Field theories invariant under one-parameter fermionic symmetry

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Corfu, Greece, Workshop on Quantum Fields and Strings,
14 - 21 September 2014

Based on

S.R. Esipova, P.M. Lavrov, O.V.R., IJMPA (2014)

Contents

Motivation

Field models with quantum action invariant under one-parameter fermion transformations which appear in several ways within modern quantum field theory.

- Faddeev -Popov action for Yang -Mills fields (L.D. Faddeev, V.N. Popov, Phys. Lett. B (1967))

$$S_{\mathrm{FP}}(\phi) = S_{YM} + S_{gf} + S_{gh} \tag{1}$$

where

$$\phi^A = (A^{\mu a}, B^a, C^a, \bar{C}^a)$$

is the set of dynamical fields in the Lagrangian formalism of the Yang-Mills theory,

$$\begin{aligned} S_{YM} &= -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, \end{aligned}$$

Motivation

$$S_{gf} = B^a \partial_\mu A^{\mu a},$$
$$S_{gh} = \bar{C}^a \partial_\mu D^{\mu ab} C^b,$$
$$D_\mu^{ab} = \delta^{ab}\partial_\mu + f^{acb} A_\mu^c.$$

This action is invariant under the remarkable nilpotent transformations known as BRST - transformations (C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. (1975); I. V. Tyutin, Preprint of Lebedev Physics Institute, No. 39 (1975))

$$\delta_{\rm B} S_{\rm FP}(\phi) = 0, \qquad (2)$$

$$\delta_{\rm B} A_\mu^a = D_\mu^{ab}(A) C^b \lambda, \qquad \delta_{\rm B} C^a = \frac{1}{2} f^{abc} C^b C^c \lambda, \qquad (3)$$
$$\delta_{\rm B} \bar{C}^a = B^a \lambda, \qquad \delta_{\rm B} B^a = 0, \qquad (4)$$

where λ is a constant Grassmann parameter ($\varepsilon(\lambda) = 1$), $\lambda^2 = 0$.

- The Curci-Ferrari model (C. Curci and R. Ferrari, Phys. Lett. B (1976)) is described by the action

$$S = S_{YM} + S_{gf} + S_m, \tag{5}$$

where S_{YM} is the Yang-Mills action,

$$S_{gf} = B^a \partial^\mu A^a_\mu + \bar{C}^a \partial^\mu D^{ab}_\mu C^b + \frac{\beta}{2} B^a B^a + \frac{\beta}{4} N^a N^a - \frac{\beta}{2} B^a N^a, \tag{6}$$

$$N^a = N^a(C, \bar{C}) = f^{abc} \bar{C}^b C^a, \tag{7}$$

β is a parameter of the model.
The action S_m

$$S_m = \frac{1}{2} m^2 A^a_\mu A^{a\mu} + \beta m^2 \bar{C}^a C^a \tag{8}$$

contains a mass m for the vector fields A^a_μ and ghost C^a and antighost $\bar{C}^a$, B^a are auxiliary fields introducing the gauge.

The action of the Curci-Ferrari model is not invariant under BRST transformation

$$\delta_B S \neq 0, \tag{9}$$

though it is invariant $\delta_{mB} S = 0$ under the modified BRST transformation

$$\delta_{mB} A^a_\mu = D^{ab}_\mu C^b \theta, \qquad \delta_{mB} C^a = \frac{1}{2} f^{abc} C^b C^c \theta,$$
$$\delta_{mB} \bar{C}^a = B^a \theta, \qquad \delta_{mB} B^a = m^2 C^a \theta$$

The modified BRST (or modified anti-BRST) transformation is not nilpotent $\delta^2_{mB} S \neq 0$

- The action appearing in generalized Hamiltonian formalism proposed by Batalin, Fradkin, Vilkovisky (E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B (1975); I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B (1977))

$$W_\psi = \int \Big[\Big(\frac{1}{2}\Big) z^i(t)\omega_{ik}\dot{z}^k(t) - H(t) \Big] \tag{10}$$

where $z^i = (q;p)$ is a complete set of canonical variables specific to the extended phase space of generalized Hamiltonian formalism,

$$\dot{z}^k(t) = \frac{dz^k(t)}{dt}, \tag{11}$$

ω_{ik} is inverse metric to Poisson structure

$$\{z^i, z^k\} = \omega^{ik} = const. \tag{12}$$

$H(t) = \mathcal{H} + \{\Omega, \psi\}$
$\mathcal{H} = H_0 + ...$

$$\{\Omega, \Omega\} = 0, \qquad \{\Omega, \mathcal{H}\} = 0, \tag{13}$$

$\Omega = cT + ...$ is BRST charge,
T is a set of first-class constraints.
The action is invariant under the transformation

$$\bar{z}^k(t) = \bar{z}^k|_{z \to z(t)} = z^k(t) + \{z^k, \Omega\}_t \mu \tag{14}$$

where μ is a constant Grassmann parameter, $\mu^2 = 0$.

Motivation

- Superextension of sigma models (S. Catterall, S. Chadab, JHEP (2004)) leads to actions again invariant under fermion transformations.
- Reformulation of Yang-Mills theories (A. Quadri, A.A. Slavnov, JHEP (2010)) in a form being free of the Gribov problem.
- A new realization of supersymmetry (A. Jourjine, Phys. Lett. (2013)), called scalar supersymmetry.

Our starting point is a theory of variables $\phi = \{\phi^i\}$ with Grassmann parities $\varepsilon(\phi^i) = \varepsilon_i$. We assume a non-degenerate action $S(\phi)$ of the theory so that the generating functional of Green functions is given by the standard functional integral

$$Z(J) = \int \mathcal{D}\phi \, \exp\left\{\frac{i}{\hbar}\big[S(\phi) + J\phi\big]\right\}.$$

Here $J\phi = J_i\phi^i$, $\varepsilon(J_i) = \varepsilon_i$.
We suppose invariance of $S(\phi)$ under fermion transformations

$$\phi^i \quad \mapsto \quad \phi^i = \varphi^i(\phi') \,, \quad \varphi^i(\phi) = \phi^i + R^i(\phi)\xi \,, \quad \xi^2 = 0 \,,$$

so that

$$S_{,i}(\phi)R^i(\phi) = 0 \,.$$

Here ξ is an odd Grassmann parameter and $R^i(\phi)$ are generators of fermion transformations with $\varepsilon(R^i) = \varepsilon_i + 1$.

One-parameter fermion symmetry

Consider some consequences of the invariance on quantum level. Making the change of variables in the functional integral we have

$$Z(J) = \int \mathcal{D}\phi \ \mathsf{sDet}\, M(\phi) \ \exp\left\{ \frac{i}{\hbar}\big[S(\varphi(\phi)) + J\varphi(\phi)\big]\right\}$$

where $\mathsf{sDet}\, M$ means the superdeterminant of supermatrix M with matrix elements

$$M^i_{\ j}(\phi) = \delta^i_{\ j} + (-1)^{\varepsilon_j} \frac{\delta R^i(\phi)}{\delta \phi^j}\xi\,, \quad \varepsilon(M^i_{\ j}) = \varepsilon_i + \varepsilon_j\ .$$

In general, for the theory under consideration this superdeterminant is not equal to unity

$$\begin{aligned} \mathsf{sDet}\, M(\phi) &= \exp\left\{ \mathsf{sTr}\ \mathsf{ln}\, M(\phi)\right\} = \exp\left\{ \frac{\delta R^i(\phi)}{\delta\phi^i}\xi\right\} = \\ &= 1 + \frac{\delta R^i(\phi)}{\delta\phi^i}\xi = 1 + R^i_{,i}(\phi)\xi\ . \end{aligned}$$

It leads to the following presentation of functional $Z(J)$

$$Z(J) = \int \mathcal{D}\phi \left(1 + R^i_{,i}(\phi)\xi + \frac{i}{\hbar} J_i R^i(\phi)\xi\right) \exp\left\{\frac{i}{\hbar}\big[S(\phi) + J\phi\big]\right\}$$

from which the identity follows

$$\int \mathcal{D}\phi \left(R^i_{,i}(\phi) + \frac{i}{\hbar} J_i R^i(\phi)\right) \exp\left\{\frac{i}{\hbar}\big[S(\phi) + J\phi\big]\right\} = 0 \, .$$

With the help of usual manipulations this identity can be written in closed form with respect to $Z(J)$

$$\left[J_i R^i\Big(\frac{\hbar}{i}\frac{\delta}{\delta J}\Big) - i\hbar R^i_{,i}\Big(\frac{\hbar}{i}\frac{\delta}{\delta J}\Big)\right] Z(J) = 0 \, .$$

This identity is nothing but the Ward identity for generating functional of Green functions. The existence of this identity is direct consequence of invariance under fermion transformation of $S(\phi)$.

One-parameter fermion symmetry

To simplify presentation of the Ward identity we define the extended generating functional of Green functions by introducing additional sources K_i with Grassmann parities opposite to fields ϕ^i, $\varepsilon(K_i) = \varepsilon_i + 1$

$$Z(J, K) = \int \mathcal{D}\phi \ \exp\left\{\frac{i}{\hbar}\left[S(\phi, K) + J\phi\right]\right\}$$

where

$$S(\phi, K) = S(\phi) + K_i R^i(\phi) \ .$$

In general, the action $S(\phi, K)$ is not invariant under fermion transformation

$$\hat{s}S(\phi, K) = K_i \ \hat{s}R^i(\phi) \neq 0$$

where the operator $\hat{s}$ of fermion transformation was used. Action of this operator on arbitrary functional X is given by

$$\hat{s}X = \frac{\delta X}{\delta\phi^i}R^i = X_{,i}R^i \ .$$

There is an evident relation

$$Z(J,K)\big|_{K=0} = Z(J)\ .$$

In terms of $Z(J,K)$ the Ward identity reads

$$J_i \frac{\delta Z(J,K)}{\delta K_i} = i\hbar R^i_{,i}\Big(\frac{\hbar}{i}\frac{\delta}{\delta J}\Big) Z(J,K)\ .$$

Note that the left side of the Ward identity has the local form while the right side is nonlocal.

Field-dependent nilpotent fermion transformations

Let us study more general type of fermion transformations when the parameter ξ is replaced by a field-dependent functional $\xi(\phi)$

$$\varphi^{i}(\phi) = \phi^{i} + R^{i}(\phi)\xi(\phi) \, , \quad \xi^{2}(\phi) = 0 \, .$$

We will referee to these transformations as field-dependent fermion transformations. Note that the action $S = S(\phi)$ remains invariant under these transformations due to nilpotency of $\xi(\phi)$

$$S(\phi) = S(\varphi(\phi^{'})) = S(\phi^{'}) \, .$$

Using the technique described in (P. Lavrov, O. Lechtenfeld, Phys.Lett. B (2013)) it is not difficult to find the explicit form of the superdeterminant of supermatrix

$$M^{i}_{\ j}(\phi) = \delta^{i}_{\ j} + R^{i}(\phi)\xi_{,j}(\phi) + (-1)^{\varepsilon_i} R^{i}_{,j}(\phi)\xi(\phi)\ ,$$

with the result

$$\mathrm{sDet}\, M(\phi) = \big(1 + \hat{s}\xi(\phi)\big)^{-1}\Big[1 + R^{i}_{,i}(\phi)\xi(\phi) - \frac{\big(\hat{s}^2\xi(\phi)\big)\xi(\phi)}{1 + \hat{s}\xi(\phi)}\Big]\ .$$

Here we took into account that the action of the square operator $\hat{s}$ on an arbitrary functional X is given by

$$\hat{s}^2 X = \frac{\delta X}{\delta \phi^i} \frac{\delta R^i}{\delta \phi^j} R^j = X_{,i} R^i_{,j} R^j \ .$$

In what follows we restrict ourselves to the case when the operator $\hat{s}$ is nilpotent, $\hat{s}^2 = 0$

$$\hat{s}^2 = 0 \qquad \rightarrow \qquad \frac{\delta R^i}{\delta \phi^j} R^j = 0 \ .$$

In particular, it means

$$\hat{s} R^i = 0$$

and we find that the action $S(\phi, K)$ is invariant under field-dependent nilpotent fermion transformations

$$S(\phi, K)_{,i} \ R^i(\phi) = 0 \ .$$

Field-dependent nilpotent fermion transformations

The invariance of $S(\phi, K)$ can be expressed in an unique form

$$\frac{\delta S(\phi, K)}{\delta \phi^i} \frac{\delta S(\phi, K)}{\delta K_i} = 0 \ .$$

This equation is nothing but the Zinn-Justin equation appearing for the first time in quantization of non-abelian gauge fields (Zinn-Justin, Lecture Notes in Physics (1975)).

Field-dependent nilpotent fermion transformations

Performing the change of variables in form of field-dependent nilpotent fermion transformations and using the explicit form of the Jacobian we have

$$\begin{aligned}\mathsf{sDet}\, M(\phi) &= \exp\left\{R^{i}_{,i}(\phi)\xi(\phi) - \ln\left(1+\hat{s}\xi(\phi)\right)\right\} = \\ &= \left(1+\hat{s}\xi(\phi)\right)^{-1}\left[1+R^{i}_{,i}(\phi)\xi(\phi)\right]\end{aligned}$$

and arrive at the following presentation of the generating functional $Z(J,K)$

$$\begin{aligned}Z(J,K) &= \int \mathcal{D}\phi\, \exp\left\{\frac{i}{\hbar}\left[S(\phi,K) + J\left(\phi + R(\phi)\xi(\phi)\right) - \right.\right. \\ &\left.\left. -i\hbar R^{i}_{,i}(\phi)\xi(\phi) + i\hbar \ln\left(1+\hat{s}\xi(\phi)\right)\right]\right\}.\end{aligned}$$

Field-dependent nilpotent fermion transformations

Due to the equivalence theorem (R.E. Kallosh, I.V. Tyutin, Sov. J. Nucl. Phys. (1973)) we can work with the following generating functional

$$Z_\xi(J,K) = \int \mathcal{D}\phi \ \exp\left\{ \frac{i}{\hbar}\big[S_\xi(\phi,K) + J\phi\big]\right\}$$

where

$$S_\xi(\phi,K) = S(\phi,K) + i\hbar \ln\left(1 + \hat{s}\xi(\phi)\right) - i\hbar R^i_{,i}(\phi)\xi(\phi) \ .$$

In its turn, in general, the action $S_\xi(\phi,K)$ is not invariant under fermion transformations

$$\hat{s}S_\xi(\phi,K) = -i\hbar \ \hat{s}\big(R^i_{,i}(\phi)\xi(\phi)\big) \neq 0 \ .$$

We consider this as an indication of the inconsistency in formulation of the model being invariant under fermion transformations. Indeed, it seems strange that a theory with the action invariant under fermion transformations is equivalently presented in the form when this symmetry looks like broken. This inconsistency can be avoided if the additional requirement is fulfilled, $R^i_{,i}(\phi) = 0$.

Special fermion transformations

We will refer *special type of theories* when generators $R^i(\phi)$ of fermion symmetry transformations are subjected to the restriction

$$R^i_{,i}(\phi) = 0 \; .$$

In the case of such theories the superdeterminant of field-dependent fermion transformations reads

$$\mathrm{sDet}\, M = (1 + \hat{s}\xi)^{-1}$$

and for the action S_ξ one has

$$S_\xi(\phi, K) = S(\phi, K) + i\hbar \ln\left(1 + \hat{s}\xi(\phi)\right) \; .$$

Special fermion transformations

There is the presentation

$$S_\xi(\phi, K) = S(\phi, K) + \hat{s}F(\phi) = S(\phi) + \hat{s}\big(K\phi + F(\phi)\big) \ ,$$

where

$$F = \xi\Big[1 - \frac{1}{2}(\hat{s}\xi) + \frac{1}{3}(\hat{s}\xi)^2 - \cdots\Big] = \xi\ (\hat{s}\xi)^{-1}\ \ln(1 + \hat{s}\xi).$$

This presentation can be very useful in theories with the action invariant under fermion transformation. In particular, it was shown (P. Lavrov, O. Lechtenfeld, Phys.Lett. (2013)) that for Yang-Mills theories the result of change of variables can be presented in the similar form and interpreted as a modification of the gauge condition. This made it possible to prove the independence of the effective action in Yang-Mills theories on the finite increment of gauge on-shell and suggest the formulation of the Gribov-Zwanziger theory (V.N. Gribov, Nucl. Phys. (1978); D. Zwanziger, Nucl.Phys. (1989)) free from the problem of gauge dependence (P. Lavrov, O. Lechtenfeld, A. Reshetnyak, JHEP, (2011); P. Lavrov, O. Lechtenfeld, Phys.Lett. (2013)).

Special fermion transformations

The action $S_\xi(\phi, K)$ is invariant under fermion transformations

$$\hat{s} S_\xi(\phi, K) = 0 \ .$$

This invariance can be expressed in the form of Zinn-Justin equation

$$\frac{\delta S_\xi}{\delta \phi^i} \frac{\delta S_\xi}{\delta K^i} = 0 \ .$$

As a consequence the generating functional $Z_\xi(J, K)$ satisfies the Ward identity

$$J_i \frac{\delta Z_\xi(J, K)}{\delta K_i} = 0$$

like $Z(J, K)$. One can rewrite the Ward identity in term of the generating functional of connected Green functions $W_\xi(J, K) = (\hbar/i) \ln Z_\xi(J, K)$ as

$$J_i \frac{\delta W_\xi(J, K)}{\delta K_i} = 0 \ .$$

Using the Legendre transformation

$$\phi^i = \frac{\delta W_\xi(J,K)}{\delta J_i}$$

and introducing the generating functional of vertex functions $\Gamma_\xi(\phi, K)$

$$\Gamma_\xi(\phi,K) = W_\xi(J,K) - J_i\phi^i \, , \qquad \frac{\delta \Gamma_\xi}{\delta K_i} = \frac{\delta W_\xi}{\delta K_i} \, , \qquad \frac{\delta \Gamma_\xi}{\delta \phi^i} = -J_i \, ,$$

the Ward identity for $\Gamma_\xi = \Gamma_\xi(\phi, K)$

$$\frac{\delta \Gamma_\xi}{\delta \phi^i}\frac{\delta \Gamma_\xi}{\delta K_i} = 0$$

has the form of the Zinn-Justin equation and repeats on quantum level the invariance of the given theory under fermion transformations. It is clear that all relations are valid for the initial theory ($\xi = 0$).

Conclusions

- From general point of view we have studied quantum properties of field theories for which an action appearing in the generating functional of Green functions is invariant under (one-parameter) fermion transformations.
- The fermion transformations can be of three types:
  1) general;
  2) nilpotent;
  3) special.

Thank You!