Mini-Superspace Quantum Supergravity and its Hidden Hyperbolic Kac-Moody Structures

Ph. Spindel
Université de Mons

work with Th. Damour (IHES)
[arXiv : 1304.6381, Class. Quantum Grav. 30 (2013) 162001
arXiv : 1406.1309, *Quantum Supersymmetric Bianchi IX Cosmology*]

Corfu Summer Institute
Workshop on Quantum Fields and Strings September 14 - 21, 2014

PLAN

1. SOME MOTIVATIONS
2. MINISUPERSPACE MODEL
   - The model
   - Bosonic Degrees of Freedom
   - Fermionic Degrees of Freedom
   - Geometry of the β-space : hints for an hidden Kac–Moody symmetry
3. HIDDEN SYMMETRIES– COSET MODEL
   - Kac–Moody algebras (*just some definitions*)
   - Gravity/Coset Conjecture
4. QUANTUM $\mathcal{N} = 1,\ d = 4$ BIANCHI IX COSMOLOGY
   - Constraint algebra
   - Hilbert Space Structure
   - Explicit Equations and Solutions (*sorry for the technicalities*)
5. CONCLUSIONS
   - About hidden symmetries
   - Some original features

Quantum Cosmology !?!
Some motivations

- The birth of our Universe seems to be of quantum nature

Quantum Cosmology !?!
Some motivations

- The birth of our Universe seems to be of quantum nature
- Near a generic cosmological singularity space–like gradients become negligible compared to the time-like ones and the Universe evolves in a chaotic way, like a Bianchi IX (or VIII) cosmological model

  V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology,' Adv. Phys. **19**, 525 (1970).
  C. W. Misner, Mixmaster Universe," Phys. Rev. Lett. **22**, 1071 (1969).
  C. W. Misner, Quantum cosmology. 1., Phys. Rev. **186**, 1319 (1969).

QUANTUM COSMOLOGY !?!
SOME MOTIVATIONS

- Susy Bianchi IX model involves only a finite number of degrees of freedom;

QUANTUM COSMOLOGY !?!
SOME MOTIVATIONS

- Susy Bianchi IX model involves only a finite number of degrees of freedom; a lot of works have been devoted on the subject, for instance the articles :
  P. D. D'Eath, Quantization of the Bianchi IX model in supergravity, Phys. Rev. D **48**, 713 (1993)
  P. D. D'Eath, S. W. Hawking and O. Obregon, Supersymmetric Bianchi models and the square root of the Wheeler-DeWitt equation, Phys. Lett. B **300**, 44 (1993).
  A. Csordas and R. Graham, Supersymmetric minisuperspace with nonvanishing fermion number, Phys. Rev. Lett. **74**, 4129 (1995) [gr-qc/9502004].
  A. Csordas and R. Graham, Hartle-Hawking state in supersymmetric minisuperspace, Phys. Lett. B **373**, 51 (1996) [gr-qc/9506074].
  R. Graham and A. Csordas, Quantum states on supersymmetric minisuperspace with a cosmological constant, Phys. Rev. D **52**, 5653 (1995) [grqc/ 9506002].
  A. D. Y. Cheng, P. D. D'Eath and P. R. L. V. Moniz, Quantization of the Bianchi type IX model in supergravity with a cosmological constant, Phys. Rev. D **49**, 5246 (1994) [gr-qc/9404008]
  A. D. Y. Cheng and P. D. D'Eath, Diagonal quantum Bianchi type IX models in N=1 supergravity, Class. Quant. Grav. **13**, 3151 (1996) [grqc/ 9610054].
  O. Obregon and C. Ramirez, Dirac like formulation of quantum supersymmetric cosmology, Phys. Rev. D **57**, 1015 (1998).

QUANTUM COSMOLOGY !?! SOME MOTIVATIONS

or the books :

P. D. D'Eath, Supersymmetric quantum cosmology, Cambridge, UK: Univ. Pr. (1996) 252 p

P. Vargas Moniz, Quantum cosmology - the supersymmetric perspective : Vol. 1: Fundamentals," Lect. Notes Phys. 803, 1 (2010); : Vol. 2: Advanced Topic," Lect. Notes Phys. 804, 1 (2010).

Quantum Cosmology !?!
Some motivations

or the books :

P. D. D'Eath, Supersymmetric quantum cosmology, Cambridge, UK: Univ. Pr. (1996) 252 p
P. Vargas Moniz, Quantum cosmology - the supersymmetric perspective : Vol. 1: Fundamentals," Lect. Notes Phys. 803, 1 (2010); : Vol. 2: Advanced Topic," Lect. Notes Phys. 804, 1 (2010).

but they are incomplete ...

QUANTUM COSMOLOGY !?!
SOME MOTIVATIONS

or the books :

P. D. D'Eath, Supersymmetric quantum cosmology, Cambridge, UK: Univ. Pr. (1996) 252 p
P. Vargas Moniz, Quantum cosmology - the supersymmetric perspective : Vol. 1: Fundamentals," Lect. Notes Phys. 803, 1 (2010); : Vol. 2: Advanced Topic," Lect. Notes Phys. 804, 1 (2010).

but they are incomplete ...

- We have used a new approach to quantize this system that treats the Rarita–Schwinger field as a fully quantum fermionic operator (instead of a "classical" Grassmanian field) and takes into account all the nonlinearities in the fermions that sugra requires to exists

Quantum Cosmology !?!
Some motivations

or the books :

P. D. D'Eath, Supersymmetric quantum cosmology, Cambridge, UK: Univ. Pr. (1996) 252 p
P. Vargas Moniz, Quantum cosmology - the supersymmetric perspective : Vol. 1: Fundamentals," Lect. Notes Phys. 803, 1 (2010); : Vol. 2: Advanced Topic," Lect. Notes Phys. 804, 1 (2010).

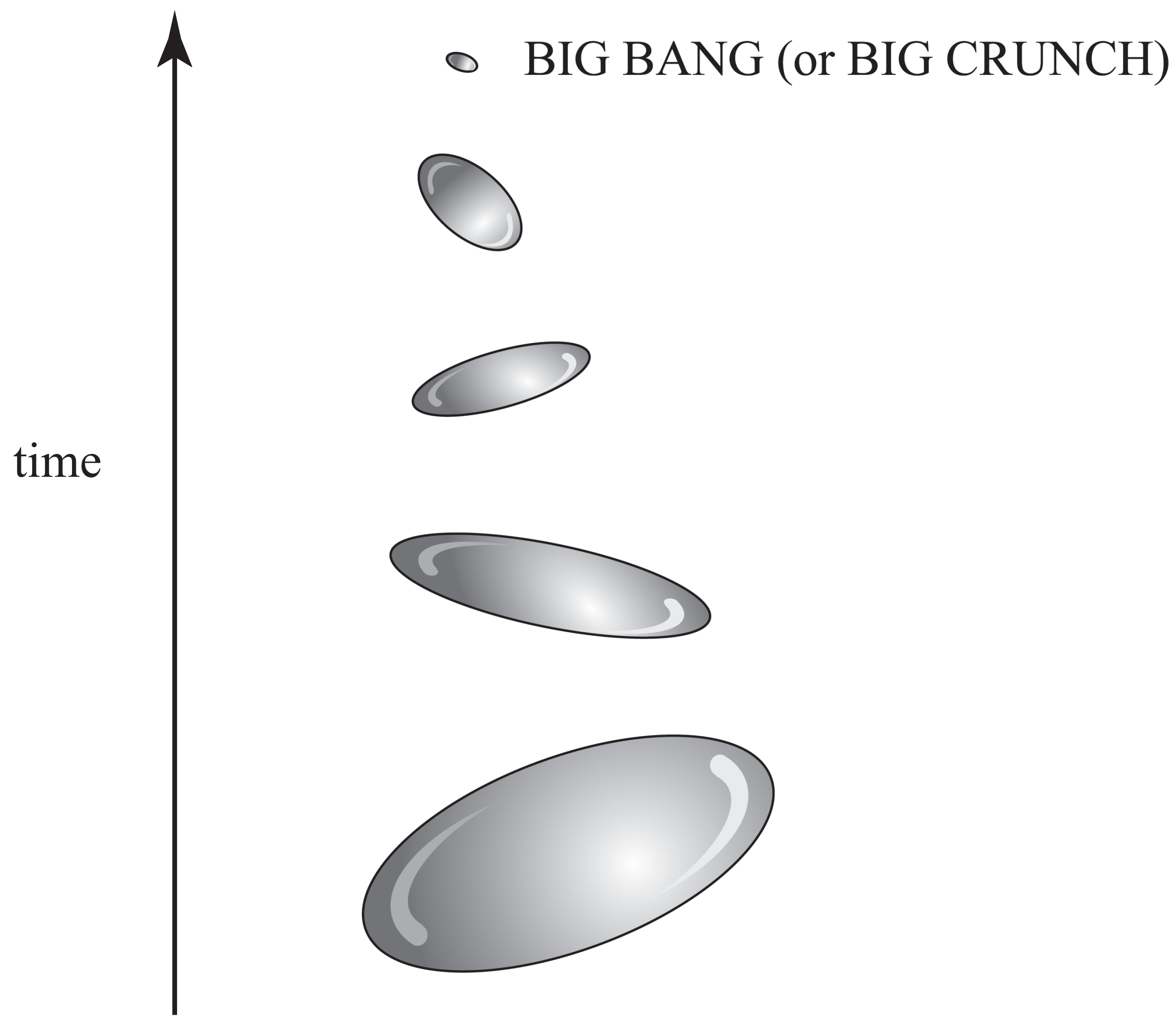
but they are incomplete ...

- We have used a new approach to quantize this system that treats the Rarita–Schwinger field as a fully quantum fermionic operator (instead of a "classical" Grassmanian field) and takes into account all the nonlinearities in the fermions that sugra requires to exists
- In this framework, we put into evidence aspects of an hyperbolic Kac–Moody structure hidden in $\mathcal{N} = 1$, d=4 supergravity.

A CONCRETE CASE STUDY

T. Damour, Ph. S. , arXiv : 1304.6381, Class. Quantum Grav. 30 (2013) 162001,
1406.1309, *Quantum Supersymmetric Bianchi IX Cosmology*

- Quantum dynamics of a supersymmetric triaxially squashed three-sphere



Evolution of the eigendirections of the curvature tensor.

Notice that the space remains homogeneous, it just loses its isotropy

Supergravity Lagrangian

- Rarita-Schwinger Lagrangian

$$\mathcal{L}_{RS} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta\mathcal{D}_\gamma\psi_\delta \quad , \quad \mathcal{D}_{\hat{\beta}}\,\psi_{\hat{\gamma}} = \partial_{\hat{\beta}}\,\psi_{\hat{\gamma}} + \mathring{\omega}_{\hat{\gamma}\hat{\sigma}\hat{\beta}}\,\psi^{\hat{\sigma}} + \frac{1}{4}\,\omega_{\hat{\rho}\hat{\sigma}\,\hat{\beta}}\,\gamma^{\hat{\rho}\hat{\sigma}}\,\psi_{\hat{\gamma}}$$

SUPERGRAVITY LAGRANGIAN

- Rarita-Schwinger Lagrangian

$$\mathcal{L}_{RS} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta\mathcal{D}_\gamma\psi_\delta \quad , \quad \mathcal{D}_{\hat\beta}\,\psi_{\hat\gamma} = \partial_{\hat\beta}\,\psi_{\hat\gamma} + \mathring{\omega}_{\hat\gamma\hat\sigma\hat\beta}\,\psi^{\hat\sigma} + \frac{1}{4}\,\omega_{\hat\rho\hat\sigma\,\hat\beta}\,\gamma^{\hat\rho\hat\sigma}\,\psi_{\hat\gamma}$$

- Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{2\kappa^2}\sqrt{{}^4 g}\,R = -\frac{1}{8\kappa^2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}\theta^{\hat\gamma}_\rho\theta^{\hat\delta}_\sigma R^{\hat\alpha\hat\beta}{}_{\mu\nu}(\omega)$$

Supergravity Lagrangian

- Rarita-Schwinger Lagrangian

$$\mathcal{L}_{RS} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta\mathcal{D}_\gamma\psi_\delta \quad , \quad \mathcal{D}_{\hat\beta}\,\psi_{\hat\gamma} = \partial_{\hat\beta}\,\psi_{\hat\gamma} + \mathring{\omega}_{\hat\gamma\hat\sigma\hat\beta}\,\psi^{\hat\sigma} + \frac{1}{4}\,\omega_{\hat\rho\hat\sigma\,\hat\beta}\,\gamma^{\hat\rho\hat\sigma}\,\psi_{\hat\gamma}$$

- Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{2\kappa^2}\sqrt{{}^4g}\,R = -\frac{1}{8\kappa^2}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}\theta^{\hat\gamma}_\rho\theta^{\hat\delta}_\sigma R^{\hat\alpha\hat\beta}{}_{\mu\nu}(\omega)$$

- Connexion with torsion

$$\omega_{\hat\alpha\hat\beta\mu} = \overset{0}{\omega}_{\hat\alpha\hat\beta\mu} + \kappa_{\hat\alpha\hat\beta\mu}$$

$$\kappa_{\hat\alpha\hat\beta\hat\gamma} = \kappa_{\hat\alpha\hat\beta\mu}\,\theta^\mu_{\hat\gamma} = \frac{\kappa^2}{4}\left(\overline{\psi}_{\hat\beta}\gamma_{\hat\alpha}\psi_{\hat\gamma} - \overline{\psi}_{\hat\alpha}\gamma_{\hat\beta}\psi_{\hat\gamma} + \overline{\psi}_{\hat\beta}\gamma_{\hat\gamma}\psi_{\hat\alpha}\right)$$

- Rarita-Schwinger Lagrangian

$$\mathcal{L}_{RS} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\overline{\psi}_\alpha\gamma_5\gamma_\beta\mathcal{D}_\gamma\psi_\delta \quad , \quad \mathcal{D}_{\hat\beta}\,\psi_{\hat\gamma} = \partial_{\hat\beta}\,\psi_{\hat\gamma} + \mathring{\omega}_{\hat\gamma\hat\sigma\hat\beta}\,\psi^{\hat\sigma} + \frac{1}{4}\,\omega_{\hat\rho\hat\sigma\,\hat\beta}\,\gamma^{\hat\rho\hat\sigma}\,\psi_{\hat\gamma}$$

- Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = \frac{1}{2\kappa^2}\sqrt{{}^4g}\,R = -\frac{1}{8\kappa^2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}\theta^{\hat\gamma}_{\rho}\theta^{\hat\delta}_{\sigma}R^{\hat\alpha\hat\beta}{}_{\mu\nu}(\omega)$$

- Connexion with torsion

$$\omega_{\hat\alpha\hat\beta\mu} = \overset{0}{\omega}_{\hat\alpha\hat\beta\mu} + \kappa_{\hat\alpha\hat\beta\mu}$$

$$\kappa_{\hat\alpha\hat\beta\hat\gamma} = \kappa_{\hat\alpha\hat\beta\mu}\,\theta^{\mu}_{\hat\gamma} = \frac{\kappa^2}{4}\left(\overline{\psi}_{\hat\beta}\gamma_{\hat\alpha}\psi_{\hat\gamma} - \overline{\psi}_{\hat\alpha}\gamma_{\hat\beta}\psi_{\hat\gamma} + \overline{\psi}_{\hat\beta}\gamma_{\hat\gamma}\psi_{\hat\alpha}\right)$$

- Total Lagrangian

$$\mathcal{L}_{Tot} = \theta\left[\frac{1}{2}\overset{0}{R} + \overset{0}{L}_{3/2} + \frac{1}{8}T^{\hat\alpha}T_{\hat\alpha} - \frac{1}{16}T^{\hat\alpha\hat\beta\hat\gamma}T_{\hat\gamma\hat\beta\hat\alpha} - \frac{1}{32}T^{\hat\alpha\hat\beta\hat\gamma}T_{\hat\alpha\hat\beta\hat\gamma}\right]$$

with

$$T_{\hat\alpha\hat\beta\hat\gamma} := \overline{\psi}_{\hat\beta}\gamma_{\hat\alpha}\psi_{\hat\gamma} \qquad , \qquad T_{\hat\alpha} = \overline{\psi}_{\hat\alpha}\gamma^{\hat\beta}\psi_{\hat\beta}$$

MINISUPERSPACE

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity built on an $SO(3)$-homogeneous (Bianchi IX) cosmological model

Minisuperspace

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity built on an $SO(3)$-homogeneous (Bianchi IX) cosmological model

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} \quad = \quad -N^2(t)dt^2 + g_{ab}(t)(\tau^a(x) + N^a(t)dt)(\tau^b(x) + N^b(t)dt)\,,$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3 : d\tau^a = \frac{1}{2}\,\varepsilon^a{}_{bc}\,\tau^b \wedge \tau^c$

Dynamical degrees of freedom

- Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^{a}} \, S^{\hat{a}}{}_{b}(\varphi_1, \varphi_2, \varphi_3) \, S^{\hat{a}}{}_{c}(\varphi_1, \varphi_2, \varphi_3)$$

Dynamical degrees of freedom

• Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^a} \, S^{\hat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) \, S^{\hat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$$

Six metric degrees of freedom :

Dynamical degrees of freedom

• Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^a} \, S^{\hat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) \, S^{\hat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$$

Six metric degrees of freedom :
cologarithms of the squashing parameters of the 3-sphere

$$\beta^a = (\beta^1(t), \beta^2(t), \beta^3(t))$$

Dynamical degrees of freedom

• Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^a} \, S^{\hat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) \, S^{\hat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$$

Six metric degrees of freedom :
cologarithms of the squashing parameters of the 3-sphere

$$\beta^a = (\beta^1(t), \beta^2(t), \beta^3(t))$$

and three Euler angles:

$$\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$$

Gravitino components $(\psi_A^{\hat{\alpha}})$ in specific gauge-fixed orthonormal frame $\theta^{\widehat{\alpha}}$ canonically associated to the Gauss-decomposition :

$$\theta^{\hat{0}} = N(t)dt\,,\ \theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt) \qquad .$$

Fermionic Degrees of Freedom

Gravitino components ($\psi_A^{\hat{\alpha}}$) in specific gauge-fixed orthonormal frame $\theta^{\widehat{\alpha}}$ canonically associated to the Gauss-decomposition :

$$\theta^{\hat{0}} = N(t)dt\,,\ \theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt) \qquad .$$

After a suitable redefinition of the gravitino field :

- 3×4 dynamical gravitino components $\Phi_A^a := (\gamma^{\hat{a}}\, g^{\frac{1}{4}}\, \psi^{\hat{a}})_A$, $a = 1,2,3; A = 1,2,3,4$

so that $g^{\frac{1}{2}}\, \bar{\psi}_{\hat{a}} \gamma^{\hat{a}\hat{0}\hat{b}} \dot{\psi}_{\hat{b}} = G_{ab}\, \Phi^a\, \dot{\Phi}^b$

Fermionic Degrees of Freedom

Gravitino components ($\psi_A^{\hat{\alpha}}$) in specific gauge-fixed orthonormal frame $\theta^{\widehat{\alpha}}$ canonically associated to the Gauss-decomposition :

$$\theta^{\hat{0}} = N(t)dt \,,\; \theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt) \qquad .$$

After a suitable redefinition of the gravitino field :

- 3×4 dynamical gravitino components $\Phi_A^a := (\gamma^{\hat{a}} g^{\frac{1}{4}} \psi^{\hat{a}})_A$, $a = 1,2,3; A = 1,2,3,4$

so that $g^{\frac{1}{2}}\, \bar{\psi}_{\hat{a}} \gamma^{\hat{a}\hat{0}\hat{b}} \dot{\psi}_{\hat{b}} = G_{ab}\, \Phi^a\, \dot{\Phi}^b$

- and four Lagrange multipliers : $\Psi_A'^{\hat{0}} := g^{\frac{1}{4}} (\psi^{\hat{0}} - \sum_a \gamma^{\hat{0}\hat{a}}\, \psi_{\hat{a}})_A$.

β-SPACE STRUCTURE

$$
\begin{aligned}
8\pi G\,\mathring{\mathcal{L}}_{\mathrm{EH}} &= \frac{1}{2} N\sqrt{g}\,\mathring{R} \\
&= \frac{1}{N}\, e^{-\sum_a \beta^a} \Big\{ -(\dot\beta^1\dot\beta^2 + \dot\beta^2\dot\beta^3 + \dot\beta^3\dot\beta^1) + (N^{\bar 1} + w^1)^2 \sinh^2[\beta^2 - \beta^3] \\
&\quad + (N^{\bar 2} + w^2)^2 \sinh^2[\beta^3 - \beta^1] + (N^{\bar 3} + w^3)^2 \sinh^2[\beta^1 - \beta^2] \Big\} \\
&\quad - N \left\{ \frac{1}{4} e^{\sum_a \beta^a} \sum_b e^{-4\beta^b} - \frac{1}{2} e^{-\sum_a \beta^a} \sum_b e^{2\beta^b} \right\} \qquad .
\end{aligned}
$$

where $w_{\bar a \bar b} := \dot S_i^{\bar a}\, S_{\bar b}^i = -w_{\bar b \bar a}, \quad w^a = \epsilon^{abc} w_{\bar b \bar c}$

$$
\begin{aligned}
8\pi G\,\mathring{\mathcal{L}}_{\rm EH} &= \frac{1}{2} N\,\sqrt{g}\,\mathring{R} \\
&= \frac{1}{N}\,e^{-\sum_a \beta^a}\Big\{-(\dot\beta^1\dot\beta^2+\dot\beta^2\dot\beta^3+\dot\beta^3\dot\beta^1)+(N^{\bar 1}+w^1)^2\sinh^2[\beta^2-\beta^3] \\
&\quad +(N^{\bar 2}+w^2)^2\sinh^2[\beta^3-\beta^1]+(N^{\bar 3}+w^3)^2\sinh^2[\beta^1-\beta^2]\Big\} \\
&\quad -N\left\{\frac{1}{4}\,e^{\sum_a\beta^a}\sum_b e^{-4\beta^b}-\frac{1}{2}e^{-\sum_a\beta^a}\sum_b e^{2\beta^b}\right\} \qquad .
\end{aligned}
$$

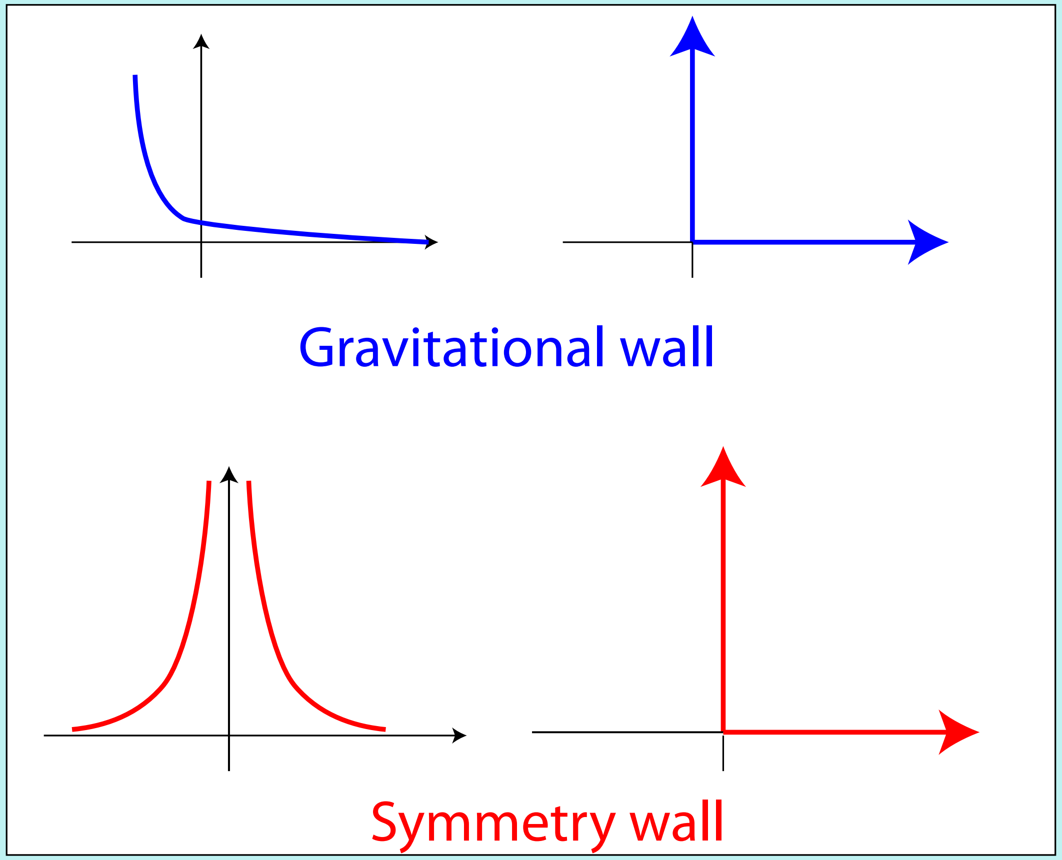
where $w_{\bar a\bar b} := \dot S^{\bar a}_i\,S^i_{\bar b} = -w_{\bar b\bar a}, \quad w^a = \epsilon^{abc} w_{\bar b\bar c}$
This is conveniently rewritten as

$$
\begin{aligned}
8\pi G\,\mathring{\mathcal{L}}_{\rm EH} &= \frac{1}{2\widetilde N}\left[\dot\beta^a\,G_{ab}\,\dot\beta^b+(N^{\bar k}+w^k)\,K_{k\ell}(N^{\bar\ell}+w^\ell)\right]-\widetilde N\,V_g(\beta) \\
&\equiv \frac{1}{\widetilde N}\left[T_\beta+T_w\right]-\widetilde N\,V_g(\beta) \qquad . \qquad (1)
\end{aligned}
$$

β-SPACE "NEAR INFINITY"

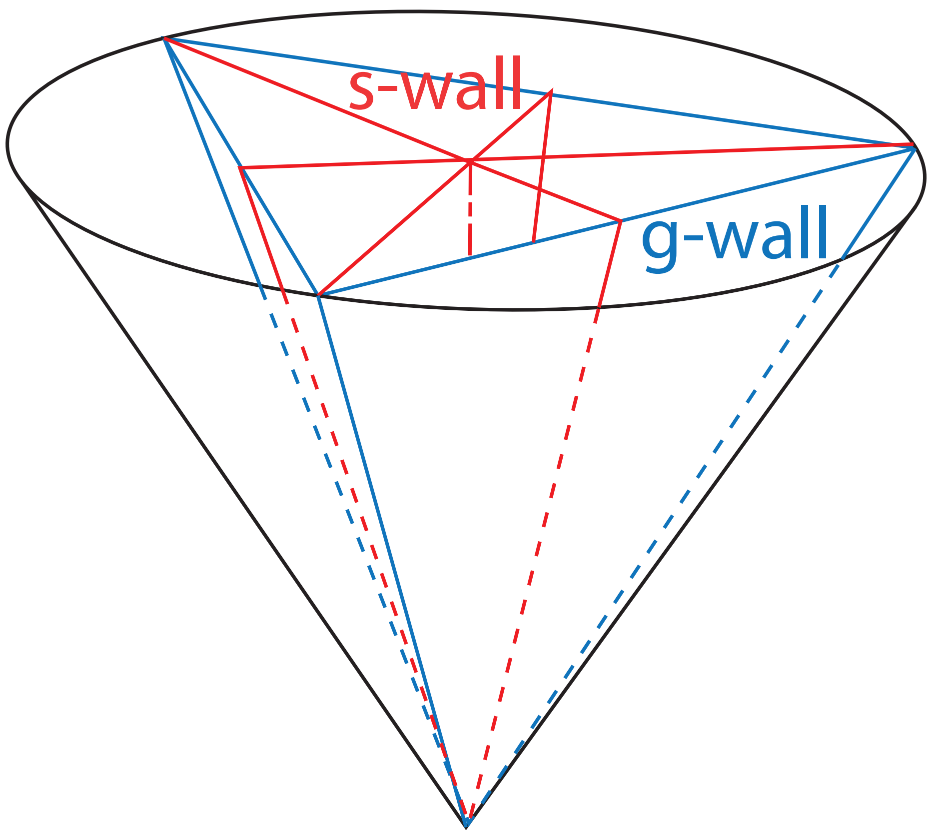
Sharp wall approximation : $\exp[-2\,w(\beta)] \mapsto \infty\theta[-w(\beta)]$

$$\coth^2[w(\beta)] \mapsto \delta[w(\beta)] \sim \infty\theta[-w(\beta)]$$



V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

β-SPACE "NEAR INFINITY"

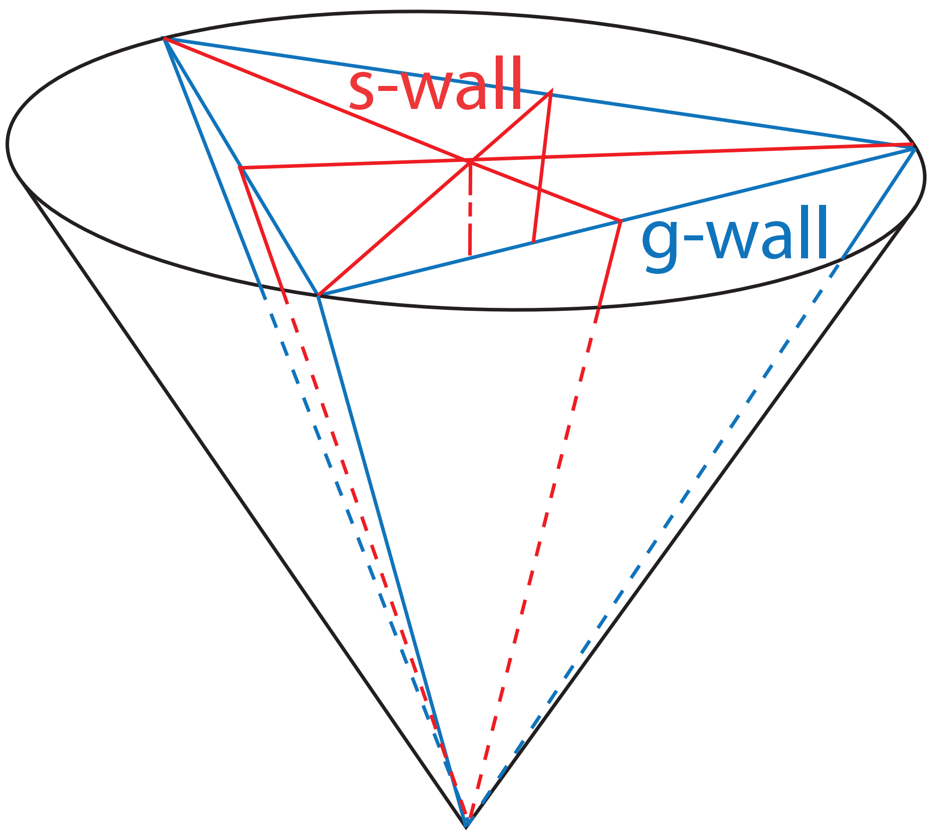


Lorentzian structure of the β-space.
$\beta^0 := \beta^1 + \beta^2 + \beta^3$ play the rôle of time

Symmetry walls : $\sinh^{-2}(\beta^i - \beta^j)$:
$w^s_1(\beta) := \beta^2 - \beta^1 = 0,\ w^s_2(\beta) := \beta^3 - \beta^2 = 0,\ w^s_3(\beta) := \beta^3 - \beta^1 = 0$

Gravitational walls : $\exp(-2\,\beta^i)$:
$w^g_1(\beta) := 2\,\beta^1 = 0,\ 2\,\beta^2 = 0,\ 2\,\beta^3 = 0$

β-SPACE "NEAR INFINITY"



Lorentzian structure of the β-space.
$\beta^0 := \beta^1 + \beta^2 + \beta^3$ play the rôle of time

Symmetry walls : $\sinh^{-2}(\beta^i - \beta^j)$:
$w^s_1(\beta) := \beta^2 - \beta^1 = 0,\ w^s_2(\beta) := \beta^3 - \beta^2 = 0,\ w^s_3(\beta) := \beta^3 - \beta^1 = 0$

Gravitational walls : $\exp(-2\,\beta^i)$:
$w^g_1(\beta) := 2\,\beta^1 = 0,\ 2\,\beta^2 = 0,\ 2\,\beta^3 = 0$

$w^s_1 > 0\ \&\ w^s_2 > 0 \Rightarrow w^s_3 > 0$
$((f,g)) := f_p G^{pq} g_q$
$((w^s_1, w^s_1)) = ((w^s_2, w^s_2)) = ((w^g_1, w^g_1)) = 2$
$((w^s_1, w^s_2)) = -1,\ ((w^s_1, w^g_1)) = -2,$
$((w^s_1, w^g_1)) = 0$

E_{10} (OR AE_n) SEEMS TO BE HIDDEN IN (SUPER)GRAVITY

T. Damour and M. Henneaux, Phys. Rev. Lett. **86**, 4749 (2001),
T. Damour, M. Henneaux, B. Julia and H. Nicolai, Phys. Lett. B **509**, 323 (2001),
T. Damour, M. Henneaux and H. Nicolai, Phys. Rev. Lett. **89**, 221601 (2002),
T. Damour, A. Kleinschmidt and H. Nicolai, Phys. Lett. B **634**, 319 (2006),
S. de Buyl, M. Henneaux and L. Paulot, JHEP **0602**, 056 (2006), T. Damour, A. Kleinschmidt and H. Nicolai, JHEP **0608**, 046 (2006), ...

[first conjectured by Julia '82; related conjectures Ganor '99, '04; West (E_{11}) 01]

Lead to Gravity/Coset conjecture (DHN, 2002)

'Duality' between $D = 11$ supergravity (or, hopefully, M-theory) and the (quantum) dynamics of a massless spinning particle on $E_{10}/K(E_{10})$

KAC-MOODY ALGEBRAS (IN VERY, VERY BRIEF)

Generalized Cartan matrix :

$$(A_{i\,i}) = 2\,,\; A_{i\,j} \in \mathbb{Z}^{-} \; \textit{if} \; i \neq j \quad , \quad A_{i\,j} = 0 \; \textit{iff} \; A_{j\,i} = 0\,,\; A = D\,S$$

with D diagonal positive and S symmetric.

KAC-MOODY ALGEBRAS (IN VERY, VERY BRIEF)

Generalized Cartan matrix :

$$(A_{ii}) = 2 \,,\; A_{ij} \in \mathbb{Z}^- \; if \; i \neq j \quad , \quad A_{ij} = 0 \; iff \; A_{ji} = 0 \,,\; A = D\,S$$

with D diagonal positive and S symmetric.

Chevalley–Serre presentation :

$$[h_i, h_j] = 0 \;\; [h_i, e_j] = A_{ij}\, e_j \; (no \; sum.) \,, [h_i, f_j] = -A_{ij} f_j \,,\; [e_i, f_j] = \delta_{ij}\, h_j \,,\; \mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0$$

Kac-Moody algebras (in very, very brief)

Generalized Cartan matrix :

$$(A_{ii}) = 2\,,\ A_{ij} \in \mathbb{Z}^- \ \textit{if}\ i \neq j \quad , \quad A_{ij} = 0\ \textit{iff}\ A_{ji} = 0\,,\ A = D\,S$$

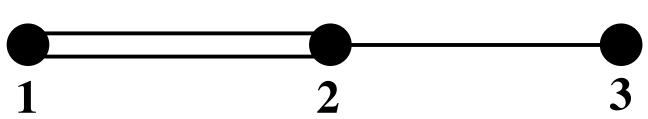
with D diagonal positive and S symmetric.

Chevalley–Serre presentation :

$$[h_i, h_j] = 0\ \ [h_i, e_j] = A_{ij}\, e_j\ (\textit{no sum.})\,, [h_i, f_j] = -A_{ij} f_j\,,\ [e_i, f_j] = \delta_{ij}\, h_j\,,\ \mathrm{ad}_{e_i}^{1-A_{ij}}(e_j) = 0$$

S positive corresponds to finite dimensionnal semi-simple Lie algebras.

AE_3: $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



Level decomposition: $\alpha = \ell\,\alpha_1 + m_2\,\alpha_2 + m_3\,\alpha_3$

Kac-Moody algebras (in very, very brief)

Chevalley involution :

$$\omega(e_i) = -f_i \quad , \quad \omega(f_i) = -e_i \quad , \quad \omega(h_i) = -h_i$$

Chevalley involution :

$$\omega(e_i) = -f_i \quad , \quad \omega(f_i) = -e_i \quad , \quad \omega(h_i) = -h_i$$

"Maximally compact" sub-algebra :

$$\omega(x) = x \quad , \quad \textit{generated by} \quad x_i = e_i - f_i$$

KAC-MOODY ALGEBRAS (IN VERY, VERY BRIEF)

Chevalley involution :

$$\omega(e_i) = -f_i \quad , \quad \omega(f_i) = -e_i \quad , \quad \omega(h_i) = -h_i$$

"Maximally compact" sub-algebra :

$$\omega(x) = x \quad , \quad \textit{generated by} \quad x_i = e_i - f_i$$

Coset element

$$\mathcal{V} = \exp[\sum_{a=1}^{10} \beta^a H_{\hat{a}}] \exp[\sum_{\alpha \in \Delta^+} \sum_{s=1}^{mult(\alpha)} \nu_{\alpha,s} E_\alpha^{(s)} \in E_{10}/K(E_{10})$$

$$\dot{\mathcal{V}}\,\mathcal{V}^{-1} =: \mathcal{P} + \mathcal{Q} \quad , \quad \mathcal{Q} \in K(E_{10}) \quad , \quad \mathcal{P} \in E_{10} \ominus K(E_{10})$$

$h = \beta^i H_i \quad , \quad \alpha = n_k \alpha^k \quad , \quad [h, E_\alpha^{(s)}] = \alpha(h)\, E_\alpha^{(s)} \quad , \quad \alpha(h) = n_k\, \alpha^k(h) = n_k\, A_{ji}\, \beta^j$

$\mathcal{P} := (\dot{\mathcal{V}}\,\mathcal{V}^{-1})^{sym}$: coset velocity

$\mathcal{Q} := (\dot{\mathcal{V}}\,\mathcal{V}^{-1})^{antisym}$: "K"–angular velocity

GRAVITY/COSET CORRESPONDENCE

$G_{\mu\nu}(t,\mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t,\mathbf{x}), \psi_\mu(t,\mathbf{x})$

$S_{11} = \int d_x^{11} \{ \frac{E}{4} R(G)$
$- \frac{E}{48} (d\mathcal{A}_3)^2 + \dots \}$

Gradient Expansion (BKL)
($\sim$ Small Tension Expansion:
$\alpha' \to \infty$)

$\partial_{x^1}^{k_1} \partial_{x^2}^{k_2} \dots \partial_{x^{10}}^{k_{10}} \ll \partial_T^{k_1+k_2+\dots+k_{10}}$

GRAVITY/COSET CORRESPONDENCE

$G_{\mu\nu}(t,\mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t,\mathbf{x}), \psi_\mu(t,\mathbf{x})$

$$S_1^{\text{COSET}} = \int dt \{ \tfrac{1}{4\,n(t)} \langle \mathcal{P}(t), \mathcal{P}(t) \rangle - \tfrac{i}{2} (\Psi(t) \mid \mathcal{D}^{\text{vs}}\, \Psi(t))_{\text{vs}} + \ldots \}$$

$$S_{11} = \int d_x^{11} \{ \tfrac{E}{4} R(G) - \tfrac{E}{48} (d\mathcal{A}_3)^2 + \ldots \}$$

Height Expansion
in Kac-Moody Algebra

$$\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$$

Gradient Expansion (BKL)
($\sim$ Small Tension Expansion: $\alpha' \to \infty$)

$$\partial_t \mathcal{P}(t) = [\mathcal{Q}(t),\ \mathcal{P}(t)] \quad , \quad \partial_t \Psi(t) = \mathcal{Q}^{vs}\, \Psi(t)$$

$$\partial_{x^1}^{k_1}\, \partial_{x^2}^{k_2} \ldots \partial_{x^{10}}^{k_{10}} \ll \partial_T^{k_1+k_2+\ldots+k_{10}}$$

Gravity/Coset Correspondence

$G_{\mu\nu}(t,\mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t,\mathbf{x}), \psi_\mu(t,\mathbf{x})$

$$S_1^{\text{COSET}} = \int dt \{ \frac{1}{4\,n(t)} \langle \mathcal{P}(t), \mathcal{P}(t) \rangle$$
$$- \frac{i}{2} (\Psi(t) \mid \mathcal{D}^{\text{vs}} \Psi(t))_{\text{vs}} + \ldots \}$$

Height Expansion
in Kac-Moody Algebra

$$S_{11} = \int d_x^{11} \{ \frac{E}{4} R(G)$$
$$- \frac{E}{48} (d\mathcal{A}_3)^2 + \ldots \}$$

$$\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$$

Gradient Expansion (BKL)
($\sim$ Small Tension Expansion:
$\alpha' \to \infty$)

$$\partial_t \mathcal{P}(t) = [\mathcal{Q}(t),\, \mathcal{P}(t)] \quad , \quad \partial_t \Psi(t) = \mathcal{Q}^{vs}\, \Psi(t)$$

$$\mathcal{Q}(t)|_\alpha \propto \cosh^{-1}[2\sqrt{E}(t - t_c)] J_\alpha$$

$$\partial_{x^1}^{k_1} \partial_{x^2}^{k_2} \ldots \partial_{x^{10}}^{k_{10}} \ll \partial_T^{k_1+k_2+\ldots+k_{10}}$$

$$E = -\frac{1}{2} G_{ab} \dot\beta_{\|}^a \dot\beta_{\|}^a \quad , \quad J_\alpha = E_\alpha - E_{-\alpha}$$

E_{10} : Damour, Henneaux, Nicolai $'02$; related: Ganor $'99$ $'04$; E_{11} : West $'01$, de Buyl, Hennaux, Paulot $'06$, Damour, Hillmann $'09$

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\theta^a} \, \dot{\theta}^a + \frac{i}{2} \, G_{ab} \, \Phi^a_A \, \dot{\Phi}^b_A + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\theta^a} \, \dot{\theta}^a + \frac{i}{2} G_{ab} \, \Phi^a_A \, \dot{\Phi}^b_A + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\theta^a} \, \dot{\theta}^a + \frac{i}{2} G_{ab} \, \Phi^a_A \, \dot{\Phi}^b_A + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers imply constraints : $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

- Bosonic dof:

$$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \; ; \quad \widehat{p}_{\varphi^a} = -i \frac{\partial}{\partial \varphi^a}$$

- Fermionic dof:

$$\widehat{\Phi}^a_A \, \widehat{\Phi}^b_B + \widehat{\Phi}^b_B \, \widehat{\Phi}^a_A = G^{ab} \, \delta_{AB}$$

This is the Clifford algebra Spin $(8^+, 4^-)$

• Bosonic dof:

$$\widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a}\ ;\quad \widehat{p}_{\varphi^a} = -i\,\frac{\partial}{\partial\varphi^a}$$

• Fermionic dof:

$$\widehat{\Phi}^a_A\,\widehat{\Phi}^b_B + \widehat{\Phi}^b_B\,\widehat{\Phi}^a_A = G^{ab}\,\delta_{AB}$$

This is the Clifford algebra Spin $(8^+, 4^-)$

• The wave function of the Universe $\Psi_\sigma(\beta^a, \theta^a)$ is a 64-dimensional spinor of Spin $(8, 4)$ and the gravitino operators Φ^a_A are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$. Hermiticity $(\ddagger)$ is defined by use of $h = \Gamma_9\Gamma_{10}\Gamma_{11}\Gamma_{12} = h^{-1}$: $\Psi^\ddagger := h\,\Psi^\dagger\,h$.

Dirac Quantization of the Constraints

$$\Psi = \Psi[\beta, \varphi]\,, \quad \widehat{\mathcal{S}}_A \Psi = 0\,, \quad \widehat{H}\Psi = 0\,, \quad \widehat{H}_a \Psi = 0$$

Dirac Quantization of the Constraints

$$\Psi = \Psi[\beta, \varphi]\,, \quad \widehat{\mathcal{S}}_A \Psi = 0\,, \quad \widehat{H}\,\Psi = 0\,, \quad \widehat{H}_a \Psi = 0$$

Diffeomorphism constraints $\Leftrightarrow \widehat{p}_{\varphi^a}\,\Psi = -i\,\dfrac{\partial}{\partial \varphi^a}\,\Psi = 0$

Dirac Quantization of the Constraints

$$\Psi = \Psi[\beta, \varphi]\,, \quad \widehat{\mathcal{S}}_A\,\Psi = 0\,, \quad \widehat{H}\,\Psi = 0\,, \quad \widehat{H}_a\,\Psi = 0$$

Diffeomorphism constraints $\Leftrightarrow \widehat{p}_{\varphi^a}\,\Psi = -i\,\dfrac{\partial}{\partial \varphi^a}\,\Psi = 0$

S-wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0\,, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0$$

DIRAC QUANTIZATION OF THE CONSTRAINTS

$$\Psi = \Psi[\beta, \varphi]\,, \quad \widehat{\mathcal{S}}_A\,\Psi = 0\,, \quad \widehat{H}\,\Psi = 0\,, \quad \widehat{H}_a\,\Psi = 0$$

Diffeomorphism constraints $\Leftrightarrow \widehat{p}_{\varphi^a}\,\Psi = -i\,\dfrac{\partial}{\partial\varphi^a}\,\Psi = 0$

S-wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0\,, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi})\,\Psi(\beta) = 0$$

$\widehat{\pi}_a = -i\,\frac{\partial}{\partial\beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's

Explicit form of the SUSY constraints

$$\begin{aligned}\widehat{\mathcal{S}}_A &= -\frac{1}{2}\sum_a \widehat{\pi}_a\,\Phi^a_A + \frac{1}{2}\sum_a e^{-2\beta^a}(\gamma^5\,\Phi^a)_A \\ &- \frac{1}{8}\coth\beta_{12}(\widehat{S}_{12}(\gamma^{12}\,\widehat{\Phi}^{12})_A + (\gamma^{12}\,\widehat{\Phi}^{12})_A\,\widehat{S}_{12}) \\ &+ \text{cyclic}_{(123)} + \frac{1}{2}(\widehat{\mathcal{S}}^{\text{cubic}}_A + \widehat{\mathcal{S}}^{\text{cubic}\,\ddagger}_A)\end{aligned}$$

where $\gamma^5 \equiv \gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}$, $\beta_{12} \equiv \beta^1 - \beta^2$, $\widehat{\Phi}^{12} \equiv \widehat{\Phi}^1 - \widehat{\Phi}^2$,

$$\begin{aligned}\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\widehat{\overline{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\widehat{\overline{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\ &+ (\widehat{\overline{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\widehat{\overline{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,, \qquad \textit{su}(2)\ \textit{subalgebra}\end{aligned}$$

Explicit form of the SUSY constraints

$$\begin{aligned}\widehat{\mathcal{S}}_A^{\text{cubic}} &= \frac{1}{4}\sum_a (\widehat{\overline{\Phi}}_0\,\gamma^{\hat{0}}\,\widehat{\Psi}_{\widehat{a}})\,\gamma^{\hat{0}}\,\widehat{\Psi}^A_{\widehat{a}} - \frac{1}{8}\sum_{a,b}(\widehat{\overline{\Psi}}_{\widehat{a}}\gamma^{\hat{0}}\,\widehat{\Psi}_{\widehat{b}})\,\gamma^{\widehat{a}}\,\widehat{\Psi}^A_{\widehat{b}} \\ &+ \frac{1}{8}\sum_{a,b}(\widehat{\overline{\Phi}}_0\,\gamma^{\widehat{a}}\,\Psi_{\widehat{b}})(\gamma^{\widehat{a}}\,\Psi^A_{\widehat{b}} + \gamma^{\widehat{b}}\,\Psi^A_{\widehat{a}}),\end{aligned}$$

where $\Phi_0 = \gamma_{\hat{0}}(\Phi^1 + \Phi^2 + \Phi^3)$,

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

This unique, hermitian ordering of $\mathcal{S}_A$ defines a unique Hamiltonian, such that :

$$\widehat{\mathcal{S}}_A\,\widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B\,\widehat{\mathcal{S}}_A = 4\,i\sum_C \widehat{L}_{AB}^{C}(\beta)\,\widehat{\mathcal{S}}_C + \frac{1}{2}\,\widehat{H}\delta_{AB}$$

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

This unique, hermitian ordering of $\mathcal{S}_A$ defines a unique Hamiltonian, such that :

$$\widehat{\mathcal{S}}_A\,\widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B\,\widehat{\mathcal{S}}_A = 4\,i\sum_C \widehat{L}^C_{AB}(\beta)\,\widehat{\mathcal{S}}_C + \frac{1}{2}\,\widehat{H}\delta_{AB}$$

$$[\widehat{\mathcal{S}}_A, \widehat{H}] = \widehat{M}^B_A\,\widehat{\mathcal{S}}_B + \widehat{N}_A\widehat{H}$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$A_a = \partial_a \ln[e^{\frac{3}{4}\beta^0}(\sinh\beta_{12}\ \sinh\beta_{23}\ \sinh\beta_{31})^{-\frac{1}{8}}] \qquad \textit{pure gauge vector potential}$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$A_a = \partial_a \ln[e^{\frac{3}{4}\beta^0}(\sinh\beta_{12}\ \sinh\beta_{23}\ \sinh\beta_{31})^{-\frac{1}{8}}] \qquad \textit{pure gauge vector potential}$$

$$\widehat{W}(\beta) = \widehat{W}_g^{\text{bos}}(\beta) + \widehat{W}_g^{\text{spin}}(\beta) + \widehat{W}_{\text{sym}}^{\text{spin}}(\beta)\,.$$

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\widehat{H} = G^{ab}(\widehat{\pi}_a + i A_a)(\widehat{\pi}_b + i A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$A_a = \partial_a \ln[e^{\frac{3}{4}\beta^0}(\sinh\beta_{12}\ \sinh\beta_{23}\ \sinh\beta_{31})^{-\frac{1}{8}}] \qquad \textit{pure gauge vector potential}$$

$$\widehat{W}(\beta) = \widehat{W}_g^{\text{bos}}(\beta) + \widehat{W}_g^{\text{spin}}(\beta) + \widehat{W}_{\text{sym}}^{\text{spin}}(\beta)\,.$$

$$\widehat{W}_g^{\text{bos}}(\beta) = W_g^{\text{bos}}(\beta) = \frac{1}{2}\,e^{-4\beta^1} - e^{-2(\beta^2+\beta^3)} + \text{cyclic}_{123}$$

KAC-MOODY STRUCTURES HIDDEN IN THE QUANTUM HAMILTONIAN

$$\widehat{W}^{\text{spin}}_{\text{sym}}(\beta) = \frac{1}{2}\,\frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha^{\text{sym}}_{12}(\beta)} + \text{cyclic}_{123}\,,$$

Linear forms $\alpha^{\text{sym}}_{12}(\beta) = \beta^1 - \beta^2$, $\alpha^{\text{sym}}_{23}(\beta) = \beta^2 - \beta^3$, $\alpha^{\text{sym}}_{31}(\beta) = \beta^3 - \beta^1 \Leftrightarrow$ three level-0 roots of AE_3

$$\begin{aligned} \widehat{W}^{\text{spin}}_{g}(\beta, \widehat{\Phi}) &= e^{-\alpha^g_{11}(\beta)}\widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha^g_{22}(\beta)}\widehat{J}_{22}(\widehat{\Phi}) \\ &+ e^{-\alpha^g_{33}(\beta)}\widehat{J}_{33}(\widehat{\Phi})\,. \end{aligned}$$

Linear forms $\alpha^g_{ab}(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

Spin dependent (Clifford) Operators coupled to AE_3 roots

$$\begin{aligned}\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\overline{\widehat{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1+\widehat{\Phi}^2))+(\overline{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1)\\ &+ (\overline{\widehat{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)-(\overline{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,\end{aligned}$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}\,[\overline{\widehat{\Phi}}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1+\widehat{\Phi}^2+\widehat{\Phi}^3)+\overline{\widehat{\Phi}}^2\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3]\,.$$

• $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}, \widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

BI-COMPLEX

$$\{\Phi^k_A\, \Phi^l_B + \Phi^l_B\, \Phi^k_A\} = G^{kl}\delta_{AB} Id_{64}$$
$$b^k_+ = \Phi^k_1 + i\,\Phi^k_2 \quad , \quad b^k_- = \Phi^k_3 - i\,\Phi^k_4$$
$$\tilde{b}^k_+ = \Phi^k_1 - i\,\Phi^k_2 \quad , \quad \tilde{b}^k_- = \Phi^k_3 + i\,\Phi^k_4$$
$$\{b^k_\epsilon\,,\, \tilde{b}^l_\sigma\} = 2\,G^{kl}\delta_{\epsilon\sigma}\, Id_{64}$$

$$\mathcal{S}_\epsilon = \frac{i}{2}\partial_{\beta_k} b^k_\epsilon + \alpha_k\, b^k_\epsilon + \frac{1}{2}\mu_{klm}\, B^{klm}_\epsilon + \rho_{klm} C^{klm}_\epsilon + \frac{1}{2}\nu_{klm}\, D^{klm}_\epsilon$$

where

$$\begin{aligned} B^{klm}_\epsilon &= b^k_\epsilon\, b^l_\epsilon\, \tilde{b}^m_\epsilon - G^{lm}\, b^k_\epsilon + G^{km} b^l_\epsilon \\ C^{klm}_\epsilon &= b^k_\epsilon\, b^l_{-\epsilon}\, \tilde{b}^m_{-\epsilon} - G^{lm}\, b^k_\epsilon \\ D^{klm}_\epsilon &= b^k_{-\epsilon}\, b^l_{-\epsilon}\, \tilde{b}^m_\epsilon \end{aligned}$$

and all the tensor components are purely imaginary (as they must be to insure hermiticity of the operators).

Fermion number operator : $\widehat{N}_F := G_{ab}\tilde{b}^a_+ b^b_+ + G_{ab}\tilde{b}^a_- b^b_- = \frac{1}{2} G_{ab}\overline{\widehat{\Phi}}^a \gamma^{\hat{1}\hat{2}\hat{3}}\widehat{\Phi}^b + 3 =: \widehat{C}_F + 3$

SOLUTIONS OF SUSY CONSTRAINTS

Overdetermined system of 4×64 Dirac-like equations

$$\widehat{\mathcal{S}}_A \, \Psi = \left(\frac{i}{2} \, \Phi^a_A \, \frac{\partial}{\partial \beta^a} + \ldots \right) \Psi = 0$$

Space of solutions splits according to the fermion number $N_F = C_F + 3)$

Depending on the fermion number there exist "discrete states" and "continuous states" (parametrized by initial data involving arbitrary *functions*, at $C_F = -1, 0, +1$)

Solution Space Structure

The wave function we are looking for are solutions of :

$$\mathcal{S}_\epsilon \, \Psi = 0 = \tilde{\mathcal{S}}_\epsilon \, \Psi$$

There is a unique lower state :

$$b^k_\epsilon \Psi_0 = 0 \quad , \quad N_F \, \Psi_0 = 0 \quad , \quad \mathcal{S}_\epsilon \; N_F = (N_F - 1) \, \mathcal{S}_\epsilon \quad , \quad \tilde{\mathcal{S}}_\epsilon \; N_F = (N_F + 1) \, \tilde{\mathcal{S}}_\epsilon$$

Solution Space Structure

The wave function we are looking for are solutions of :

$$\mathcal{S}_\epsilon \, \Psi = 0 = \tilde{\mathcal{S}}_\epsilon \, \Psi$$

There is a unique lower state :

$$b^k_\epsilon \Psi_0 = 0 \quad , \quad N_F \, \Psi_0 = 0 \quad , \quad \mathcal{S}_\epsilon \; N_F = (N_F - 1) \, \mathcal{S}_\epsilon \quad , \quad \tilde{\mathcal{S}}_\epsilon \; N_F = (N_F + 1) \, \tilde{\mathcal{S}}_\epsilon$$

- Level 0 : Ψ_0 (dim=1)
- Level 1 : $\{\tilde{b}^k_+, \; \tilde{b}^l_-\}\Psi_0$ (dim=6=2 $\times$ 3)
- Level 2 : $\{\tilde{b}^{(k}_+\tilde{b}^{l)}_-, \; \tilde{b}^{[k}_+\tilde{b}^{l]}_+, \; \tilde{b}^{[k}_+\tilde{b}^{l]}_-, \; \tilde{b}^{[k}_-\tilde{b}^{l]}_-\}\Psi_0$ (dim=15= 6+3+3+3)
- Level 3 : $\{\frac{1}{2}\epsilon^{[a}{}_{k\,l}\tilde{b}^{m]}_-\tilde{b}^k_+\tilde{b}^l_+, \; \tilde{b}^1_-\tilde{b}^2_-\tilde{b}^3_-, \; \frac{1}{2}\epsilon^{(a}{}_{k\,l}\tilde{b}^{m)}_-\tilde{b}^k_+\tilde{b}^l_+\}\Psi_0$, (dim=20=2 $\times$ $(3+1+6)$)
- Level 4 : $\ldots$ (dim=15)
- Level 5 : $\ldots$ (dim=6)
- Level 6 : $\tilde{b}^1_+\tilde{b}^2_+\tilde{b}^3_+\tilde{b}^1_-\tilde{b}^2_-\tilde{b}^3_-\Psi_0$ (dim=1)

Explicit Equations and Solutions

With $x := e^{2\beta_1},\ y := e^{2\beta_2},\ z := e^{2\beta_3},\ :$

- Level 0 :

$$\frac{i}{2}\partial_{\beta_k} f - \phi_k f = 0$$

$$\phi_k = -i\left\{\frac{1}{2} - \frac{1}{2\,x} - \frac{3}{8}\frac{y(x-z)+z(x-y)}{(x-y)(x-z)}, \textit{cyclic perm.}\right\}$$

$$f = f_0\, Exp[-\frac{1}{2}(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})](x\,y\,z)^{-\frac{5}{4}}\,((x-y)(y-z)(z-x))^{3/8}$$

Explicit Equations and Solutions

With $x := e^{2\beta_1},\ y := e^{2\beta_2},\ z := e^{2\beta_3},\ :$

- Level 0 :

$$\frac{i}{2}\partial_{\beta_k} f - \phi_k f = 0$$

$$\phi_k = -i\left\{\frac{1}{2} - \frac{1}{2x} - \frac{3}{8}\frac{y(x-z)+z(x-y)}{(x-y)(x-z)}, cyclic\ perm.\right\}$$

$$f = f_0\, Exp[-\frac{1}{2}(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})](x\,y\,z)^{-\frac{5}{4}}\ \underline{((x-y)(y-z)(z-x))^{3/8}}$$

Differs from previously obtained solutions by the factors vanishing on the symmetry walls

Explicit Equations and Solutions

- Level 1 : $\Psi = f_k^\sigma \, \tilde{b}_\sigma^k \, \Psi_0$

$$\mathcal{S}_\epsilon \Psi = 0 \Leftrightarrow \frac{i}{2} \partial_{\beta_k} f_\epsilon^k + \varphi_k f_\epsilon^k = 0$$

$$\tilde{\mathcal{S}}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \tilde{\nu}_{[kl]}{}^m f_m^\epsilon = 0 \\ \left(\frac{i}{2} \partial_{\beta_{[k}} f_{l]}^\epsilon + \tilde{\varphi}_{[k} f_{l]}^\epsilon - 2\, \tilde{\mu}_{[kl]}{}^m f_m^\epsilon = 0\right) \\ \frac{i}{2} \partial_{\beta_k} f_l^\epsilon + \tilde{\varphi}_k f_l^\epsilon - 2\, \tilde{\rho}_{k\,l}{}^m f_m^\epsilon = 0 \end{cases}$$

$$f_k^\epsilon = f^\epsilon \, \{x(y-z),\, y(z-x),\, z(x-y)\}$$

with

$$f^\epsilon = f_0^\epsilon \, Exp[-\frac{1}{2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})](x\,y\,z)^{-\frac{3}{4}} \, ((x-y)(y-z)(z-x))^{-3/8}$$

A two-dimensionnal subspace of solutions

Square integrable singularity

Explicit Equations and Solutions

- Level 1 : $\Psi = f_k^{\sigma}\, \tilde{b}^k_{\sigma}\, \Psi_0$

$$\mathcal{S}_{\epsilon}\Psi = 0 \Leftrightarrow \frac{i}{2}\partial_{\beta_k} f^k_{\epsilon} + \varphi_k f^k_{\epsilon} = 0$$

$$\tilde{\mathcal{S}}_{\epsilon}\Psi = 0 \Leftrightarrow \begin{cases} \tilde{\nu}_{[kl]}{}^{m} f^{\epsilon}_m = 0 \\ \left(\frac{i}{2}\partial_{\beta_{[k}} f^{\epsilon}_{l]} + \tilde{\varphi}_{[k} f^{\epsilon}_{l]} - 2\,\tilde{\mu}_{[kl]}{}^{m} f^{\epsilon}_m = 0\right) \\ \frac{i}{2}\partial_{\beta_k} f^{\epsilon}_l + \tilde{\varphi}_k f^{\epsilon}_l - 2\,\tilde{\rho}_{k\,l}{}^{m} f^{\epsilon}_m = 0 \end{cases}$$

$$f^{\epsilon}_k = f^{\epsilon}\,\{x(y-z),\, y(z-x),\, z(x-y)\}$$

with

$$f^{\epsilon} = f^{\epsilon}_0\, Exp[-\frac{1}{2}(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})](x\,y\,z)^{-\frac{3}{4}}\,((x-y)(y-z)(z-x))^{-3/8}$$

A two-dimensionnal subspace of solutions

Previous works concluded to the absence of states with odd fermion numbers

Explicit Equations and Solutions

- Level 2 : $\Psi = \frac{1}{2} f_{p\,q}^{\epsilon,\epsilon'}\, \tilde{b}^p_\epsilon \tilde{b}^q_{\epsilon'}\, \Psi_0$, with $f_{p\,q}^{\epsilon,\,\epsilon'} = -f_{q\,p}^{\epsilon',\,\epsilon})$

$$\mathcal{S}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,\epsilon} + \varphi^k f_{k\,n}^{\epsilon,\epsilon} - \mu^{k\,l}{}_n f_{k\,l}^{\epsilon,\epsilon} - \nu^{k\,l}{}_n f_{k\,l}^{-\epsilon,-\epsilon} = 0 \\ \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,-\epsilon} + \varphi^k f_{k\,n}^{\epsilon,-\epsilon} - 2\,\rho^{k\,l}{}_n f_{k\,l}^{\epsilon,-\epsilon} = 0 \end{cases}$$

$$\tilde{\mathcal{S}}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \left[\frac{i}{2}\partial_{\beta_n} f_{p\,q}^{\epsilon,\epsilon} + \tilde{\varphi}_n f_{p\,q}^{\epsilon,\epsilon} + 2\,\tilde{\mu}_{p\,q}{}^s f_{n\,s}^{\epsilon,\epsilon}\right] \varepsilon^{n\,p\,q} = 0 \\ \frac{i}{2}\partial_{\beta_{[k}} f_{l]\,q}^{\epsilon,-\epsilon} + \tilde{\varphi}_{[k} f_{l]q}^{\epsilon,-\epsilon} - \tilde{\mu}_{k\,l}{}^s f_{s\,q}^{\epsilon,-\epsilon} - 2\,\tilde{\rho}_{[k\,|q|}{}^s f_{l]\,s}^{\epsilon,-\epsilon} = 0 \\ \frac{i}{2}\partial_{\beta_k} f_{p\,q}^{-\epsilon,-\epsilon} + \tilde{\varphi}_k f_{p\,q}^{-\epsilon,-\epsilon} + 4\,\tilde{\rho}_{k\,[p}{}^s f_{q]\,s}^{-\epsilon,-\epsilon} + 2\,\tilde{\nu}_{p\,q}{}^s f_{k\,s}^{\epsilon,\epsilon} = 0 \\ \tilde{\nu}_{[p\,q}{}^s f_{n]\,s}^{-\epsilon,\epsilon} = 0 \end{cases}$$

Explicit Equations and Solutions

- Level 2 : $\Psi = \frac{1}{2} f_{p\,q}^{\epsilon,\epsilon'}\, \tilde{b}^p_\epsilon \tilde{b}^q_{\epsilon'}\, \Psi_0$, with $f_{p\,q}^{\epsilon,\,\epsilon'} = -f_{q\,p}^{\epsilon',\,\epsilon}$)

$$\mathcal{S}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,\epsilon} + \varphi^k f_{k\,n}^{\epsilon,\epsilon} - \mu^{k\,l}{}_n f_{k\,l}^{\epsilon,\epsilon} - \nu^{k\,l}{}_n f_{k\,l}^{-\epsilon,-\epsilon} = 0 & \star \\ \frac{i}{2}\partial_{\beta_k} G^{kp} f_{p\,n}^{\epsilon,-\epsilon} + \varphi^k f_{k\,n}^{\epsilon,-\epsilon} - 2\,\rho^{k\,l}{}_n f_{k\,l}^{\epsilon,-\epsilon} = 0 & \bullet \end{cases}$$

$$\tilde{\mathcal{S}}_\epsilon \Psi = 0 \Leftrightarrow \begin{cases} \left[\frac{i}{2}\partial_{\beta_n} f_{p\,q}^{\epsilon,\epsilon} + \tilde{\varphi}_n f_{p\,q}^{\epsilon,\epsilon} + 2\,\tilde{\mu}_{p\,q}{}^s f_{n\,s}^{\epsilon,\epsilon}\right] \varepsilon^{n\,p\,q} = 0 & \star \\ \frac{i}{2}\partial_{\beta_{[k}} f_{l]\,q}^{\epsilon,-\epsilon} + \tilde{\varphi}_{[k} f_{l]q}^{\epsilon,-\epsilon} - \tilde{\mu}_{k\,l}{}^s f_{s\,q}^{\epsilon,-\epsilon} - 2\,\tilde{\rho}_{[k\,|q|}{}^s f_{l]\,s}^{\epsilon,-\epsilon} = 0 & \bullet \\ \frac{i}{2}\partial_{\beta_k} f_{p\,q}^{-\epsilon,-\epsilon} + \tilde{\varphi}_k f_{p\,q}^{-\epsilon,-\epsilon} + 4\,\tilde{\rho}_{k\,[p}{}^s f_{q]\,s}^{-\epsilon,-\epsilon} + 2\,\tilde{\nu}_{p\,q}{}^s f_{k\,s}^{\epsilon,\epsilon} = 0 & \star \\ \tilde{\nu}_{[p\,q}{}^s f_{n]\,s}^{-\epsilon,\epsilon} = 0 & \bullet \end{cases}$$

Equations for the $f_{p\,q}^{\epsilon\,\epsilon}$ and $f_{p\,q}^{\epsilon\,-\epsilon}$ components decouple.

Explicit Equations and Solutions

- Level 2 : *A three dimensional subspace of discrete modes* ($\epsilon = \pm$)

$$\{f^{\epsilon\,\epsilon}_{1\,2}, f^{\epsilon\,\epsilon}_{2\,3}, f^{\epsilon\,\epsilon}_{3\,1} \quad =: \quad f^{\epsilon\,\epsilon}\{2(x\,y - y\,z - x\,z) + x\,y\,z,\ cyclic\ perm.\}$$

with

$$f^{\epsilon\,\epsilon} = Exp[-\frac{1}{2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})](x\,y\,z)^{-3/4}(x-y)^{-1/8}(x-z)^{-1/8}(y-z)^{-1/8}$$
$$\left(C_1\,(x-z)^{-1/2} + \epsilon\,C_2\,(y-z)^{-1/2}\right)$$

$$\{f^{+-}_{[1\,2]}, f^{+-}_{[2\,3]}, f^{+-}_{[3\,1]}\} = f^{+-}\{2(x\,y - y\,z - x\,z) + x\,y\,z,\ cyclic\ perm.\}$$

$$f^{+-} = C_3\,Exp[-\frac{1}{2}(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})]\,(x\,y\,z)^{-3/4}((x-y)(y-z)(z-x))^{-1/8}\ (x-y)^{-1/2}$$

Unphysical solution!

- Level 2 : The six modes $f^{+-}_{(k\,l)} =: k_{k\,l}$ are propagating modes

  They obey Maxwell-like equations :

$$\begin{aligned} \delta k \sim 0 \quad &: \quad \frac{i}{2}\partial^p k_{pa} + \phi^p\, k_{pa} - 2\,\rho^{pq}\ k_{pq} = 0 \\ d\,k \sim 0 \quad &: \quad \frac{i}{2}\partial_{[a} k_{b]c} - \phi_{[a}\, k_{b]c} + \mu_{ab}{}^{p}\, k_{pc} + 2\,\rho_{[a|c|}{}^{p}\, k_{b]p} = 0 \end{aligned}$$

  Their compatibility is guaranteed by “Bianchi” identities : $d^2 = 0 = \delta^2$.

Explicit Equations and Solutions

- Level 2 : The six modes $f^{+-}_{(k\,l)} =: k_{k\,l}$ are propagating modes

  They obey Maxwell-like equations :

$$\delta k \sim 0 \quad : \quad \frac{i}{2}\partial^p k_{pa} + \phi^p\, k_{pa} - 2\,\rho^{pq}\, k_{pq} = 0$$

$$d\,k \sim 0 \quad : \quad \frac{i}{2}\partial_{[a} k_{b]c} - \phi_{[a}\, k_{b]c} + \mu_{ab}{}^p\, k_{pc} + 2\,\rho_{[a|c|}{}^p\, k_{b]p} = 0$$

  Their compatibility is guaranteed by “Bianchi” identities : $d^2 = 0 = \delta^2$.
  These equations split into :
  • five constraint equations (no “time derivative”)
  • six evolution equations (with respect to the “time” $\beta^0 = \beta^1 + \beta^2 + \beta^3$)

Explicit Equations and Solutions

The general solution is parametrized by two arbitrary functions of two variables (leaving in a plane $\beta^0 = Cte$) from which we compute (via an Euler-Darboux-Poisson equation) the Cauchy data for the six k_{ab}, which then propagate thanks to the six evolution equations : $\partial_0 k_{ab} = \ldots$.

Explicit Equations and Solutions

The general solution is parametrized by two arbitrary functions of two variables (leaving in a plane $\beta^0 = Cte$) from which we compute (via an Euler-Darboux-Poisson equation) the Cauchy data for the six k_{ab}, which then propagate thanks to the six evolution equations : $\partial_0 k_{ab} = \ldots$.

They also may be described as modes superposition: far from the walls (in terms of plane waves), or when bouncing on a wall, far from the corners (in terms of special functions of Legendre or Kummer).

Explicit Equations and Solutions

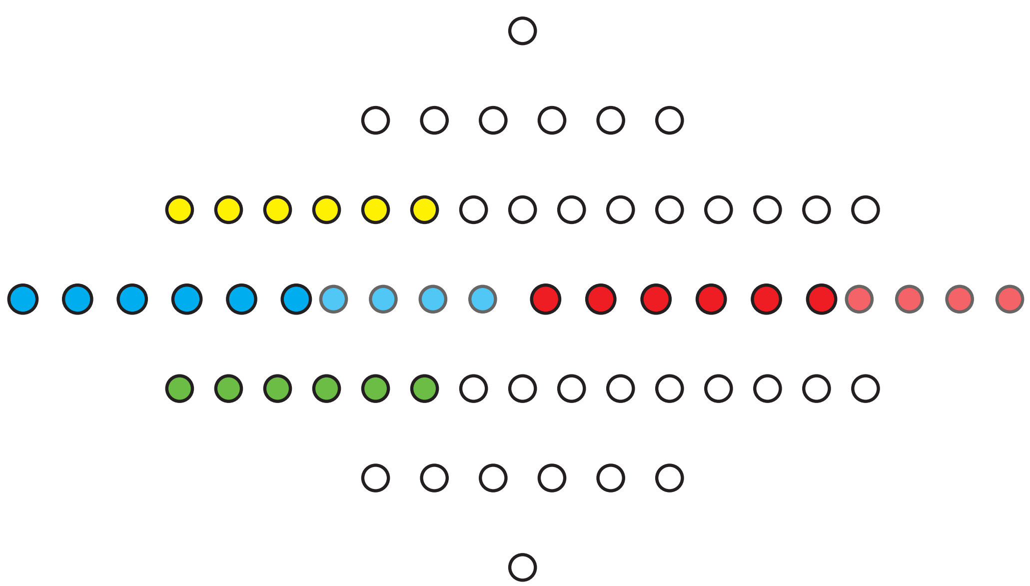
The general solution is parametrized by two arbitrary functions of two variables (leaving in a plane $\beta^0 = Cte$) from which we compute (via an Euler-Darboux-Poisson equation) the Cauchy data for the six k_{ab}, which then propagate thanks to the six evolution equations : $\partial_0 k_{ab} = \ldots$.

They also may be described as modes superposition: far from the walls (in terms of plane waves), or when bouncing on a wall, far from the corners (in terms of special functions of Legendre or Kummer).

- Level 3 : A similar analysis can be done. The 20 components of $\Psi = \frac{1}{\sqrt{2}} \sum_\epsilon \frac{1}{3!} f^\epsilon \eta_{pqr} \tilde{b}^p_\epsilon \tilde{b}^q_\epsilon \tilde{b}^r_\epsilon + \frac{1}{2} h^\epsilon_{pq,r} \tilde{b}^p_{-\epsilon} \tilde{b}^q_{-\epsilon} \tilde{b}^r_\epsilon$ split into 10 +10 that decouple. Defining the dual components $h^\epsilon{}_{ab} = \frac{1}{2} \eta_a{}^{pq} h^\epsilon_{pq,b}$, all modes may be expressed in terms of $h^\epsilon{}_{(ab)}$.

Hilbert Space Structure

The solution space diamond :

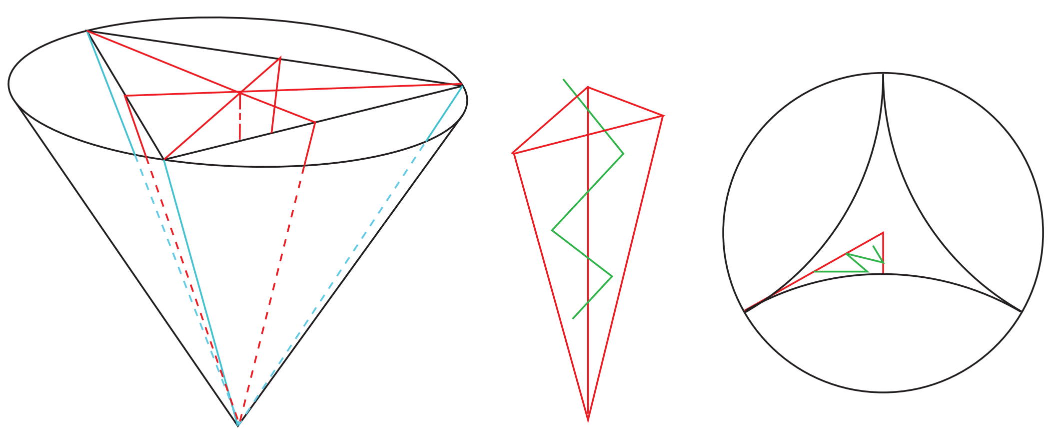


C_F	dim
3	1
2	6
1	15
0	20=(3+1+6)+(3+1+6)
-1	15=6+3+3+3
-2	6=3+3
-3	1

$$\mathcal{V}^{(0)} = V_1^{(0)}, \quad \mathcal{V}^{(1)} = V_2^{(1)}, \quad \mathcal{V}^{(2)} = V_3^{(2)} \oplus V_{1,\infty^2}^{(2)}, \quad \mathcal{V}^{(3)} = V_{2,\infty^2}^{(3)} \oplus V_{2,\infty^2}^{(2)}$$

Quantum Supersymmetric Billiard

The spinorial wave function of the Universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and "reflects" on the walls (= simple roots of AE_3). In the small-wavelength limit, the "reflection operators" define a *spinorial extension of the Weyl group of* AE_3 (Damour, Hillmann '09) defined within some subspaces of $\mathrm{Spin}(8,4)$



$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i\,\frac{\pi}{2}\,\widehat{\varepsilon}_{\alpha_i}\widehat{J}_{\alpha_i}\right)$$

with $\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\}$ and $\widehat{\varepsilon}^2_{\alpha_i} = \mathrm{Id}$

THE "SQUARED-MASS" QUARTIC OPERATOR $\widehat{\mu}^2$ IN $\widehat{H}$

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$) :

$$2\,\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated quartic-in-fermions terms (including $\sum \widehat{S}_{ab}^2$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

- $\widehat{\mu}^2 \in$ *Center* of the algebra generated by the $K(AE_3)$ generators $\widehat{S}_{ab}$, $\widehat{J}_{ab}$

- $\widehat{\mu}^2$ is $\sim$ the square of a very simple operator $\in$ Center

$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2$$

where $\widehat{C}_F := \frac{1}{2}\, G_{ab}\,\overline{\widehat{\Phi}}^a\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^b$.

THE "SQUARED-MASS" QUARTIC OPERATOR $\widehat{\mu}^2$ IN $\widehat{H}$

$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2 = -\frac{59}{8}, -3, -\frac{3}{2}, +\frac{1}{2}$$

Conclusions

• The case studied of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures in supergravity. [Here, AE_3 and $K(AE_3)$]

Conclusions

• The case studied of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures in supergravity. [Here, AE_3 and $K(AE_3)$]

• The wave function of the Universe $\Psi(\beta^1, \beta^2, \beta^3)$ is a 64-dimensional spinor of $\mathrm{Spin}(8,4)$ which satisfies Dirac-like, and Klein-Gordon-like, wave equations describing the propagation of a "quantum spinning particle" reflecting off spin-dependent potential walls which are built from quantum operators $\widehat{S}_{12}, \widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ that generate a 64-dim representation of $K(AE_3)$. The squared-mass term $\widehat{\mu}^2$ in the KG equation belongs to the center of this algebra.

CONCLUSIONS

• The case studied of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures in supergravity. [Here, AE_3 and $K(AE_3)$]

• The wave function of the Universe $\Psi(\beta^1, \beta^2, \beta^3)$ is a 64-dimensional spinor of $\mathrm{Spin}(8,4)$ which satisfies Dirac-like, and Klein-Gordon-like, wave equations describing the propagation of a "quantum spinning particle" reflecting off spin-dependent potential walls which are built from quantum operators $\widehat{S}_{12}, \widehat{S}_{23}, \widehat{S}_{31}$, $\widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ that generate a 64-dim representation of $K(AE_3)$. The squared-mass term $\widehat{\mu}^2$ in the KG equation belongs to the center of this algebra.

• This result might help in clarifying the extent to which the gravity/coset correspondence holds (here for the coset $AE_3/K(AE_3)$, and more interestingly for $E_{10}/K(E_{10})$).

Conclusions

We recover, in the framework of simple (D=4) supergravity, elements of the hidden hyperbolic Kac–Moody structures.

Our dynamical model differs significantly from those obtained in previous works. We obtain modes for all values of N_F (all previous works agreed on the inexistence of solutions for odd values of N_F).

P. D. D'Eath, S. W. Hawking and O. Obregon, Phys. Lett. B **300**, 44 (1993).
P. D. D'Eath, Phys. Rev. D **48**, 713 (1993).
A. Csordas and R. Graham, Phys. Rev. Lett. **74**, 4129 (1995).
A. Csordas and R. Graham, Phys. Lett. B **373**, 51 (1996).
O. Obregon and C. Ramirez, Phys. Rev. D **57**, 1015 (1998).

Conclusions

We recover, in the framework of simple (D=4) supergravity, elements of the hidden hyperbolic Kac–Moody structures.

Our dynamical model differs significantly from those obtained in previous works. We obtain modes for all values of N_F (all previous works agreed on the inexistence of solutions for odd values of N_F).

P. D. D'Eath, S. W. Hawking and O. Obregon, Phys. Lett. B **300**, 44 (1993).
P. D. D'Eath, Phys. Rev. D **48**, 713 (1993).
A. Csordas and R. Graham, Phys. Rev. Lett. **74**, 4129 (1995).
A. Csordas and R. Graham, Phys. Lett. B **373**, 51 (1996).
O. Obregon and C. Ramirez, Phys. Rev. D **57**, 1015 (1998).

In the short wave limit (wave packets), some components of the wave function of the Universe behave like tachyonic particle. They may "bounce near the past", and thus escape from the singularity.