}_\alpha + (\mathcal{D}^\gamma\sigma)\mathcal{M}_{\gamma\alpha} - (\mathcal{D}_\alpha\sigma)\mathcal{J} , \quad \dots$$

It suffices to require $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$, which implies

$$\xi^\alpha = -\frac{\mathrm{i}}{6}\bar{\mathcal{D}}_\beta \xi^{\beta\alpha} , \qquad K_{\alpha\beta} = \mathcal{D}_{(\alpha}\xi_{\beta)} - \bar{\mathcal{D}}_{(\alpha}\bar{\xi}_{\beta)} - 2\xi_{\alpha\beta}\mathcal{S}$$

$$\sigma = \frac{1}{2}\big(\mathcal{D}_\alpha\xi^\alpha + \bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha\big) , \qquad \tau = -\frac{\mathrm{i}}{4}\big(\mathcal{D}_\alpha\xi^\alpha - \bar{\mathcal{D}}^\alpha\bar{\xi}_\alpha\big)$$

All parameters ξ^α, $K_{\alpha\beta}$, σ and τ are expressed in terms of ξ^a.

Conformal isometries

The remaining vector parameter ξ^a satisfies

$$\mathcal{D}_{(\alpha}\xi_{\beta\gamma)} = 0 \qquad (\star)$$

and its conjugate.
Implication: superfield analogue of the conformal Killing equation

$$\mathcal{D}_a\xi_b + \mathcal{D}_b\xi_a = \frac{2}{3}\eta_{ab}\mathcal{D}^c\xi_c \ .$$

Eq. $(\star)$ is fundamental in the sense that it implies $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_A \equiv 0$ provided the parameters ξ^α, $K_{\alpha\beta}$, σ and τ are defined as above.
The conformal Killing supervector field is a real supervector field

$$\xi = \xi^A E_A \ , \qquad \xi^A \equiv (\xi^a, \xi^\alpha, \bar{\xi}_{\dot\alpha}) = \Big(\xi^a, -\frac{\rm i}{6}\bar{\mathcal{D}}_{\dot\beta}\xi^{\dot\beta\alpha}, -\frac{\rm i}{6}\mathcal{D}^\beta\xi_{\beta\dot\alpha}\Big)$$

which obeys the master equation $(\star)$.
If ξ_1 and ξ_2 are two conformal Killing supervector fields, their Lie bracket $[\xi_1, \xi_2]$ is a conformal Killing supervector field.

Conformal isometries

Equation $(\delta_{\mathcal{K}} + \delta_\sigma)\mathcal{D}_\alpha = 0$ implies some additional results that have not been discussed above. Define

$$\Upsilon := (\xi^B, K^{\beta\gamma}, \tau)$$

It turns out that

- $\mathcal{D}_A \Upsilon$ is a linear combination of Υ, σ and $\mathcal{D}_C \sigma$;
- $\mathcal{D}_A \mathcal{D}_B \sigma$ can be represented as a linear combination of Υ, σ and $\mathcal{D}_C \sigma$.

The super Lie algebra of the conformal Killing vector fields on $\mathcal{M}^{3|4}$ is finite dimensional. The number of its even and odd generators cannot exceed those in the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{osp}(2|4)$.

Charged conformal Killing spinors

Look for curved superspace backgrounds admitting at least one conformal supersymmetry. Such a superspace must possess a conformal Killing supervector field ξ^A with the property

$$\xi^a| = 0 , \qquad \epsilon^\alpha := \xi^\alpha| \neq 0 .$$

All other bosonic parameters are assumed to vanish, $\sigma| = \tau| = K_{\alpha\beta}| = 0$.
Bosonic superspace backgrounds without covariant fermionic fields:

$$\mathcal{D}_\alpha \mathcal{S}| = 0 , \qquad \mathcal{D}_\alpha R| = 0 , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma}| = 0 .$$

These conditions mean that the gravitini can completely be gauged away

$$\mathcal{D}_a| = \mathbf{D}_a := e_a + \frac{1}{2}\omega_a{}^{bc}\mathcal{M}_{bc} + \mathrm{i} b_a \mathcal{J} \equiv \mathfrak{D}_a + \mathrm{i} b_a \mathcal{J} , \qquad e_a := e_a{}^m \partial_m .$$

Introduce scalar and vector fields associated with the superfield torsion:

$$s := \mathcal{S}| , \qquad r := R| , \qquad c_a := \mathcal{C}_a| .$$

S-supersymmetry parameter: $\eta_\alpha := \mathcal{D}_\alpha \sigma|$.

Charged conformal Killing spinors

Q-supersymmetry parameter $\epsilon^\alpha := \xi^\alpha|$ obeys the equation

$$\mathbf{D}_a\epsilon^\alpha + \frac{\mathrm{i}}{2}(\tilde\gamma_a\bar\eta)^\alpha + \mathrm{i}\varepsilon_{abc}\, c^b(\tilde\gamma^c\epsilon)^\alpha - s(\tilde\gamma_a\epsilon)^\alpha - \mathrm{i}r(\tilde\gamma_a\bar\epsilon)^\alpha = 0 ,$$

which is equivalent to

$$\Big(\mathbf{D}_{(\alpha\beta} - \mathrm{i}c_{(\alpha\beta}\Big)\epsilon_{\gamma)} = \Big(\mathfrak{D}_{(\alpha\beta} - \mathrm{i}(b+c)_{(\alpha\beta}\Big)\epsilon_{\gamma)} = 0 ,$$

$$\bar\eta_\alpha = -\frac{2\mathrm{i}}{3}\Big((\gamma^a\mathbf{D}_a\epsilon)_\alpha + 2\mathrm{i}(\gamma^a\epsilon)_\alpha c_a + 3s\epsilon_\alpha + 3\mathrm{i}r\bar\epsilon_\alpha\Big) .$$

This follows by bar-projecting the equation ($\boldsymbol{C}_{\alpha\beta\gamma} = -\mathrm{i}\mathcal{D}_{(\alpha}\mathcal{C}_{\beta\gamma)}$)

$$0 = \mathcal{D}_a\xi_\alpha + \frac{\mathrm{i}}{2}(\gamma_a)_\alpha{}^\beta\bar{\mathcal{D}}_\beta\sigma - \mathrm{i}\varepsilon_{abc}(\gamma^b)_\alpha{}^\beta\mathcal{C}^c\xi_\beta - (\gamma_a)_\alpha{}^\beta(\xi_\beta\mathcal{S} + \bar\xi_\beta R)$$
$$-\frac{1}{2}\varepsilon_{abc}\xi^b(\gamma^c)^{\beta\gamma}\Big(\mathrm{i}\bar{\boldsymbol{C}}_{\alpha\beta\gamma} - \frac{4\mathrm{i}}{3}\varepsilon_{\alpha(\beta}\bar{\mathcal{D}}_{\gamma)}\mathcal{S} - \frac{2}{3}\varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma)}R\Big) ,$$

which is one of the implications of $(\delta_\mathcal{K} + \delta_\sigma)\mathcal{D}_a = 0$.

Supersymmetric backgrounds

Rigid supersymmetry transformations (in super-Weyl gauge $\phi = 1$) are characterised by

$$\sigma[\xi] = 0 \quad \Longrightarrow \quad \eta_\alpha = 0 ,$$

The conformal Killing spinor equation turns into

$$\mathbf{D}_a \epsilon^\alpha = -\mathrm{i}\varepsilon_{abc} c^b (\tilde{\gamma}^c \epsilon)^\alpha + s(\tilde{\gamma}_a \epsilon)^\alpha + \mathrm{i} r(\tilde{\gamma}_a \bar{\epsilon})^\alpha .$$

$$\mathbf{D}_a = e_a + \frac{1}{2}\omega_a{}^{bc}\mathcal{M}_{bc} + \mathrm{i} b_a \mathcal{J} = \mathfrak{D}_a + \mathrm{i} b_a \mathcal{J} .$$

$$[\mathbf{D}_a, \mathbf{D}_b] = \frac{1}{2}\mathcal{R}_{ab}{}^{cd}\mathcal{M}_{cd} + \mathrm{i}\mathcal{F}_{ab}\mathcal{J} = [\mathfrak{D}_a, \mathfrak{D}_b] + \mathrm{i}\mathcal{F}_{ab}\mathcal{J} .$$

Supersymmetric backgrounds with four supercharges

The existence of rigid supersymmetries imposes non-trivial restrictions on the background fields. In the case of four supercharges, these are

$$\mathfrak{D}_a s = 0 , \qquad \mathfrak{D}_a r = 2\mathrm{i} b_a r , \qquad \mathfrak{D}_a c_b = 2\varepsilon_{abc} c^c s ,$$
$$r\, s = 0 , \qquad r\, c_a = 0 .$$

c_a is a Killing vector field,

$$\mathfrak{D}_a c_b + \mathfrak{D}_b c_a = 0 .$$

The $U(1)_R$ field strength proves to vanish, $\mathcal{F}_{ab} = 0$.
The Einstein tensor $\mathcal{G}_{ab} := \mathcal{R}^{ab} - \frac{1}{2}\eta^{ab}\mathcal{R}$ is

$$\mathcal{G}_{ab} = 4\Big[c_a c_b + \eta_{ab}\big(s^2 + \bar{r} r\big)\Big] .$$

For the Cotton tensor $\mathcal{W}_{ab} := \frac{1}{2}\varepsilon_{acd}\mathcal{W}^{cd}{}_b = \mathcal{W}_{ba}$, with $\mathcal{W}_{abc} = 2\mathfrak{D}_{[a}\mathcal{R}_{b]c} + \frac{1}{2}\eta_{c[a}\mathfrak{D}_{b]}\mathcal{R}$, we obtain

$$\mathcal{W}_{ab} = -24 s\Big[c_a c_b - \frac{1}{3}\eta_{ab} c^2\Big] .$$

Compensators

Type I supergravity: Chiral compensator

$$\bar{\mathcal{D}}_\alpha \Phi = 0 , \qquad \delta_\sigma \Phi = \frac{1}{2}\sigma\Phi , \qquad \mathcal{J}\Phi = -\frac{1}{2}\Phi .$$

The freedom to perform the super-Weyl and local $\mathrm{U}(1)_R$ transformations can be used to impose the gauge $\Phi = 1$.
Consistency conditions:

$$\mathcal{S} = 0 , \qquad \Phi_\alpha = 0 , \qquad \Phi_{\alpha\beta} = \mathcal{C}_{\alpha\beta} .$$

Supersymmetric backgrounds with four supercharges:

$$r\, c_a = 0 , \qquad \mathfrak{D}_a r = 0 , \qquad \mathfrak{D}_a c_b = 0 .$$

Such spacetimes are necessarily conformally flat,

$$\mathcal{W}_{ab} = 0 .$$

Solution with $c_a = 0$ corresponds to (1,1) AdS superspace.

Compensators

Type II supergravity: Real linear compensator

$$(\bar{\mathcal{D}}^2 - 4R)\mathbb{G} = 0 \ , \qquad \delta_\sigma \mathbb{G} = \sigma \mathbb{G} \ .$$

Super-Weyl invariance allows us to choose the gauge $\mathbb{G} = 1$.
Consistency conditions:

$$R = \bar{R} = 0 \ .$$

All supersymmetric backgrounds with four supercharges:

$$\mathfrak{D}_a s = 0 \ , \qquad \mathfrak{D}_a c_b = 2\varepsilon_{abc} c^c s \ .$$

The Cotton tensor

$$\mathcal{W}_{ab} = -24 s \Big[c_a c_b - \frac{1}{3} \eta_{ab} c^d c_d \Big] = -6 s \Big[\mathcal{R}_{ab} - \frac{1}{3} \eta_{ab} \mathcal{R} \Big]$$

Solution with $c_a = 0$ corresponds to (2,0) AdS superspace.

General feature of maximally supersymmetric backgrounds

For any background admitting four supercharges, if there exists a tensor superfield T such that its bar-projection vanishes, $T| = 0$, and this condition is supersymmetric, then the entire superfield is zero, $T = 0$.

Supersymmetric conditions

$$\mathcal{D}_\alpha \mathcal{S}| = 0 \ , \qquad \mathcal{D}_\alpha R| = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma}| = 0 \ .$$

imply

$$\mathcal{D}_\alpha \mathcal{S} = 0 \ , \qquad \mathcal{D}_\alpha R = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_{\beta\gamma} = 0 \ .$$

Example: Maximally supersymmetric backgrounds in Type I SUGRA

Dimension-1 torsion superfields

$$\mathcal{S} = 0 \ , \qquad R\,\mathcal{C}_a = 0 \ , \qquad \mathcal{D}_A R = 0 \ , \qquad \mathcal{D}_A \mathcal{C}_b = 0$$

are covariantly constant.

Algebra of covariant derivatives:

$$\begin{aligned}
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} &= -4\bar{R}\mathcal{M}_{\alpha\beta} \ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 4R\mathcal{M}_{\alpha\beta} \ , \\
\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} &= -2\mathrm{i}(\gamma^c)_{\alpha\beta}\mathcal{D}_c - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} \ , \\
[\mathcal{D}_a, \mathcal{D}_\beta] &= \mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\mathcal{D}_\gamma - \mathrm{i}(\gamma_a)_{\beta\gamma}\bar{R}\bar{\mathcal{D}}^\gamma \ , \\
[\mathcal{D}_a, \bar{\mathcal{D}}_\beta] &= -\mathrm{i}\varepsilon_{abc}(\gamma^b)_\beta{}^\gamma\mathcal{C}^c\bar{\mathcal{D}}_\gamma - \mathrm{i}(\gamma_a)_\beta{}^\gamma R\mathcal{D}_\gamma \ , \\
[\mathcal{D}_a, \mathcal{D}_b] &= 4\varepsilon_{abc}\Big(\delta^c{}_d\bar{R}R + \mathcal{C}^c\mathcal{C}_d\Big)\mathcal{M}^d \ .
\end{aligned}$$

4 different superalgebras: (a) $R \neq 0, \ C_a = 0$ (1,1) AdS superspace;
(b) $R = 0, \ C_a$ is time-like; (c) $R = 0, \ C_a$ is space-like;
(d) $R = 0, \ C_a$ is null.

Example: Maximally supersymmetric backgrounds in Type II SUGRA

Dimension-1 torsion superfields:

$$R = 0 \ , \qquad \mathcal{D}_A \mathcal{S} = 0 \ , \qquad \mathcal{D}_\alpha \mathcal{C}_b = 0 \ , \qquad \mathcal{D}_a \mathcal{C}_b = 2\varepsilon_{abc}\mathcal{C}^c \mathcal{S}$$

$$\mathcal{C}^b \mathcal{C}_b = \text{const} \ .$$

The algebra of covariant derivatives is

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0 \ , \qquad \{\bar{\mathcal{D}}_\alpha, \bar{\mathcal{D}}_\beta\} = 0$$
$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_\beta\} = -2\mathrm{i}\mathcal{D}_{\alpha\beta} - 2\mathcal{C}_{\alpha\beta}\mathcal{J} - 4\mathrm{i}\varepsilon_{\alpha\beta}\mathcal{S}\mathcal{J} + 4\mathrm{i}\mathcal{S}\mathcal{M}_{\alpha\beta} - 2\varepsilon_{\alpha\beta}\mathcal{C}^{\gamma\delta}\mathcal{M}_{\gamma\delta} \ .$$

Solution with $\mathcal{C}_a = 0$ corresponds to (2,0) AdS superspace.

Spacetime is of type N (for c_a null), type $\mathrm{D_s}$ (for c_a spacelike) or $\mathrm{D_t}$ (for c_a timelike) in the Petrov-Segre classification.

D. Chow, C. Pope & E. Sezgin (2010)

$d = 5$, $\mathcal{N} = 1$ supergravity

5D $\mathcal{N} = 1$ curved superspace, $\mathcal{M}^{5|8}$, parametrized by local coordinates $z^{\hat{M}} = (x^{\hat{m}}, \theta^{\hat{\mu}}_i)$, $\hat{m} = 0, 1, 2, 3, 4$ and $\hat{\mu} = 1, 2$

Superspace structure group $\mathrm{SO}(4,1) \times \mathrm{SU}(2)_R$

Superspace covariant derivatives

$$\mathcal{D}_{\hat{A}} = (\mathcal{D}_{\hat{a}}, \mathcal{D}^i_{\hat{\alpha}}) = E_{\hat{A}} + \Omega_{\hat{A}} + \Phi_{\hat{A}}$$

Algebra of covariant derivatives

$$\begin{aligned}\{\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}^j_{\hat{\beta}}\} = & -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat{\alpha}\hat{\beta}} - \mathrm{i}\,\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}X^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} + \frac{\mathrm{i}}{4}\varepsilon^{ij}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Gamma_{\hat{a}})_{\hat{\alpha}\hat{\beta}}N_{\hat{b}\hat{c}}M_{\hat{d}\hat{e}} \\ & -\frac{\mathrm{i}}{2}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}}C_{\hat{c}}{}^{ij}M_{\hat{d}\hat{e}} + 4\mathrm{i}\,S^{ij}M_{\hat{\alpha}\hat{\beta}} + 3\mathrm{i}\,\varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon^{ij}S^{kl}J_{kl} \\ & -\mathrm{i}\,\varepsilon^{ij}C_{\hat{\alpha}\hat{\beta}}{}^{kl}J_{kl} - 4\mathrm{i}\Big(X_{\hat{\alpha}\hat{\beta}} + N_{\hat{\alpha}\hat{\beta}}\Big)J^{ij}\ ,\end{aligned}$$

All torsion and curvature tensors are given in terms of four dimension-1 superfields:

$$S^{ij} = S^{ji}\ , \quad X_{\hat{a}\hat{b}} = -X_{\hat{b}\hat{a}}\ , \quad N_{\hat{a}\hat{b}} = -N_{\hat{b}\hat{a}}\ , \quad C_{\hat{a}}{}^{ij} = C_{\hat{a}}{}^{ji}\ .$$

5D (conformal) isometries

The conformal Killing supervector fields obey the equation

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathcal{D}_{\hat{A}} = 0 ,$$

where

$$\delta_{\mathcal{K}}\mathcal{D}_{\hat{A}} = [\mathcal{K}, \mathcal{D}_{\hat{A}}] , \qquad \mathcal{K} = \xi^{\hat{C}}\mathcal{D}_{\hat{C}} + \frac{1}{2}K^{\hat{c}\hat{d}}M_{\hat{c}\hat{d}} + K^{kl}J_{kl} ,$$

and

$$\delta_\sigma \mathcal{D}^i_{\hat{\alpha}} = \frac{1}{2}\sigma\mathcal{D}^i_{\hat{\alpha}} + 2(\mathcal{D}^{\hat{\gamma}i}\sigma)M_{\hat{\gamma}\hat{\alpha}} - 3(\mathcal{D}_{\hat{\alpha}k}\sigma)J^{ki} , \qquad \dots$$

In a super-Weyl gauge $\phi = 1$, the Killing supervector fields obey the equation

$$\delta_{\mathcal{K}[\xi]}\mathcal{D}_{\hat{A}} = 0 .$$

Classification of maximally supersymmetric backgrounds I

All backgrounds with eight supercharges are characterized by

$$\mathcal{D}^i_{\hat\alpha} S^{kl} = 0 \ , \qquad \mathcal{D}^i_{\hat\alpha} C_{\hat a}{}^{kl} = 0 \ , \qquad \mathcal{D}^i_{\hat\alpha} X_{\hat a\hat b} = 0 \ , \qquad \mathcal{D}^i_{\hat\alpha} N_{\hat a\hat b} = 0 \ .$$

Case 1 $\qquad S = \sqrt{\frac{1}{2} S^{ij} S_{ij}} \neq 0$

Implication:

$$C_{\hat a}{}^{ij} = 0 \ , \qquad X_{\hat a\hat b} = 0 \ , \qquad N_{\hat a\hat b} = 0 \ .$$

5D AdS superspace $\mathrm{AdS}^{5|8}$

Algebra of covariant derivatives

$$\{\mathcal{D}^i_{\hat\alpha}, \mathcal{D}^j_{\hat\beta}\} = -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta} + 4\mathrm{i}\,S^{ij} M_{\hat\alpha\hat\beta} + 3\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij} S^{kl} J_{kl} \ ,$$

$$[\mathcal{D}_{\hat a}, \mathcal{D}^j_{\hat\beta}] = \frac{1}{2}(\Gamma_{\hat a})_{\hat\beta}{}^{\hat\gamma} S^j{}_k \mathcal{D}^k_{\hat\gamma}$$

Classification of maximally supersymmetric backgrounds II

Case 2 $C_{\hat{a}}{}^{kl} \neq 0$

Implication:

$$S^{ij} = 0 \ , \qquad X_{\hat{a}\hat{b}} = 0 \ , \qquad N_{\hat{a}\hat{b}} = 0 \ .$$

Superspace geometry is described by a single covariantly constant tensor $C_{\hat{a}}{}^{ij}$, $\mathcal{D}_{\hat{B}} C_{\hat{a}}{}^{ij} = 0$.

$$\begin{aligned}
\{\mathcal{D}^i_{\hat{\alpha}}, \mathcal{D}^j_{\hat{\beta}}\} &= -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat{\alpha}\hat{\beta}} - \frac{\mathrm{i}}{2}\varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}(\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}} C_{\hat{c}}{}^{ij} M_{\hat{d}\hat{e}} - \mathrm{i}\,\varepsilon^{ij} C_{\hat{\alpha}\hat{\beta}}{}^{kl} J_{kl} \ , \\
[\mathcal{D}_{\hat{a}}, \mathcal{D}^j_{\hat{\beta}}] &= \frac{1}{2}(\Sigma_{\hat{a}}{}^{\hat{b}})_{\hat{\beta}}{}^{\hat{\gamma}} C_{\hat{b}}{}^j{}_k \mathcal{D}^k_{\hat{\gamma}} \ , \\
[\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{b}}] &= \frac{1}{4}\Big(\delta^{[\hat{c}}_{[\hat{a}} C_{\hat{b}]kl} C^{\hat{d}]kl} - \frac{1}{2}\delta^{\hat{c}}_{[\hat{a}}\delta^{\hat{d}}_{\hat{b}]} C^{\hat{e}kl} C_{\hat{e}kl}\Big) M_{\hat{c}\hat{d}} \ .
\end{aligned}$$

Integrability condition $C_{\hat{a}}{}^{(i}{}_k C_{\hat{b}}{}^{j)k} = 0$.

Solution:

$$C_{\hat{b}}{}^{ij} = C_{\hat{b}} C^{ij} \ , \qquad C^{ij} C_{ij} = 2 \ , \qquad \mathcal{D}_{\hat{A}} C_{\hat{b}} = 0 \ , \qquad \mathcal{D}_{\hat{A}} C^{ij} = 0 \ .$$

Three different superalgebras

Classification of maximally supersymmetric backgrounds III

Case 3 $\mathcal{S}^{ij} = 0 \;\; \& \;\; C_{\hat{a}}{}^{kl} = 0$

The superspace geometry is determined by the tensors $X_{\hat{a}\hat{b}}$ and $N_{\hat{a}\hat{b}}$ obeying the differential constraints

$$\mathcal{D}^i_{\hat{\alpha}} X_{\hat{a}\hat{b}} = 0 \,, \qquad \mathcal{D}^i_{\hat{\alpha}} N_{\hat{a}\hat{b}} = 0 \,,$$
$$\mathcal{D}_{\hat{a}} X_{\hat{m}\hat{n}} = \frac{1}{2} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}[\hat{m}} X_{\hat{n}]}{}^{\hat{b}} N^{\hat{c}\hat{d}} \,,$$
$$\mathcal{D}_{\hat{a}} N_{\hat{m}\hat{n}} = \frac{1}{2} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}[\hat{m}} N_{\hat{n}]}{}^{\hat{b}} N^{\hat{c}\hat{d}} = -\frac{1}{8} \eta_{\hat{a}[\hat{m}} \varepsilon_{\hat{n}]\hat{b}\hat{c}\hat{d}\hat{e}} N^{\hat{b}\hat{c}} N^{\hat{d}\hat{e}}$$

and the algebraic ones

$$X_{[\hat{a}}{}^{\hat{c}} N_{\hat{b}]\hat{c}} = 0 \,, \qquad X_{[\hat{a}\hat{b}} X_{\hat{c}]\hat{d}} = N_{[\hat{a}\hat{b}} N_{\hat{c}]\hat{d}} \,.$$

Compensators

So far no compensators for 5D $\mathcal{N}=1$ supergravity have been specified. To describe (gauged) supergravity, two compensators are required.

- Vector multiplet compensator $W=\bar{W}\neq 0$

$$\mathcal{D}^{(i}_{\hat\alpha}\mathcal{D}^{j)}_{\hat\beta}W-\frac{1}{4}\varepsilon_{\hat\alpha\hat\beta}\mathcal{D}^{\hat\gamma(i}\mathcal{D}^{j)}_{\hat\gamma}W=\frac{\rm i}{2}C_{\hat\alpha\hat\beta}{}^{ij}W~.$$

The super-Weyl transformation law of W

$$\delta_\sigma W=\sigma W~.$$

- $\mathcal{O}(2)$ multiplet compensator H^{ij} $\qquad H^2:=\frac{1}{2}H^{ij}H_{ij}>0$.

$$H^{ij}=\varepsilon^{ik}\varepsilon^{jl}~,\qquad \mathcal{D}^{(i}_{\hat\alpha}H^{jk)}=0$$

The super-Weyl transformation law of H^{ij}

$$\delta_\sigma H^{ij}=3\sigma H^{ij}~.$$

Compensators

- Super-Weyl invariance may be fixed by setting $W = \text{const}$ or $H = \text{const}$.
- Bosonic conditions

$$\mathcal{D}^i_{\hat\alpha} W| = 0 \ , \qquad \mathcal{D}^i_{\hat\alpha} H^{jk}| = 0$$

are supersymmetric for backgrounds with eight supercharges iff

$$W = \text{const} \ , \qquad \mathcal{D}^i_{\hat\alpha} H^{jk} = 0$$

- Consistency conditions

$$C_{\hat a}{}^{kl} = 0 \ , \qquad S^{ij} = S\, H^{ij} \ , \qquad N_{\hat a\hat b} = -X_{\hat a\hat b}$$

It also holds that $X_{\hat a\hat b} = F_{\hat a\hat b}$, the field strength of the vector multiplet.

Maximally supersymmetric off-shell supergravity backgrounds

- Case $S \neq 0$

$$C_{\hat{a}}{}^{kl} = 0 \,, \qquad , \qquad N_{\hat{a}\hat{b}} = X_{\hat{a}\hat{b}} = 0$$

5D AdS Superspace

- Case $S = 0$

$$C_{\hat{a}}{}^{kl} = 0 \,, \qquad S^{ij} = 0 \,, \qquad N_{\hat{a}\hat{b}} = -X_{\hat{a}\hat{b}}$$

Superspace geometry is formulated in terms of a single superfield obeying the equation

$$\mathcal{D}_{\hat{a}} X_{\hat{m}\hat{n}} = \frac{1}{8} \eta_{\hat{a}[\hat{m}} \varepsilon_{\hat{n}]\hat{b}\hat{c}\hat{d}\hat{e}} X^{\hat{b}\hat{c}} X^{\hat{d}\hat{e}}$$

Supersymmetric solutions in Poincaré and anti-de Sitter supergravities

Supergravity equations of motion

$$H - W^3 = 0 \,, \qquad \mathbb{H}^{ij} + \chi H^{ij} = 0 \,, \qquad \mathbb{W} + 3\chi W = 0 \,,$$

with χ the cosmological constant.
$\mathbb{H}^{ij}$ composite $\mathcal{O}(2)$ multiplet

$$\mathbb{H}^{ij} = \mathrm{i}\,\mathcal{D}^{\hat{\alpha}(i} W \mathcal{D}^{j)}_{\hat{\alpha}} W + \frac{\mathrm{i}}{2} W \mathcal{D}^{ij} W - 2S^{ij} W^2$$

$\mathbb{W}$ composite vector multiplet

$$\mathbb{W} = \frac{\mathrm{i}}{4} H \big(\mathcal{D}^{ij} + 12\mathrm{i} S^{ij}\big)\Big(\frac{H_{ij}}{H^2}\Big)$$

Using the above equations of motion, one naturally reproduces all supersymmetric solutions of minimal (gauged) supergravity in 5D

J. Gauntlett, J. Gutowski, C. Hull, S. Pakis & H. Reall (2003)

J. Gauntlett & J. Gutowski (2003)

Most general nonlinear sigma model in 5D AdS superspace

5D AdS superspace $\mathrm{AdS}^{5|8}$

$$\{\mathcal{D}^i_{\hat\alpha}, \mathcal{D}^j_{\hat\beta}\} = -2\mathrm{i}\,\varepsilon^{ij}\mathcal{D}_{\hat\alpha\hat\beta} + 4\mathrm{i}\,S^{ij}M_{\hat\alpha\hat\beta} + 3\mathrm{i}\,\varepsilon_{\hat\alpha\hat\beta}\varepsilon^{ij}S^{kl}J_{kl} ,$$

Matter couplings are most naturally described in $\mathrm{AdS}^{5|8} \times \mathbb{C}P^1$,
with $\mathbb{C}P^1$ parametrized by homogeneous coordinates v_i.
Off-shell sigma model

$$L^{(2)} = S^{ij} v_i v_j K(\Upsilon, \breve{\Upsilon})$$

SMK & G. Tartaglino-Mazzucchelli (2007)

$K(\Phi, \bar\Phi)$ Kähler potential of a real analytic Kähler manifold $\mathcal{M}^n$.
$\Upsilon(v)$ covariant arctic hypermultiplet.

σ-model target space:

(Open domain of the zero section of) the cotangent bundle of $\mathcal{M}^n$.
This target space is hyperkähler, for any real analytic Kähler space $\mathcal{M}^n$.

For the most general supersymmetric nonlinear sigma model in 4D $\mathcal{N} = 2$ AdS and 5D $\mathcal{N} = 1$ AdS, its target space is a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations on the two-sphere of complex structures.

D. Butter & SMK, arXiv:1105.3111

J. Bagger & C. Xiong, arXiv:1105.4852