TWO "DOUBLE STRING" THEORY ACTIONS:
covariance vs non-covariance

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Portrait of Fra Luca Pacioli
(Jacopo de' Barbari)
Capodimonte Museum, Naples

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Workshop on Quantum Fields and Strings

PLAN OF THE TALK

- Looking for a manifestly T-dual symmetric formulation of String Theory: motivation.
- Simple model: manifestly dual invariant two-dimensional free scalar field. Floreanini-Jackiw Lagrangian and quantization.
- T-duality symmetric string action with the loss of manifest 2D covariance (Tseytlin).
  Quantization.
- T-duality symmetic string action with manifest 2D covariance (Hull).
  Quantization.
- Comparison between the two actions.
- Conclusion and perspectives.

MOTIVATION

In String Theory, the presence of compact dimensions implies the existence of the following modes for (bosonic) closed strings:

momentum modes ⇒ $$p_a = \frac{k_a}{R} \quad , \quad k_a \in Z$$ quantized along the compact directions.

winding modes ⇒ $$X^a(\tau,\sigma+\pi) = X^a(\tau,\sigma) + 2\pi R w^a \; ; \; w^a \in Z$$

dual coordinates
$$\left\{ \begin{array}{l} X^a \; , \; p_a \\ \tilde{X}_a \; , \; w^a \end{array} \right.$$

representing the number of times the string winds around the compact dimension

WHAT PROVIDES THIS "NEW" COORDINATE?

T-DUALITY.

T-duality is an old subject in String Theory. It implies that in many cases two different geometries for the extra-dimensions are physically equivalent.

T-duality is a symmetry for the bosonic closed string theory and, in the case of a compactification on a circle of radius *R*, it is encoded by the following transformations:

$$R \leftrightarrow \frac{\alpha'}{R} \quad ; \quad k \leftrightarrow w$$

leaving the mass spectrum invariant.

The interchange of *w* and *k* means that the momentum excitations in one description correspond to winding mode excitations in the dual description and *viceversa.*

The winding mode *w* becomes the momentum associated with the dual coordinate obtained by a transformation on the string coordinate *X* along the compact dimension: :

$$X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma) \leftrightarrow \tilde{X}(\tau,\sigma) = X_L(\tau+\sigma) - X_R(\tau-\sigma)$$

$$X_L \rightarrow X_L \;;\; X_R \rightarrow -X_R$$

$$\partial_\sigma X = \partial_\tau \tilde{X} \quad ; \quad \partial_\tau X = \partial_\sigma \tilde{X}$$

T-duality symmetry is a clear indication that ordinary geometric concepts can break down in string theory at the string scale.

String Theory compactified on a *d*-dimensional torus admits the T-duality group $O(d,d;Z)$ (Giveon, Porrati, Rabinovici '94)

T-DUALITY $\Longleftrightarrow$ $O(d,d;Z)$

reminescent of the duality $O(d,d)$ appearing already at the classical level in a very natural way in the Hamiltonian formalism and that can be extracted from the string constraints.

$$S = \frac{T}{2}\int_{\Sigma}\left[G_{ab} dX^a \wedge {*dX^b} + B_{ab} dX^a \wedge dX^b \right]$$

$$h = \mathrm{diag}(-1,1)$$

Hodge operator with respect to h

Equation of motion for X^a

$$d * dX^a + \Gamma^a_{bc} dX^b \wedge {*dX^c} = \frac{1}{2} G^{am} H_{mbc} dX^b \wedge dX^c$$

$$H = dB$$

$$\Gamma^a_{bc} = \frac{1}{2} G^{am}(\partial_b G_{mc} + \partial_c G_{mb} - \partial_m G_{bc})$$

coefficients of the Levi-Civita connection on *TM* $\rightarrow$ Tangent space of the target space

The equation of motion for the world-sheet metric implies the vanishing of the energy-momentum tensor:

$$T_{\alpha\beta} = 0 \quad \Rightarrow \quad \left.\begin{array}{c} G_{ab}(\dot{X}^a \dot{X}^b + X'^a X'^b) = 0 \\ G_{ab}\dot{X}^a X'^b = 0 \end{array}\right\} \quad \boxed{\text{constraints}}$$

Canonical momentum conjugate to X^a

$$P_a = \frac{\partial L}{\partial \dot{X}^a} = T(G_{ab}\dot{X}^b + B_{ab}X'^b)$$

The Hamiltonian density can be expressed in terms of a *generalized vector:*

$$TM \oplus T^* M$$

$$A_p = \begin{pmatrix} \partial_\sigma X^a \\ 2\pi\alpha' P \end{pmatrix} \quad \Rightarrow \quad H(X;G,B) = \frac{1}{4\pi\alpha'} A_P^t M A_P$$

2dx2d GENERALIZED METRIC

$$M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

GENERALIZED METRIC

The constraints can be expressed in terms of the generalized vector as well:

this constraint sets the Hamiltonian density to zero

$$A_P^t M A_p = 0 \quad , \quad A_P^t \Omega A_P = 0$$

the constrained dynamics Is completely governed by it

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(F. Rennecke, 2014)

defines the group *O(d,d)*

A matrix *T* is an element of *O(d,d)* if and only if $T^t \Omega T = \Omega$, i.e. if it leaves the matrix Ω invariant.

The generalized metric is an element of *O(d,d)* $\longrightarrow$ $M^t \Omega M = \Omega$

All the admissible generalized vectors solving the constraints are related by *O(d,d)* transformations.

In the presence of constant background, the world-sheet equations of motion for the string coordinates are set of conservation laws on the world-sheet :

$$\partial_\alpha J_a^\alpha = 0 \longrightarrow J_a^\alpha = G_{ab}\partial^\alpha X^b + \varepsilon^{\alpha\beta} B_{ab}\partial_\beta X^b$$

(momentum corrents)

Locally, one can express the currents as:

$$G_{ab}\partial^\alpha X^b + \varepsilon^{\alpha\beta} B_{ab}\partial_\beta X^b \equiv -\varepsilon^{\alpha\beta}\partial_\beta \tilde{X}_a \longrightarrow \text{Dual coordinates}$$

(M. J. Duff, 1990)

$$B = 0 \Rightarrow G_{ab}\partial^\alpha X^b \equiv -\varepsilon^{\alpha\beta}\partial_\beta \tilde{X}_a$$

$$S = \frac{T}{2}\int_\Sigma \left[\tilde{G}_{ab} d\tilde{X}^a \wedge * d\tilde{X}^b + \tilde{B}_{ab} d\tilde{X}^a \wedge d\tilde{X}^b \right]$$

$$\tilde{G} = G - BG^{-1}B \qquad M \to M^{-1} \qquad \tilde{B} = -\tilde{G}B^{-1}G$$

The equations of motion for the coordinates $(X, \tilde{X})$ can be combined into a single equation invariant under $O(d,d)$: $M\partial_\alpha \chi = \Omega \varepsilon_{\alpha\beta}\partial^\beta \chi$ $\qquad \chi \equiv \begin{pmatrix} X \\ \tilde{X} \end{pmatrix}$

For *B=0*, the equations of motion become the duality condition: $\partial_\alpha X^a = -\varepsilon_{\alpha\beta}\partial^\beta \tilde{X}^a$
and the action and its dual are obtained the one from the other under the exchange
$X \leftrightarrow \tilde{X}$ and $G \leftrightarrow \tilde{G} \quad B \leftrightarrow \tilde{B}.$

If the closed string coordinates are defined on a compactified target manifold then the dual coordinates will satisfisfy the same periodicity conditions and then T-duality maps two theories of the same type into one another $\Longrightarrow$ symmetry.

For closed strings, toroidal compactification means:

$$X^a(\sigma,\tau) \equiv X^a(\sigma+2\pi,\tau)+2\pi L^a \qquad L^a = \sum_{i=1}^{d} w_i R_i e_i^a$$

basis on T^d

windings

In the compact space

$$O(d,d) \rightarrow O(d,d\,|\,Z)$$

T-duality group of the toroidal compactification

For compactified closed strings, this group becomes a symmetry not only of the mass spectrum and the vacuum partition function but also of the scattering amplitudes.

MAIN GOAL

To have a more fundamental action of the (closed) bosonic string theory in which T-duality is manifest.

Introduction in the sigma model of both compact coordinates $(X,\tilde{X})$:

such doubling of coordinates leads to a *DOUBLE STRING THEORY.*

SOME CONSIDERATIONS ON THE "DOUBLING"…

The winding modes which appear in the spectrum of a closed string compactified on a torus have to be created by the vertex operators which have to involve $(X, \tilde{X})$

Perturbations of the torus vacuum by such operators may lead (along the RG trajectory) to other possible vacua which correspond to 2-dim field theories with interactions depending on $(X, \tilde{X})$

In order to describe such 2-dim models as local QFT's one has to treat $(X, \tilde{X})$ as independent 2-dim fields which appear to be dual to each other only on the mass shell and only in the absence of interactions.

Starting from the general action $S = S_0[X, \tilde{X}] + S_{\mathrm{int}}[X, \tilde{X}]$ the dynamical equations (RG flow) could determine which particular interactions correspond to string vacua. In that way one may get more vacua than one would find within the standard formulation!

Hence, in the extended formulation, the symmetry under $X \leftrightarrow \tilde{X}$ becomes an off-shell symmetry of the world-sheet action. This makes the duality invariance of the scattering amplitudes and of the effective action manifest.

The correspondence principle with the standard formulation is satisfied since the “free” action S_o still describes *d* and not *2d* degrees of freedom.

If S_{int} does not depend on $\tilde{X}$ then one can integrate it out and reproduce the usual results for the partition function, scattering amplitudes, ecc.

Assuming that the compactification radius $R >> \sqrt{\alpha'}$, then the winding modes are very massive so the relevant interactions are $S_{int}(X)$.

At intermediate scales $R \approx \sqrt{\alpha'}$ the interactions are dependent on both the coordinates.

At small scales $R << \sqrt{\alpha'}$ the relevant interactions are $S_{\text{int}}(\tilde{X})$.

THE T-DUALITY SYMMETRIC FORMULATION MAY BE CONSIDERED AS A NATURAL GENERALIZATION OF THE STANDARD ONE AT THE STRING SCALE.

If interested in writing down the complete effective field theory of a compactified bosonic closed string, one has to include both *momentum* excitations and *winding* excitations or, equivalently, X^a *and* $\tilde{X}_a$.

The fields associated with the string states will depend on

$$X^i = (X^a, \tilde{X}_a, X^\mu) .$$

The effective closed string field theory would look like:

$$S = \int dX^a d\tilde{X}_a \, dX^\mu \, L\left(X^a, \tilde{X}_a, X^\mu\right).$$

Hence, the DOUBLE STRING effective field theory is a DOUBLE FIELD THEORY which should provide a T-duality invariant formulation of Supergravity giving a way to go beyond the usual Supergravity limit of String Theory by introducing a mere stringy feature into low-energy physics (C. Hull and B. Zwiebach, 2009).

Also: String description of non-geometric backgrounds in which the transition functions between different patches involve not only diffeomorphims but also duality transformations.

SIMPLE MODEL: THE FREE SCALAR FIELD THEORY in 2D

A general free scalar field in 2D Minkowski space described by the usual Lagrangian density:

$$L = -\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi = \frac{1}{2}\dot{\phi}^2 - \phi'^2 \qquad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

has a dual defined by: $\partial_\alpha\partial^\alpha\phi = 0 \rightarrow \quad \partial_\alpha\phi = -\varepsilon_{\alpha\beta}\,\partial^\beta\tilde{\phi} \qquad \varepsilon^{01} = 1$

with

$$\partial_\sigma\phi = \partial_\tau\tilde{\phi}\;,\; \partial_\tau\phi = \partial_\sigma\tilde{\phi} \qquad \longrightarrow \qquad \partial_\sigma X = \partial_\tau\tilde{X} \quad ; \quad \partial_\tau X = \partial_\sigma\tilde{X}$$

$$*d\phi = d\tilde{\phi}$$

It is possible to rewrite *L* in such a way that the two fields appear on an equal footing, with manifest invariance under: $\phi \leftrightarrow \tilde{\phi}$.

Procedure

- First step

Rewrite L in a “first-order” form:

$$L[p,\phi] = p\dot{\phi} - \frac{1}{2}p^2 - \frac{1}{2}\phi'^2 \quad \Rightarrow \quad p = \dot{\phi}$$

auxiliary field

- Second step

Introduce a Lagrange multiplier b $\Rightarrow$ $p = \tilde{\phi}'$

$$L[p,\phi] = p\dot{\phi} - \frac{1}{2}p^2 - \frac{1}{2}\phi'^2 + b(p - \tilde{\phi}')$$

$$L[\phi,\tilde{\phi}] = \dot{\phi}\tilde{\phi}' - \frac{1}{2}\tilde{\phi}'^2 - \frac{1}{2}\phi'^2$$

$$L_{sym} = \frac{1}{2}\dot{\phi}\,\tilde{\phi}' + \frac{1}{2}\phi'\,\dot{\tilde{\phi}} - \frac{1}{2}\phi'^2 - \frac{1}{2}\tilde{\phi}'^2$$

invariant under

$$\phi \leftrightarrow \tilde{\phi}$$

equations of motion:

$$\phi \rightarrow \partial_\sigma\left[\partial_\sigma\phi - \partial_\tau\tilde{\phi}\right] = 0 \rightarrow \partial_\sigma\phi - \partial_\tau\tilde{\phi} = f(\tau)$$

$$\tilde{\phi} \rightarrow \partial_\sigma\left[\partial_\sigma\tilde{\phi} - \partial_\tau\phi\right] = 0 \rightarrow \partial_\sigma\tilde{\phi} - \partial_\tau\phi = \tilde{f}(\tau)$$

The invariances under

$$\phi \rightarrow \phi + g(\tau)$$

$$\tilde{\phi} \rightarrow \tilde{\phi} + \tilde{g}(\tau)$$

allow to gauge away $f(\tau) = \tilde{f}(\tau) = 0$

$$\partial_\sigma\phi = \partial_\tau\tilde{\phi}\ ,\ \partial_\tau\phi = \partial_\sigma\tilde{\phi}$$

They are dual to each other on the mass-shell

$$\partial_\alpha\phi = -\varepsilon_{\alpha\beta}\,\partial^\beta\tilde{\phi} \quad \longleftrightarrow \quad \partial_\alpha X^a = -\varepsilon_{\alpha\beta}\partial^\beta\tilde{X}^a$$

(when *B=0*)

Furthermore L_{sym} can be diagonalized by introducing a pair of “new” scalar fields:

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_+ + \phi_-\right)$$

$$\tilde{\phi} = \frac{1}{\sqrt{2}}\left(\phi_+ - \phi_-\right)$$

$$L_{sym}(\phi,\tilde{\phi}) = L_+(\phi_+) + L_-(\phi_-)$$

$$L_\pm(\phi_\pm) = \pm\frac{1}{2}\dot{\phi}_\pm \phi'_\pm - \frac{1}{2}\phi'^2_\pm$$

No Lorentz invariance!

$\boxed{\text{Floreanini-Jackiw density Lagrangians for chiral and antichiral fields}}$

(Floreanini, Jackiw 1973)

Equations of motion

$$\phi_+ \to \partial_\sigma\left[\dot{\phi}_+ - \phi'_+\right] = 0$$

$$\phi_- \to \partial_\sigma\left[\dot{\phi}_+ + \phi'_-\right] = 0$$

$$\dot{\phi}_+ - \phi'_+ = f(\tau)$$

$$\dot{\phi} + \phi'_- = \tilde{f}(\tau)$$

$$\phi_+ \to \phi_+ + g(\tau)$$

$$\phi_- \to \phi_- + \tilde{g}(\tau)$$

$$\dot{\phi}_+ = \phi'_+$$

$$\dot{\phi}_- = -\phi'_-$$

on-shell they become respectively functions of

$$\sigma \pm \tau$$

chiral, anti-chiral

SYMMETRIES

L_{sym} , $L_{\pm}$ invariant under space-time translations $\delta_\tau \phi = \dot{\phi}$, $\delta_\sigma \phi = \phi'$

$L_{\pm}$ invariant under

$\delta\phi_{\pm} = f(\tau \pm \sigma)\phi'_{\pm}$ → "conformal"

$\delta\phi_{\pm} = \tau\dot{\phi}_{\pm} + \sigma\phi_{\pm}'$ → "scale"

$\delta_L\phi_{\pm} = (\tau \pm \sigma)\phi'_{\pm}$ → generalized Lorentz transformations

L_{sym} invariant under $\delta_L \phi = \tau\phi' + \sigma\tilde{\phi}'$, $\delta_L \tilde{\phi} = \tau\tilde{\phi}' + \sigma\phi'$

on-shell $\delta\phi = \tau\,\phi' + \sigma\,\dot{\phi}$

usual 2-dim Lorentz rotation

Local Lorentz invariance is recovered on-shell.

The FJ Lagrangians belong to a general class of first-order Lagrangians:

$$L = \frac{1}{2} q^i c_{ij} \dot{q}^j - V(q) \qquad i,j = 1,\ldots,N \qquad \det c_{ij} \neq 0$$

characterized by *N* primary constraints $\quad T_j \equiv p_j - \frac{1}{2} q^i c_{ij} \approx 0$

canonically conjugate momentum to q^j .

$$Q_{jk} \equiv \{T_j, T_k\}_{PB} = c_{jk} \neq 0$$

$\boxed{\text{All constraints are second class.}}$

Dirac bracket for any two functions of the phase-space variables:

$$\{f, g\}_{DB} \equiv \{f, g\}_{PB} - \{f, T_j\}_{PB} (Q^{-1})_{jk} \{T_k, g\}_{PB}$$

Transition to the quantum theory: $\quad i\{f, g\}_{DB} \rightarrow [f, g]$

$$[\phi_\pm(\sigma,\tau),\ \phi_\pm(\sigma',\tau')] = \pm i \frac{\varepsilon}{2} (\sigma - \sigma')$$

$\phi_\pm$ or $\phi, \tilde{\phi}$

$$[q_i, q_j] = i(c^{-1})_{ij}$$

$$[q_i, p_j] = \frac{1}{2} i \delta_{ij}$$

$$[p_i, p_j] = -\frac{1}{4} i c_{ij}$$

behave like "non-commuting" phase-space type coordinates.

This theory has all the characteristics that a T-duality invariant action should have in the absence of the Kalb-Ramond field

STRING GENERALIZATION

Non-covariant action *(Tseytlin)*

(Tseytlin, 1991)

first-order time derivatives

$$S\left[e^a_\alpha,\chi^i\right]=-\frac{1}{2}\int d^2\xi\, e\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j+M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]$$

$i,j=1,\ldots,N$

$$\nabla_a\chi^i\equiv e^\alpha_a\,\partial_\alpha\chi^i\ ;\ e\equiv\det e^a_\alpha$$

$$C_{ij}=C_{ji}\ ;\ M_{ij}=M_{ji}$$

$$\xi^0=\tau\ ;\quad \xi^1=\sigma$$

e^a_α: a → flat index; α → curved index

invariant

- under diffeomorphisms

$$\xi^\alpha\rightarrow\xi'^\alpha(\xi)$$

acting as

$$\chi'^i(\xi'^\alpha)=\chi^i(\xi)\ ;\ e'^a_\alpha=e^a_\beta\frac{\partial\xi^\beta}{\partial\xi'^\alpha}$$

- under Weyl transformations:

$$e^a_\alpha\rightarrow e'^a_\alpha=\lambda(\xi)e^a_\alpha$$

Not invariant under the group SO(1,1) of local Lorentz transformations. But such invariance has to hold since physical observables are independent on the choice of the vielbein:

SO(1,1) matrix

$$e'^{a}_{\alpha} \to \Lambda^{a}_{b}(\xi) e^{b}_{\alpha}$$

$$\delta e^{a}_{\alpha} = \alpha(\xi) \varepsilon^{a}_{b} e^{b}_{\alpha}$$

Requirement of on-shell local Lorentz invariance has to be made:

$$\delta_e S = \alpha(\xi) \frac{\delta S}{\delta e^{a}_{\alpha}} \varepsilon^{a}_{b} e^{b}_{\alpha} = 0 \text{ if } \varepsilon^{ab} t_{ab} = 0$$

with

$$t_{a}^{\ b} = -\frac{2}{e} \frac{\delta S}{\delta e^{a}_{\alpha}} e^{b}_{\alpha}$$

condition to be imposed

$\boxed{\text{Weyl invariance}}$ $\Rightarrow$ $\mathrm{Tr}\left[t_a^{\ b}\right] = 0$

Equations of motion for $e_a^{\ \alpha}$ $\rightarrow$ $t_a^{\ b} = 0$ constraints to be imposed at classical and quantum levels

analogously to what happens in the ordinary formulation $T_{\alpha\beta} = -\dfrac{2}{T\sqrt{g}}\dfrac{\delta S}{\delta g^{\alpha\beta}} = 0$

$\Rightarrow$ on the solutions of these equations the local Lorentz invariance holds.

Reparametrization + Weyl + Local Lorentz inv. $\longrightarrow$ gauge choice for $e^a_{\ \alpha}$

In particular, $e^a_{\ \alpha} = \delta^a_{\ \alpha}$ $\boxed{\text{flat gauge}}$

Equations of motion with *C* and *M* costant for χ^i

$$\partial_1\left[C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right] = 0$$

$$C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j = g_i(\tau)$$

arbitrary function

Further local gauge symmetry of the action:

$$\chi^i \to \chi'^i = \chi^i + f^i(\tau,\sigma) \quad \text{with} \quad \nabla_1 f^i = 0 \quad \Rightarrow \quad \text{fix} \quad g = 0$$

$$C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j = 0$$

Boundary conditions dictated by:

surface integral $\displaystyle \frac{1}{2}\int_{-\infty}^{+\infty} d\tau C_{ij}\left[\partial_0\chi^j\delta\chi^i\right]\Big|_{\sigma=0}^{\sigma=\pi} = 0$

open strings Dirichlet b.c. $\partial_0\chi^i = 0 \quad \sigma = 0,\pi$

closed strings $\chi^i(\tau,\sigma) = \chi^i(\tau,\sigma+\pi) + k^i$

The constraint coming from the vanishing of the ε-trace of t_{ab} can be explicitly solved. along the solutions of the equations of motion:

$$\varepsilon^{ab}\, t_{ab} = 0 = \nabla_1 \chi^i (C - MC^{-1}M)_{ij} \nabla_1 \chi^j$$

In the flat gauge and along the solutions of the eqs. of motion for χ^i

$$C = MC^{-1}M$$

The matrix C can be always put in the following form after suitably rotating and rescaling χ^i :

$$C = \mathrm{diag}\,(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q}) \qquad \Rightarrow C = C^{-1}$$

$$C = MCM$$

defines the indefinite orthogonal group *O(p,q)* of *NxN* matrices *M* with *N=p+q.*

$$S = -\frac{1}{2}\int d^2\xi e\left[\sum_{\mu=1}^{p}\nabla_0\chi_-^\mu \nabla_1\chi_-^\mu - \sum_{\nu=1}^{q}\nabla_0\chi_+^\nu \nabla_1\chi_+^\nu + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]$$

p two-dimensional chiral scalar fields
q two-dimensional antichiral scalar fields

reproducing F-J Lagrangians.

Requiring the absence of a quantum Lorentz anomaly implies *p=q=d* with *2d=N.*

C becomes the *O(d,d)* metric in the *2d*-dimensional target space with coordinates

$$\chi^j = (\chi_+^\mu, \chi_-^\mu) \longrightarrow ds^2 = d\chi^i C_{ij} d\chi^j$$

S describes a mixture of *D* chiral scalars and *D* antichiral scalars which are the components of

$$\chi^i = \begin{pmatrix} \chi_-^\mu \\ \chi_+^\nu \end{pmatrix} \qquad i = 1, \ldots, 2d \qquad \mu, \nu = 1, \ldots, d$$

$$C_{ij}\nabla_0\chi^j + M_{ij}\nabla_1\chi^j = 0$$

$$C = MC^{-1}M$$

can be put in a covariant form:

$$-\varepsilon_{ab}C_{ij}\nabla^b\chi^j + M_{ij}\nabla_a\chi^j = 0$$

constraints imposed in the covariant formulation.

In the "non-chiral" basis $\chi^i = \begin{pmatrix} X^\mu \\ \tilde{X}_\mu \end{pmatrix}$

$$X^\mu = \frac{1}{\sqrt{2}}\left(\chi_+^\mu + \chi_-^\mu\right)$$

$$\tilde{X}_\mu = \frac{1}{\sqrt{2}}\delta_{\mu\nu}\left(\chi_+^\nu - \chi_-^\nu\right)$$

$$C_{ij} = -\Omega_{ij} \equiv -\begin{pmatrix} 0_{\mu\nu} & \mathrm{I}_\mu{}^\nu \\ \mathrm{I}^\mu{}_\nu & 0^{\mu\nu} \end{pmatrix}$$

$$C = MC^{-1}M \Rightarrow M^{-1} = \Omega^{-1}M\Omega^{-1}$$

M parametrized by $d^2 = \frac{d(d+1)}{2} + \frac{d(d-1)}{2}$ independent elements

Symmetric *d*-dim matrix *G*

Antisymmetric *d*-dim matrix *B*

$$M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

generalized 2*d*x2*d* metric
(defined up to a sign)

element of the *O(d,d)* group.

$$S = \frac{1}{2}\int d\xi e\left[\Omega_{ij}\nabla_0\chi^i\nabla_1\chi^j - M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]$$

invariant under

$$\chi' = \Re\chi \qquad M' = \Re^{-t}M\Re^{-1} \qquad \Re^t C\Re = C \qquad \Re\in O(d,d)$$

background trasformation

In particular, for $\Re = \Omega$

$$S = -\frac{1}{2}\int d^2\xi\, e\Big[\nabla_0 X^\mu\nabla_1\tilde{X}_\mu + \nabla_0\tilde{X}^\mu\nabla_1 X_\mu - (G - BG^{-1}B)_{\mu\nu}\nabla_1 X^\mu\nabla_1 X^\nu$$
$$-(BG^{-1})_\mu{}^\nu\nabla_1 X^\mu\nabla_1\tilde{X}_\nu + (G^{-1}B)^\mu{}_\nu\nabla_1\tilde{X}_\mu\nabla_1\tilde{X}^\nu - (G^{-1})^{\mu\nu}\nabla_1\tilde{X}_\mu\nabla_1\tilde{X}^\nu\Big]$$

one gets the T-duality invariance under

$$X \Leftrightarrow \tilde{X} \quad \text{and} \quad M \Leftrightarrow M^{-1}$$

$$S = -\frac{T}{2}\int d^2\xi\, e\left[C_{ij}\,\nabla_0\chi^i\,\nabla_1\chi^j + M_{ij}\,\nabla_1\chi^i\,\nabla_1\chi^j\ \right]$$

$$T = \frac{1}{2\pi l^2}$$

It exhibits a manifest T-duality invariance *O(d,d)*.

It is candidate to describe a bosonic string in a constant background, made of *G* and *B,* and compactified on a torus T^{d}.

χ^i interpreted as string coordinates on a double torus T^{2d} defined by

$$X^{\mu}(\tau,\sigma+\pi) = X^{\mu}(\tau,\sigma) + 2\pi l\omega^{\mu}$$

$$\tilde{X}_{\mu}(\tau,\sigma+\pi) = \tilde{X}_{\mu}(\tau,\sigma) + 2\pi l^2 p_{\mu}$$

$\begin{pmatrix}\omega^{\mu}\\ lp_{\mu}\end{pmatrix}$ being a vector spanning a Lorentzian lattice $\Lambda^{d.d}$

on the torus $O(d,d) \longrightarrow O(d,d;Z)$

In order to reconduce the action to a sum of Floreanini-Jackiw Lagrangians, it is convenient to block-diagonalize simultaneously C and M through the matrix:

$$(\boldsymbol{T}^{-1})^{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} (G^{-1})^{\mu\nu} & (G^{-1})^{\mu\nu} \\ (-E^t G^{-1})_\mu{}^\nu & (EG^{-1})_\mu{}^\nu \end{pmatrix} \qquad E = G + B$$

$$C \to \boldsymbol{C}^{-1} \equiv \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix} = \boldsymbol{T}^{-t} C \boldsymbol{T}^{-1} \qquad M \to \boldsymbol{G}^{-1} \equiv \begin{pmatrix} G^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix} = \boldsymbol{T}^{-t} M \boldsymbol{T}^{-1}$$

generalized metric

introducing new coordinates in which the R and the L sectors are decoupled even at the presence of B:

$$\Phi_i = \boldsymbol{T}_{ij}\chi^j \equiv (X_{R\mu}, X_{L\mu})$$

they satisfy the equations of motion:

$$*dX_R = -dX_R \quad ; \quad *dX_L = dX_L$$

with the identifications:

$$X_R[\tau - (\sigma + \pi)] = X_R(\tau - \sigma) - 2\pi l^2 p_R$$
$$X_L[\tau + (\sigma + \pi)] = X_L(\tau + \sigma) + 2\pi l^2 p_L$$

$$\begin{pmatrix} -lp_R \\ lp_L \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} w \\ lp \end{pmatrix} \qquad (p_R, p_L) \in \Lambda^{(d,d)}$$

Lorentzian lattice

The solutions of the (duality) equations of motions with the torus identification are:

$$X_L(\tau+\sigma) = x_L + 2l^2 p_L(\tau+\sigma) + il\sum_{n\neq 0}\frac{\overline{\alpha}_n}{n}e^{-2in(\tau+\sigma)}$$

$$X_R(\tau-\sigma) = x_R + 2l^2 p_R(\tau-\sigma) + il\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-2in(\tau-\sigma)}$$

formally identical to the usual expansion of the right and left coordinates.

It is convenient to introduce the world-sheet light-cone coordinates:

$$\sigma^+ = \tau+\sigma \ ; \ \sigma^- = \tau+\sigma$$

in terms of which the components of the tensor *t* turn out to be:

$$t_{++} = \partial_+ X_R^t G^{-1}\partial_+ X_R + \partial_+ X_L^t G^{-1}\partial_+ X_L - 2\partial_+ X_L^t G^{-1}\partial_- X_L$$

$$t_{--} = \partial_- X_R^t G^{-1}\partial_- X_R + \partial_+ X_L^t G^{-1}\partial_- X_L - 2\partial_+ X_R^t G^{-1}\partial_- X_R$$

$$t_{+-} = -\frac{1}{4}\varepsilon^{ab}t_{ab} = \partial_- X_L^t G^{-1}\partial_- X_L - \partial_+ X_R^t G^{-1}\partial_+ X_R$$

Weyl invariance imposes:

$$t_{+-} = -t_{-+}$$

On-shell t_{+-} =0 while t_{++}, t_{--} look like the light-components of the energy-momentum tensor of the standard formulation leading to the Virasoro algebra.

QUANTIZATION

The Lagrangian is linear in time derivative, then the conjugate momenta define the primary constraints.

$$S = \frac{T}{2}\int d^2\xi\left[L_R + L_L\right]$$

Flat gauge

$$L_{L,R} = \pm\frac{1}{2}\partial_0 X^t_{L,R}\, G^{-1}\,\partial_1 X_{L,R} - \frac{1}{2}\partial_1 X^t_{L,R}\, G^{-1}\,\partial_1 X_{L,R}$$

$$P_R \equiv -\frac{T}{2}G^{-1}\partial_1 X_R$$

$$P_L \equiv \frac{T}{2}G^{-1}\partial_1 X_L$$

Linear in the time derivative

primary constraints

$$\Psi_R(P_R, X_R) = P_R + \frac{T}{2}G^{-1}\partial_1 X_R \approx 0$$

$$\Psi_L(P_L, X_L) = P_L - \frac{T}{2}G^{-1}\partial_1 X_L \approx 0$$

satisfying the equal “time” algebra

$$\left\{\Psi_{L,R}(\tau,\sigma), \Psi^t{}_{L,R}(\tau,\sigma')\right\}_{PB} = \pm T G^{-1}\delta'(\sigma - \sigma')$$

second class constraints

The Dirac constraint analysis has to be applied.

By analogy with the standard procedure followed in string theory, all the constraints are evaluated on the solutions of the equations of motion for (X_R, X_L)

The constraint $t_{+-} \approx 0$ is already satisfied on them

while the others become

$$\Psi_R = P_R - \frac{T}{2} G^{-1} \partial_- X_R \approx 0 \quad ; \quad \Psi_L = P_L - \frac{T}{2} G^{-1} \partial_+ X_L \approx 0$$

$$t_{++} = \partial_+ X_L^t G^{-1} \partial_+ X_L \approx 0 \quad ; \quad t_{--} = \partial_- X_R^t G^{-1} \partial_- X_R \approx 0$$

These coincide with the components of the energy-momentum tensor of the bosonic string theory.

On the equations of motion the constraints algebra becomes:

$$\left\{ \Psi_R(\tau,\sigma), t_{--}(\tau,\sigma') \right\}_{PB} = \delta'(\sigma - \sigma') G^{-1} \partial_- X_R(\tau - \sigma) \approx 0$$

$$\left\{ \Psi_L(\tau,\sigma), t_{++}(\tau,\sigma') \right\}_{PB} = \delta'(\sigma - \sigma') G^{-1} \partial_+ X_L(\tau - \sigma) \approx 0$$

coming from $t_{\pm\pm} \approx 0$

The presence of second class constraints leads to the introduction of Dirac brackets.

$$\left\{.\,,.\right\}_{DB}=\left\{.\,,.\right\}_{PB}-\int d\sigma d\sigma'\left\{.\,,\Psi^{t}_{R;L}\right\}_{PB}\left\{\Psi_{R;L},\Psi'_{R;L}\right\}^{-1}\left\{\Psi'_{R;L},.\right\}$$

where

$$\Psi_{R;L}\equiv\Psi_{R;L}(\sigma,\tau)\qquad\Psi'_{R;L}\equiv\Psi'_{R;L}(\sigma',\tau)$$

$$\left\{X_{L,R}(\tau,\sigma)\;,\;X^{t}{}_{L,R}(\tau,\sigma')\right\}_{DB}=\pm\frac{G}{T}\varepsilon(\sigma-\sigma')$$

$$\left\{P_{L,R}(\tau,\sigma)\;,\;X^{t}{}_{L,R}(\tau,\sigma')\right\}_{DB}=-\frac{1}{2}\delta(\sigma-\sigma')I$$

$$\left\{P_{R,L}(\tau,\sigma)\;,\;P^{t}{}_{R.L}(\tau,\sigma')\right\}_{DB}=\pm\frac{T}{4}G^{-1}\delta'(\sigma-\sigma')$$

$$\left\{P_{L,R}(\sigma),X^{t}_{L,R}(\sigma')\right\}_{PB}\equiv\begin{bmatrix}\left\{P^{1}_{L,R}(\sigma),X_{L,R\,1}(\sigma')\right\} & \cdots & \left\{P^{1}_{L,R}(\sigma),X_{L,R\,D}(\sigma')\right\}\\ \cdots & \cdots & \cdots\\ \left\{P^{D}_{L,R}(\sigma),X_{L,R\,1}(\sigma')\right\} & \cdots & \left\{P^{D}_{L,R}(\sigma),X_{L,R\,D}(\sigma')\right\}\end{bmatrix}$$

The double world-sheet sigma-model is quantized by the usual replacement:

$$\{.,.\}_{DB} \rightarrow -i[.,.]$$

$$\left[X(\tau,\sigma)\,,\,\tilde{X}^t(\tau,\sigma')\right] = \frac{i}{T}\varepsilon(\sigma-\sigma')I$$

non-commuting tori

The commutators satisfied by the Fourier modes of the coordinates turn out to be the standard ones in the usual string formulation:

$$\left[p_{R,L}\,,\,x^t{}_{R,L}\right] = iG\;\;;\;\left[\alpha_n\,,\,\alpha^t{}_m\right] = \left[\tilde{\alpha}_m\,,\,\tilde{\alpha}^t_n\right] = mG\delta_{m+n,0}\;;\;\left[\alpha_m\,,\,\tilde{\alpha}_n\right] = 0$$

$$\alpha_0 \equiv lp_R\;,\tilde{\alpha}_0 \equiv lp_L$$

These relations can be used in the algebra generated by the constraints $t_{\pm\pm}$ and one recovers, in the R,L-sectors the usual Virasoro algebra with vanishing quantum conformal anomaly in

$\Rightarrow$ d=26.

For this non-covariant T-dual manifest action one can conclude that:

DOUBLE STRING THEORY IS AN EXTENSION OF THE USUAL BOSONIC STRING THEORY, WITH THE MAIN DIFFERENCE GIVEN BY THE DOUBLING OF THE COMPACT DIMENSIONS DUE TO THE PRESENCE OF WINDING MODES.

$\Downarrow$

THE STRING COORDINATES ON T^{2d} ARE NON-COMMUTING PHASE SPACE BUT GENERATE THE USUAL COMMUTATION RELATION FOR THE FOURIER MODES.

HULL COVARIANT ACTION

(Hull, 2005)

Sigma-model defined by the coordinates:

$$(Y(\tau,\sigma),\chi(\tau,\sigma))$$

non compact

$$Y \equiv (Y^I) \quad I = 1,..,D-1$$

$$\chi \equiv (\chi^j) \quad j = 1,...,2d$$

mapping the world-sheet in the target space locally as

$$R^{1,D-1} \otimes T^{2d}$$

$$S = -\frac{T}{4}\int M_{ij}(Y) d\chi^i \wedge {}^* d\chi^j$$

The action, when supplemented by the torus identifications, is manifestly invariant under *GL(2d;Z)* group.

Since the number of the coordinates on the torus has been doubled, a self-duality constraint halving them has to be imposed.

$${}^*M_{ij}\, d\chi^j = -\Omega_{ij}\, d\chi^j \qquad \Omega_{ij} \equiv \begin{pmatrix} 0_{\mu\nu} & I_\mu{}^\nu \\ I^\mu{}_\nu & 0^{\mu\nu} \end{pmatrix}$$

O(d,d) invariant metric

The constraint breaks the *GL(2d;Z)* invariance to *O(d,d;Z)* invariance and it coincides with the ε-trace constraint of the non-covariant action. This latter also implies the eqs. of motion of the non-covariant action.

Energy-momentum tensor:

$$T_{\alpha\beta} = -\frac{4}{T\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = \frac{1}{2}\partial_{(\alpha}\chi^t M \partial_{\beta)}\chi - \frac{1}{2}g_{\alpha\beta}\partial_\gamma \chi^t M \partial^\gamma \chi = 0$$

Weyl invariance $\longrightarrow$ $\mathrm{tr}\, T_{\alpha\beta} = 0$

+

Invariance under reparametrizations $\Longrightarrow$ gauge fix $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$

Equations of motion for χ

$$d * (M d\chi) = 0$$ (satisfied on the constraint surface)

with boundary conditions given by the vanishing of the surface integral:

$$-\frac{T}{2}\int d\tau \delta\chi^t M \partial_1 \chi \Big|_{\sigma=0}^{\sigma=\pi}$$

satifisfied by periodicity conditions, peculiar of closed strings.

In analogy with the non-covariant formulation, one can introduce right and left coordinates:

$$\Phi_i = \left[(X_R)_m, (X_L)_m\right] = \boldsymbol{T}_{ij}\chi^j \ , \ \boldsymbol{T}_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} E^t & -I \\ E & I \end{pmatrix}$$

In these coordinates the generalized metric is:

$$\boldsymbol{G}^{-1} = \begin{pmatrix} G^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix}$$

and the action becomes:

$$S = -\frac{T}{4}\int d\Phi^t\, \boldsymbol{G}^{-1} \wedge *d\Phi$$

In this frame any dependence on *B* disappears

In this new basis the constraints imposed by hand are the “duality” conditions:

$$\frac{2}{T}\Psi_R \equiv dX_R + *dX_R = 0 \ ; \ \frac{2}{T}\Psi_L \equiv dX_L - *dX_L = 0$$

identical to the second-class constraints in the non-covariant approach and so they satisfy the same algebra.

DYNAMICS DETERMINED BY DIRAC BRACKETS

The previous constraints can be incorporated in the action following PST

(Pasti, Sorokin and Tonin, Phys. Rev. D55 (1997) 6292)

$$S = -\frac{T}{4}\int d\Phi^t \boldsymbol{G}^{-1} \wedge * d\Phi + \frac{1}{T}\int d^2\sigma \frac{u^\alpha}{u^2}\Psi^t_\alpha \boldsymbol{G}^{-1}\Psi_\beta u^\beta$$

introducing an auxiliary one-form field

$$u_\alpha = \partial_\alpha a$$

$$\Psi \equiv (\Psi_R, \Psi_L)$$

auxiliary scalar (Stueckelberg) field

The action is invariant under the local transformations:

$$\delta u_\alpha = \partial_\alpha \varphi \Leftrightarrow \ \delta a = \varphi \ ; \ \delta\Phi = \frac{2}{T}\frac{\varphi}{u^2} u_\alpha \Psi^\alpha$$

These symmetries allow to choose the gauge $\Rightarrow$ $u_\alpha = \delta^0_\alpha \qquad u_\alpha u^\alpha = -1$

In this gauge the PST action reproduces the non-covariant!

$\boxed{\text{Equivalence between the covariant and non-covariant approach}}$

This gauge breaks the Lorentz invariance of the action.

However, there exists a linear combination of Lorentz and gauge trasnformations preserving the gauge and transforming the coordinate as:

$$\delta\phi = v\xi_\alpha \varepsilon^{\alpha\beta}\partial_\beta\phi + \frac{2v\sigma}{T}\frac{u_\alpha\Psi^\alpha}{u_\alpha u^\alpha}$$

generalized Lorentz transformations

infinitesimal Lorentz transformation with parameter *v*

infinitesimal gauge rotation with $\phi = v\sigma$

- The energy-momentum tensor on the surface constraint coincides with the one of the bosonic string.
- The Dirac brackets between the canonical coordinates coincide with the ones of the non-covariant approach.
- The two double string theory actions are equivalent both at the classical and quantum level.

CONCLUSION AND PERSPECTIVES

◆ Two T-duality manifest formulations of string theory have been compared. In absence of interaction, they are equivalent at classical and quantum level providing a generalization of the usual bosonic string theory.

◆ A doubling of the coordinates in the compact space is required and the quantization requires a non-commuting geometry.

Interesting to explore:

- Interaction.
- Quantization of the open sector.
- Supersymmetric extension.
- Supersymmetric formulation of Double Field Theory.
- Connection with Generalized Complex Geometry.

WORTH TO CONTINUE….