Lattice gauge theory insights

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Quantum Chromo Dynamics (QCD)

with **action for quarks and gluons**:

$$S = \int \mathrm{d}^4x \left\{ \frac{1}{4}\mathrm{tr} F_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_\mathrm{f}} \bar{q}_f [i\gamma D + m_f] q_f \right\}$$

is a **promising candidate theory of the strong interactions**

Perturbative Asymptotic Freedom (*)

→**many HE (hard) processes can be computed by renormalized perturbation theory and are in good agreement with experiment**

(*) Nobel Prize 2004, Gross, Politzer, Wilczek

- **Does pure Yang-Mills theory have a mass gap?**
- **Are quarks and gluons confined?**

US$ 1 Million Prize by Cray Mathematics Institute!

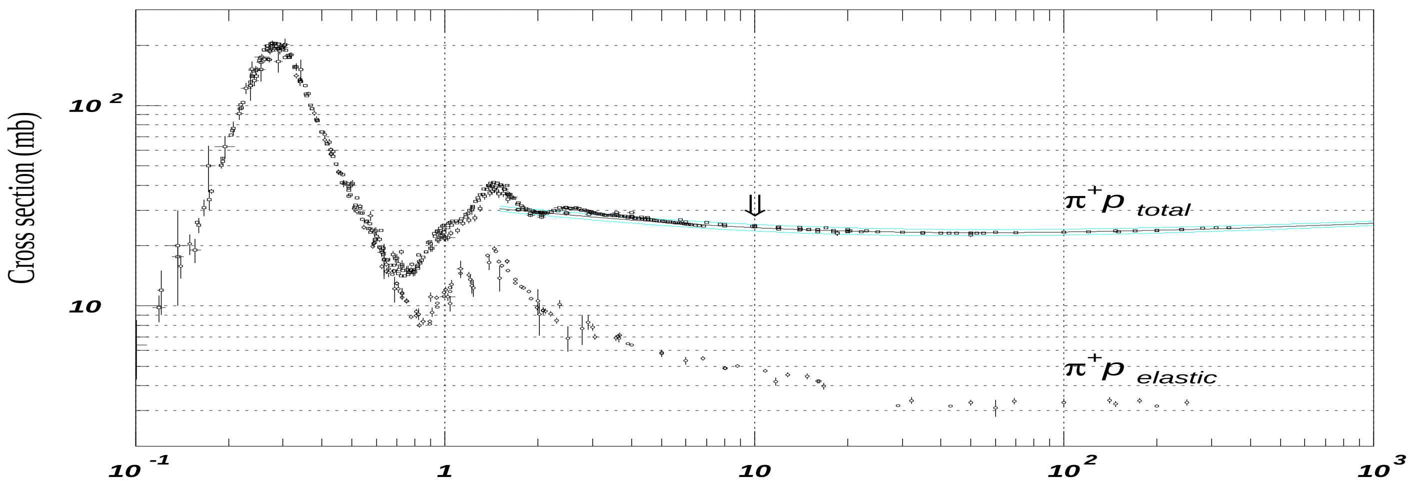
Can we show that QCD correctly describes hadronic physics?

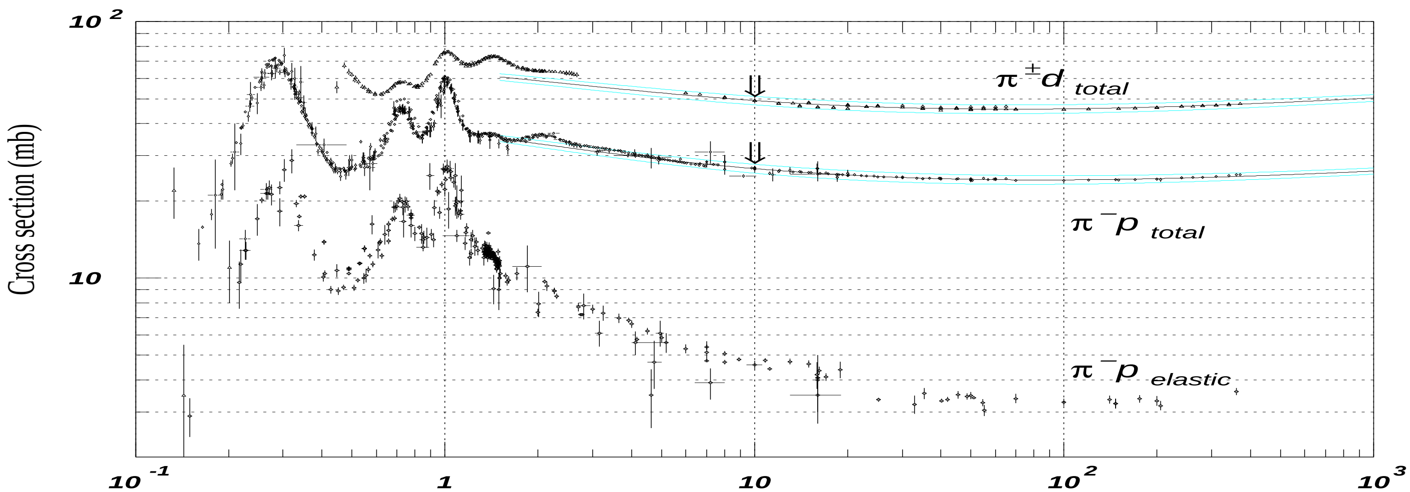
IF YES, THIS WOULD BE A GREAT ACHIEVEMENT!

Spectrum: neglecting weak & electromagnetic interactions
some hadrons are stable e.g. proton, neutron, pions,...

vast amount of data on scattering and resonances
(see the Particle Data Book)

e.g. does QCD reproduce Pion–Proton Scattering data?





For LE properties of QCD need non-perturbative regularization!

LATTICE REGULARIZATION OF PATH INTEGRAL

(Wilson, Smit, 1974)

quark fields $q(x)$: Grassmann variables

at points x on a hypercubic lattice Λ

gauge fields associated with links $\underset{x}{\bullet}\!\longrightarrow\!\underset{x+a\hat{\mu}}{\bullet}$ (a: lattice spacing)

$$U_\mu(x) \sim P \exp \int_0^1 \mathrm{d}t \, A_\mu(x + ta\hat{\mu}) \in \mathrm{SU}(3)$$

$\hat{\mu}$: unit vector in μ direction

Measure: $\int[\mathrm{d}U\mathrm{d}\bar{q}\mathrm{d}q] \equiv \int \prod_{x\in\Lambda} \mathrm{d}U_\mu(x)\mathrm{d}\bar{q}(x)\mathrm{d}q(x)$

compact $\rightarrow$ a **non-perturbative definition** of Euclidian path integral

Expectation values:

$$\langle\mathcal{O}\rangle = Z^{-1}\int[\mathrm{d}U\mathrm{d}\bar{q}\mathrm{d}q]\;\mathrm{e}^{-S[U,\bar{q},q]}\,\mathcal{O}[U,\bar{q},q]$$

well defined for a **finite** lattice Λ

Lattice Action $S[U,\bar{q},q]$ chosen invariant under

gauge transformations: $U_\mu(x) \rightarrow g(x)U_\mu(x)g(x+a\hat{\mu})^{-1}$

Gauge fixing not required for gauge invariant observables

ACTION: $S(\mathrm{Fields}(x), g_0, ...)$, Finite lattice: $N^3 \times N_0$ points

INPUT

$g_0\,, N\,\ldots$

SOFTWARE

$\longrightarrow$

Feynman path integral

$$\langle \mathcal{O} \rangle = \int_{\text{field space}} \mathcal{O}\ \mathrm{e}^{-S}$$

by Numerical Simulation

COMPUTER

CPU time $\downarrow$

OUTPUT

$am = \text{result}$

$+$ statistical

$+$ systematic

PROCESSING

$\longleftarrow$

Raw data for observables

CONTROL ERRORS!

Restricting first to pure gauge theory, evaluation of

$$\langle\mathcal{O}\rangle \propto \int [\mathrm{d}U] \exp(-S[U])\mathcal{O}[U]$$

reduces to **evaluation of an enormous integral!**

e.g. for **SU(3) gauge theory,** 10^4 **lattice**,

dimension of integral $= 8 \times 4 \times 10^4 = 320000$

Even if approx 2 pts/dimension, the sum has $2^{320000} = 10^{96329}$ terms.

integral done by importance sampling (Monte Carlo)

methods introduced in statistical mechanics

expectation value: $\langle \mathcal{O} \rangle = \int [\mathrm{d}U] p[U] \mathcal{O}[U]\,,$

$$p[U] = \exp(-S[U]) / \int [\mathrm{d}V] \exp(-S[V]) > 0\,, \;\; \int [\mathrm{d}U] p[U] = 1$$

Representative ensembles: $\{U_1, U_2, ..., U_N\}$
chosen with probability $p[U][\mathrm{d}U]$

estimate: $\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i] + \mathrm{O}(N^{-1/2})$

Usually representative ensembles are **Markov chains** where U_k obtained from U_{k-1} by a stochastic process

The chain depends on U_1 and a **transition probability** $T(U \to U')$

sufficient conditions for a valid algorithm::

1 $T(U \to U') \geq 0 \; \forall \, U, U' \, , \quad \sum_{U'} T(U \to U') = 1$

2. $\sum_U p[U] T(U \to U') = p[U']$

3. $T(U \to U) > 0 \; \forall \, U$

4. For a given region of configuration space $\mathcal{S}$ one can find $U \in \mathcal{S}, U' \notin \mathcal{S}$ with $T(U \to U') > 0$

link algorithms: e.g Metropolis

1. chose a link at random

2. chose $X \in sU(3)$ randomly in a ball with uniform distribution

3. generate a random number r uniformly in the region $[0,1]$,

   accept $U' = \mathrm{e}^X U$ if $p[\mathrm{e}^X U] > rp[U]$

   otherwise set $U' = U$

clearly need excellent random number generators

e.g. program RANLUX (Lüscher; James (1994))

other link algorithms: e,g. heat-bath...

Computation of masses

e.g. Nucleon **correlation function:** ($N \sim \epsilon_{ijk} q^i q^j C\gamma_5 \tau^2 q^k$)

$$\langle N(t=na, \mathbf{p}=0)\overline{N}(0)\rangle \propto \int [\mathrm{d}U\mathrm{d}\bar{q}\mathrm{d}q]\mathrm{e}^{-S[A,\bar{q},q]} N(t,\mathbf{0})\overline{N}(0)$$

$$\sim_{n\,\mathrm{large}} \exp\{-n/\xi_N(g_0,\ldots)\}$$

assuming the theory has an (effective) **transfer matrix**

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assuming the theory has an (effective) **transfer matrix**

g_0: **bare coupling**; $\xi_N(g_0, \ldots)$: nucleon **correlation length**

value of lattice spacing a is not an input!

setting scale e.g. declare $a/\xi_N(g_0, \ldots) =$ physical proton mass

$$M_N = 946\text{MeV} \rightarrow a(g_0, \ldots) \text{ in physical units}$$

Note: value in fm also depends on quantity chosen to set scale

Important to understand **PHASE DIAGRAM** of a given regularization, i.e. existence of critical points, or lines,...

Continuum limit: $g_0 \to g_{\rm crit}$ s.t. $\xi \to \infty$

i.e. $a \to 0$ if consider M_N fixed

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"Conventional Wisdom" (CW)

- $g_{\rm crit} = 0$
- Continuum limit of lattice QCD is asymptotically free

CW is plausible but not yet rigorously proven

Important but unproven **conjecture of UNIVERSALITY:**

a large class of actions depending on the same set of fields (and symmetries) have identical continuum limits

$\rightarrow$ **huge freedom in choosing lattice action**

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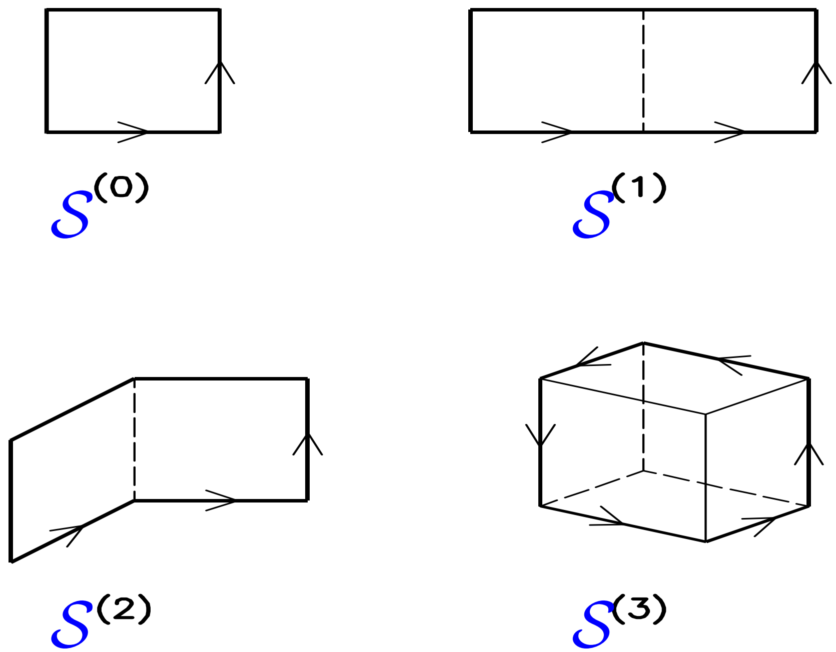
$\rightarrow$ **huge freedom in choosing lattice action**

many different lattice actions *maintaining physical locality* **in use by various collaborations**

- have their respective advantages & disadvantages
- agreement in the continuum limit gives confidence in results

e.g. for pure gauge theory action can chose: $S = \sum_{i=0}^{3} c_i(g_0) S_i$

$$S_i = \sum_{\mathcal{S}^{(i)}} \mathrm{tr}\left[\prod_{l \in \mathcal{S}^{(i)}} U_l\right] . \qquad \textbf{Wilson action}: \ c_i = 0 \text{ for } i \neq 0.$$



classical continuum limit: $S_i = a_i + b_i a^4 \sum_{x,\mu\nu} \mathrm{tr}\left[F_{\mu\nu}(x)^2\right] + \ldots$

Universality class may be larger than expected: e.g. contain actions which do not have the naive classical continuum limit
see e.g. Bietenholz, Gerber, Pepe, Wiese, arXiv:1009.2146

FERMION ACTIONS

Action is **bilinear** in the quark fields $\rightarrow$ "integrate" out exactly:

$$\langle \mathcal{O}[U,\bar{q},q] \rangle \propto \int [\mathrm{d}U] \exp(-S[U]) \mathcal{O}[U, D[U]]$$

$$S[U] = S_{\mathrm{gauge}}[U] - \ln \det(iD[U] + m)$$

huge saving in CPU-time if neglect the fermion determinant

$\rightarrow$ "Quenched approximation" $\rightarrow$ **uncontrolled systematic errors**

"serious" dynamical simulations since ~ 2000

Chiral symmetry for massless quarks and its supposed spontaneous breaking is an important property of QCD

NIELSEN-NINOMIYA THEOREM: Cannot construct local (free) lattice D with $\{D, \gamma_5\} = 0$ without "doubling" the quark "spectrum"

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Wilson-Dirac operator: $D = \frac{1}{2}\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^*\nabla_\mu\}$

∇_μ, ∇_μ^*: gauge covariant forward & backward difference operators

many good properties,
BUT the "Wilson term" $-\frac{1}{2}a\nabla_\mu^*\nabla_\mu$ violates chiral symmetry

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Old suggestion by Wilson and Ginsparg (1982),
avoid NN Theorem by demanding only $\{D, \gamma_5\} = aD\gamma_5 D$

Revived 1997: Hasenfratz, Laliena, Niedermayer; Lüscher; Neuberger

Fermion actions differ mainly on how chiral symmetry is treated

Action	χ–symmetry	(dis)advantages
Wilson (O(a) improved)	($m_0 = 0$), broken	conceptually simple
Staggered, Kogut, Susskind	"too much"	relatively cheap
Overlap Neuberger Ginsparg-Wilson, Hasenfratz	exact Lüscher	expensive
Domain wall quarks, Kaplan	approximate	
Perfect action Hasenfratz, Niedermayer	" "	
Twisted mass QCD Frezzotti, Sint, P.W	weakly broken	a compromise

Claims of lattice phenomenological successes using **staggered quarks**

but problems when $N_{\mathrm{f}}/4 \neq$ integer

"Rooting trick": $\mathrm{Det}(iD+m) \to \mathrm{Det}(iD+m)^{1/4}$ **by hand**!

tampering with functional integral intuitively dangerous
- can lead to violation of accepted principles: locality,...

but attempts (e.g. by Creutz) to prove this so far unsuccessful

"Rooted staggered fermions, good, bad or ugly?"

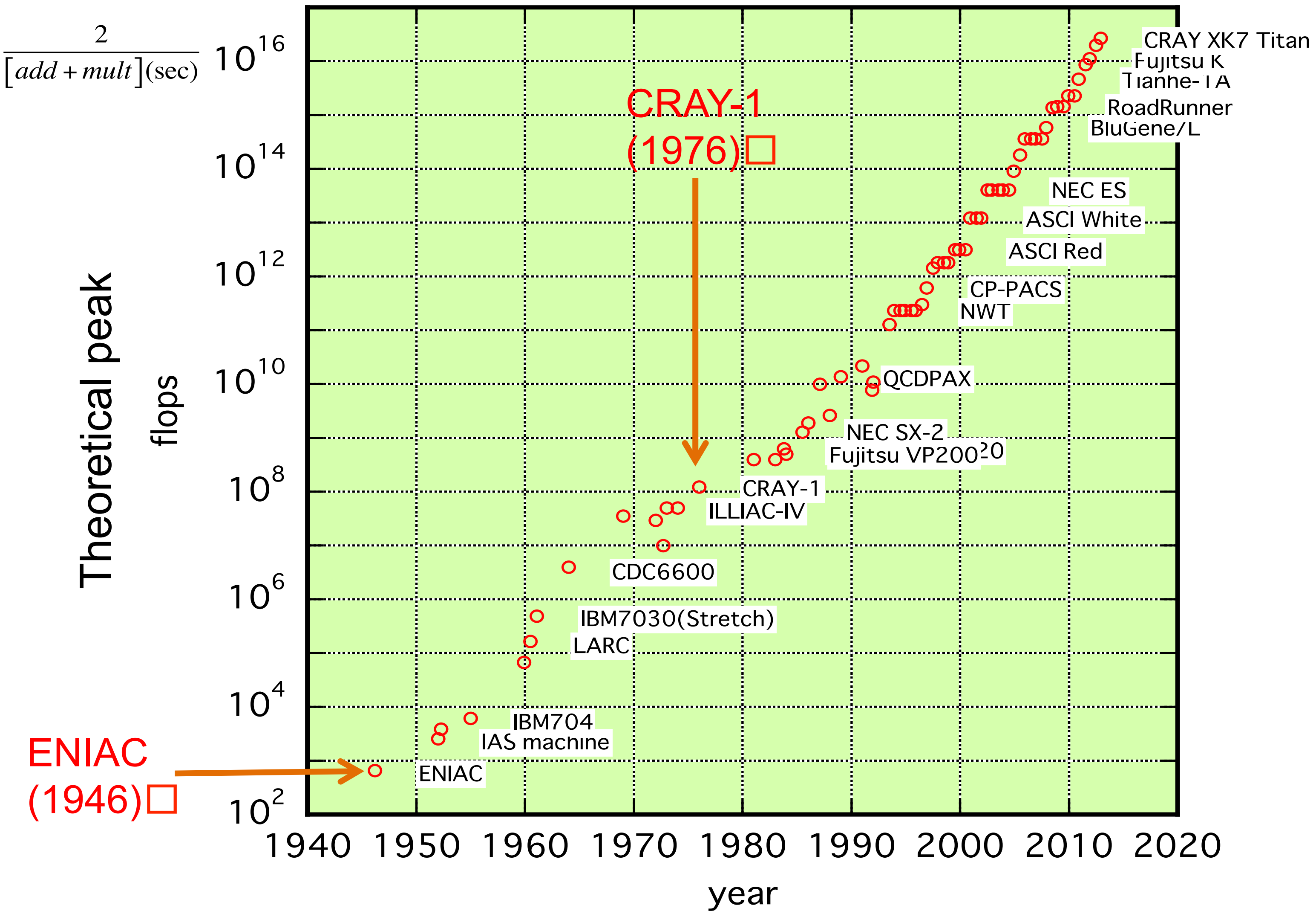
"At least ugly" (Sharpe, LAT06)

Advances particularly since ~ 2000 due to...

- **Exponential growth in computer power**.
  Access of big collaborations to Peta-flop supercomputers
- **Improved simulation algorithms**
- **Effort to control/reduce lattice artifacts & finite size effects**
- Use of better–suited fermion actions
- Inclusion of extrapolations using chiral perturbation theory
  & heavy quark effective theory

Growth of computer speed over 7 decades

$$flop \equiv \frac{2}{[add + mult](\mathrm{sec})}$$

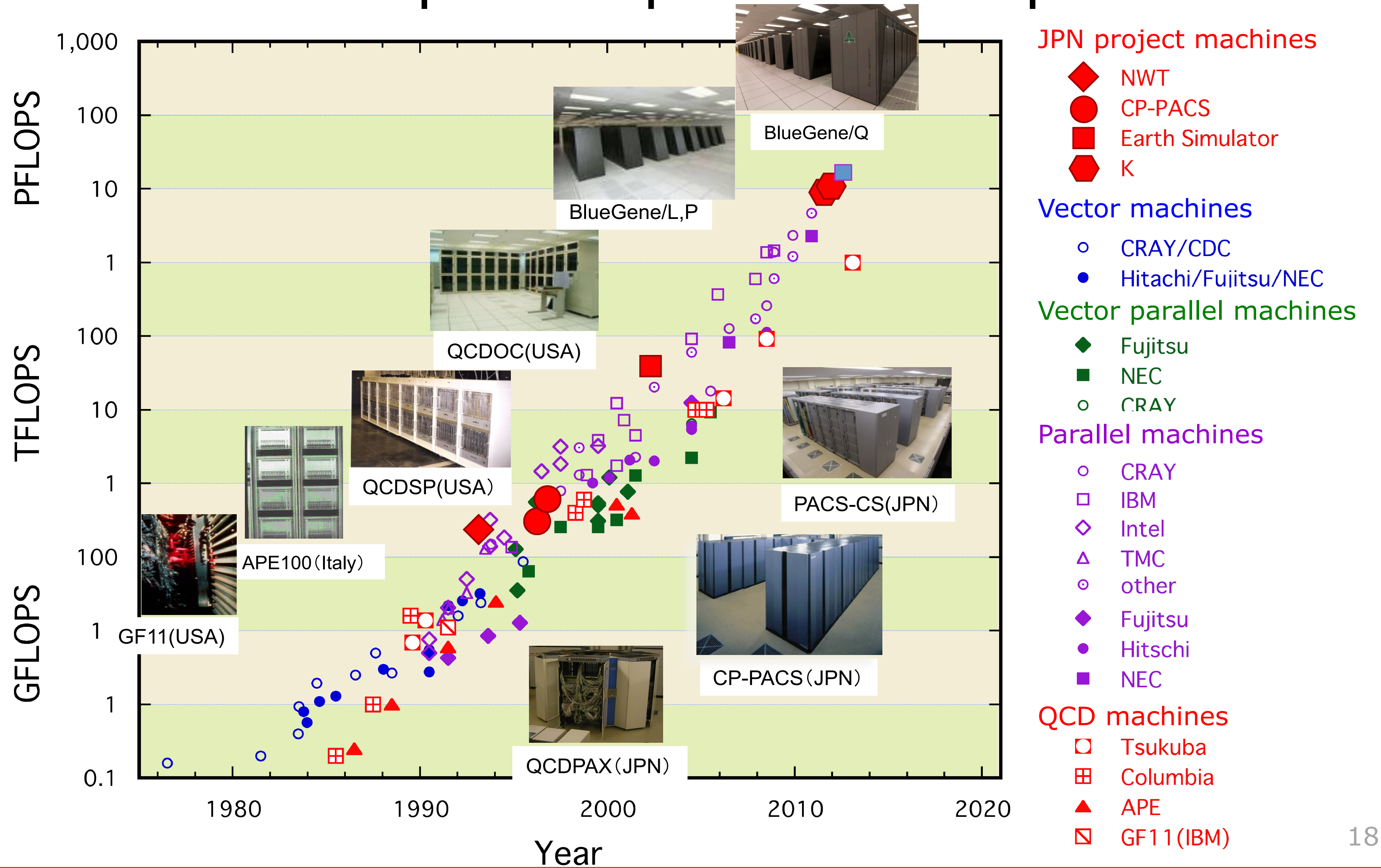


Exponential increase since ENIAC in 1946

-14-

plot from Akira Ukawa

Lattice QCD machines and the supercomputer development



plot from Akira Ukawa

Is there a limit to exponential growth?

reply from Ukawa Akira Aug 29, 2014:

"... said that **below 5nm, semiconductors will cease to function** .."

"Current technology is 20nm, & 10 nm technology expected ~ 2020. Perhaps 5nm in ~ 2030,.. that is one possible limit."

issue of power consumption vs computing power: SIMD or MIMD

"**..big issue: how many people (besides minorities like us) in the world needs such a huge amount of computing power?"**

" making a test chip for checking design & functions now costs $\sim 10^7\$$.

"..assuming you burn your chip twice, once for checking and second for actual fabrication, no project can start with a budget below $2\times10^7\$$."

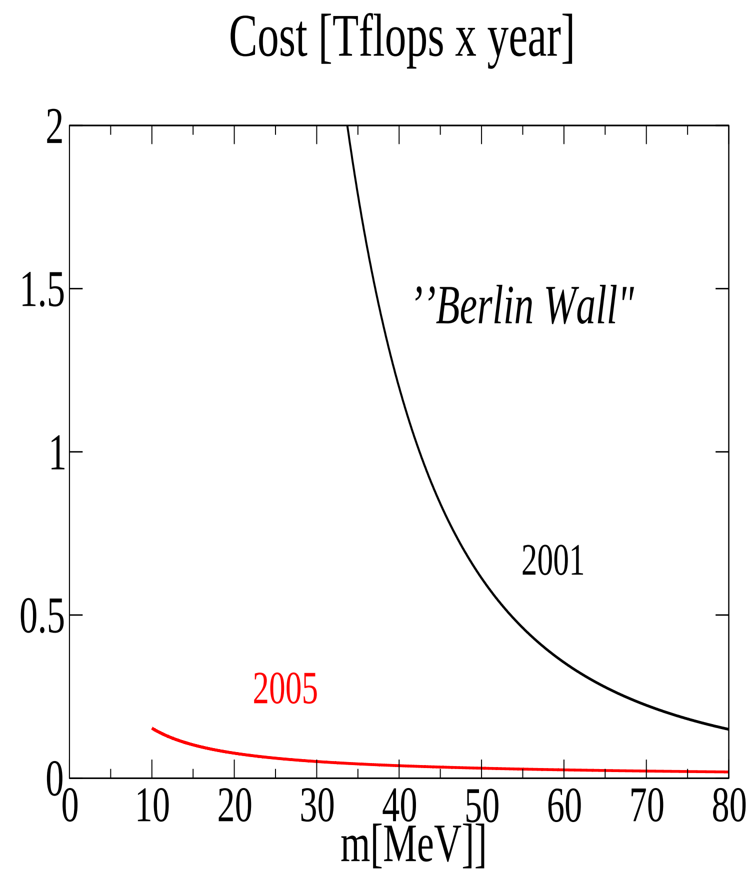
Wilson quarks not as expensive as previously thought!

Ops in TFlop Yr req. for an ensemble of 100 gauge fld. configs.

$N_\mathrm{f} = 2$, $V = 2L \times L^3$, O(a)-improved Wilson quarks, $m = m_{\overline{\mathrm{MS}}}(2\mathrm{GeV})$

$$5.00 \left[\frac{20\mathrm{MeV}}{m}\right]^3 \left[\frac{0.1\mathrm{fm}}{a}\right]^7 \left[\frac{L}{3\mathrm{fm}}\right]^5, \quad \mathrm{Ukawa, LAT01}$$

$$0.05 \left[\frac{20\mathrm{MeV}}{m}\right]^1 \left[\frac{0.1\mathrm{fm}}{a}\right]^6 \left[\frac{L}{3\mathrm{fm}}\right]^5, \quad \mathrm{Giusti, LAT06}$$



Example

64×32^3 lattice

$L \simeq 2.5\mathrm{fm}$

$a \simeq 0.08\mathrm{fm}$

Acceleration due to progress in algorithms

most frequently used is the **HMC algorithm** where

new configurations generated by solution of molecular dynamics eqns:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(x,\mu) = U(x,\mu)\pi(x,\mu)\,, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\pi(x,\mu) = -F(x,\mu)$$

Force: $F = \frac{\partial S}{\partial U} = F_{\mathrm{glue}} + F_{\mathrm{quark},UV} + F_{\mathrm{quark},IR}$

Find $\quad F_{\mathrm{glue}} \gg F_{\mathrm{quark},UV} \gg F_{\mathrm{quark},IR}$

multi-time step integration; enlarge $\delta\tau_{\mathrm{quark},IR}$ which costs most time!

Hasenbusch '01, Lüscher '03, Urbach et al '05, Clark & Kennedy '06

various QCD simulation programs publically available
e.g. openQCD-1.4 (June 2014): J. Bulava, L. Del Debbio, L. Giusti, B. Leder, **M. Lüscher**, F. Palombi, **S. Schaefer**

Configurations stored: **International Lattice Data Grid**

Also improved ways of solving the Dirac equation
needed for computing correlation functions

Time required for solving depends on the
condition number = ratio of highest and lowest eigenvalues,

reduced by preconditioning, low-mode deflation,...

Simulations of lattice QCD to study static properties of hadrons with physical pion masses now feasible

They include u, d, s sea quarks and have $a < 0.1\mathrm{fm.}, L > 2.5\mathrm{fm}$

Many results on hadron phenomenology

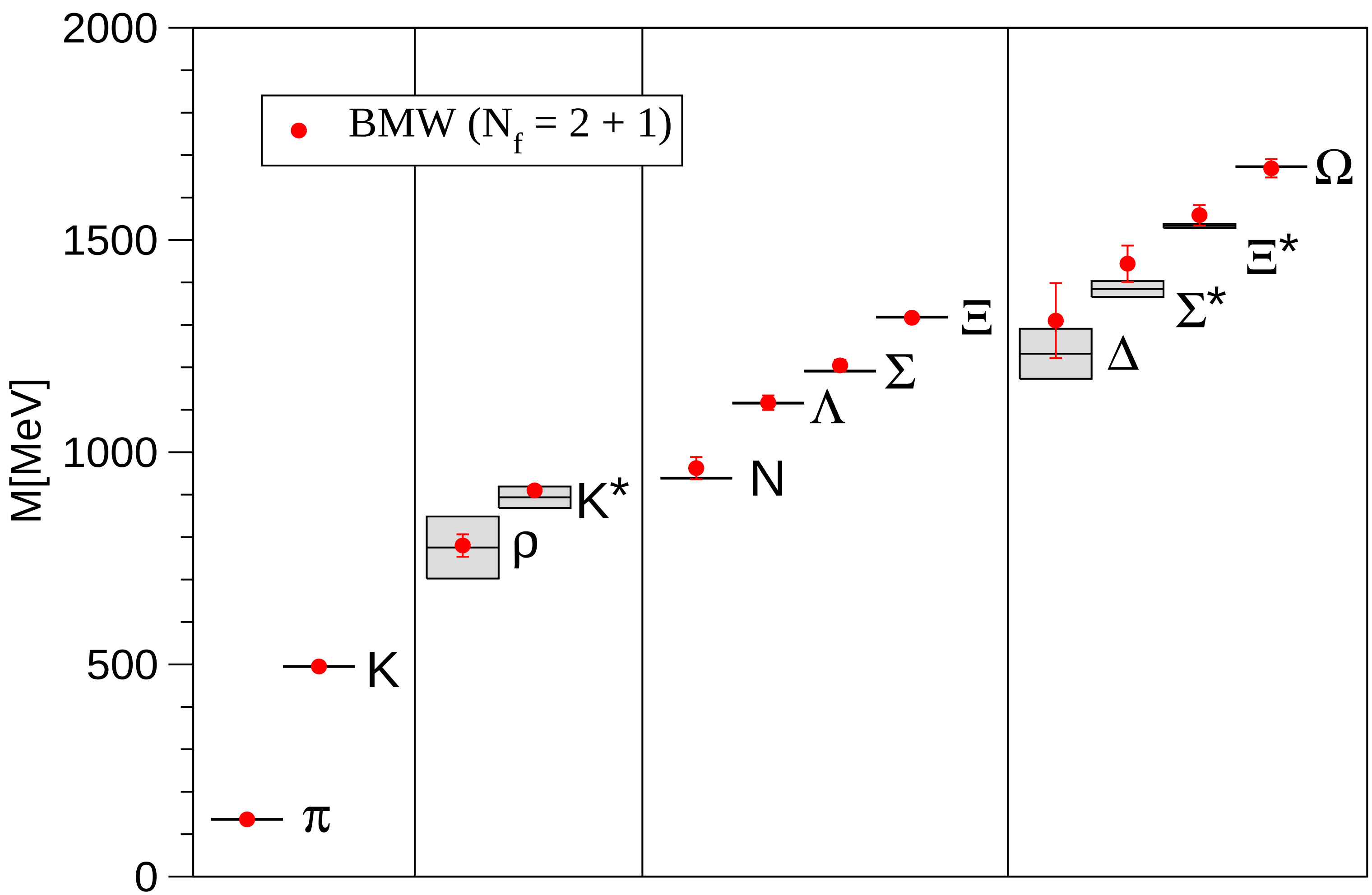
- Low-lying spectrum
- resonances and phase shifts (from finite volume effects Lüscher)
- Running couplings, and running quark masses

- decay constants, B–parameters, ...
- Hadronic contributions to $g - 2$
- Meson distribution amplitudes
- Elastic and transition form factors
- Moments of (generalized) structure functions, ...

- QCD at finite temperature, finite density

Low-lying hadron spectrum

plot from Budapest-Marseille-Wuppertal collaboration (Science 2009)



Many other groups e.g. PACS-CS obtain similar results;
it is a success of the whole lattice community

Systematic error sources

- Lattice spacing a effects
- Finite volume effects, box size L
- Unphysical quark masses
  input from (& now output to) **Chiral Perturbation Theory**
- Neglect of certain sea quarks
  use of **Heavy Quark Effective Theory**
- Noisy correlation functions

Lattice artifacts

consider e.g. mass spectrum: $m_k = 1/(a\xi_k)$, a=lattice spacing

presently ξ attained not so large $\rightarrow$ need extrapolations

usually near the continuum limit attempt fit of form:

$$[m_k/m_1](g_0) = [m_k/m_1](g_{\mathrm{crit}}) + c_k \mathrm{O}((am_1)^p)\,, \ \ k > 1$$

what is the value of p?? integer??? log corrections ???

Most of our knowledge concerning **renormalization of quantum field theories** stems from perturbation theory

few rigorous proofs: many results are *structural* and hence considered to carry over to non-perturbative formulations

supporting evidence from integrable models in 2d, and axiomatic work in 3d, and in framework of $1/n$ expansions

the same situation holds for **lattice artifacts**

Symanzik ('80) conjecture: leading artifacts of lattice correlation functions at widely separated points x_k are given by

$$Z_\varphi^{r/2}\langle\phi(x_1)\ldots\phi(x_r)\rangle_{\mathbf{latt}} = \langle\varphi_0(x_1)\ldots\varphi_0(x_r)\rangle_{\mathbf{cont}} + a^p T_1 + a^p T_2 + ..$$

$$T_1 = -\int \mathrm{d}^d y\,\langle\mathcal{L}_1^{\mathrm{eff}}(y)\varphi_0(x_1)\ldots\varphi_0(x_r)\rangle_{\mathbf{cont}}$$

Symanzik's effective Lagrangian: $\mathcal{L}_1^{\mathrm{eff}} = \sum_j c_j\mathcal{O}_j$

$\mathcal{O}_j$: **local operator dimension** $d+p$**,**
with same symmetries as lattice action

coefficients c_j are a–dependent - but thought to be weak (log)

T_2 appears if ϕ is a composite field

If Symanzik's conjecture is true one expects

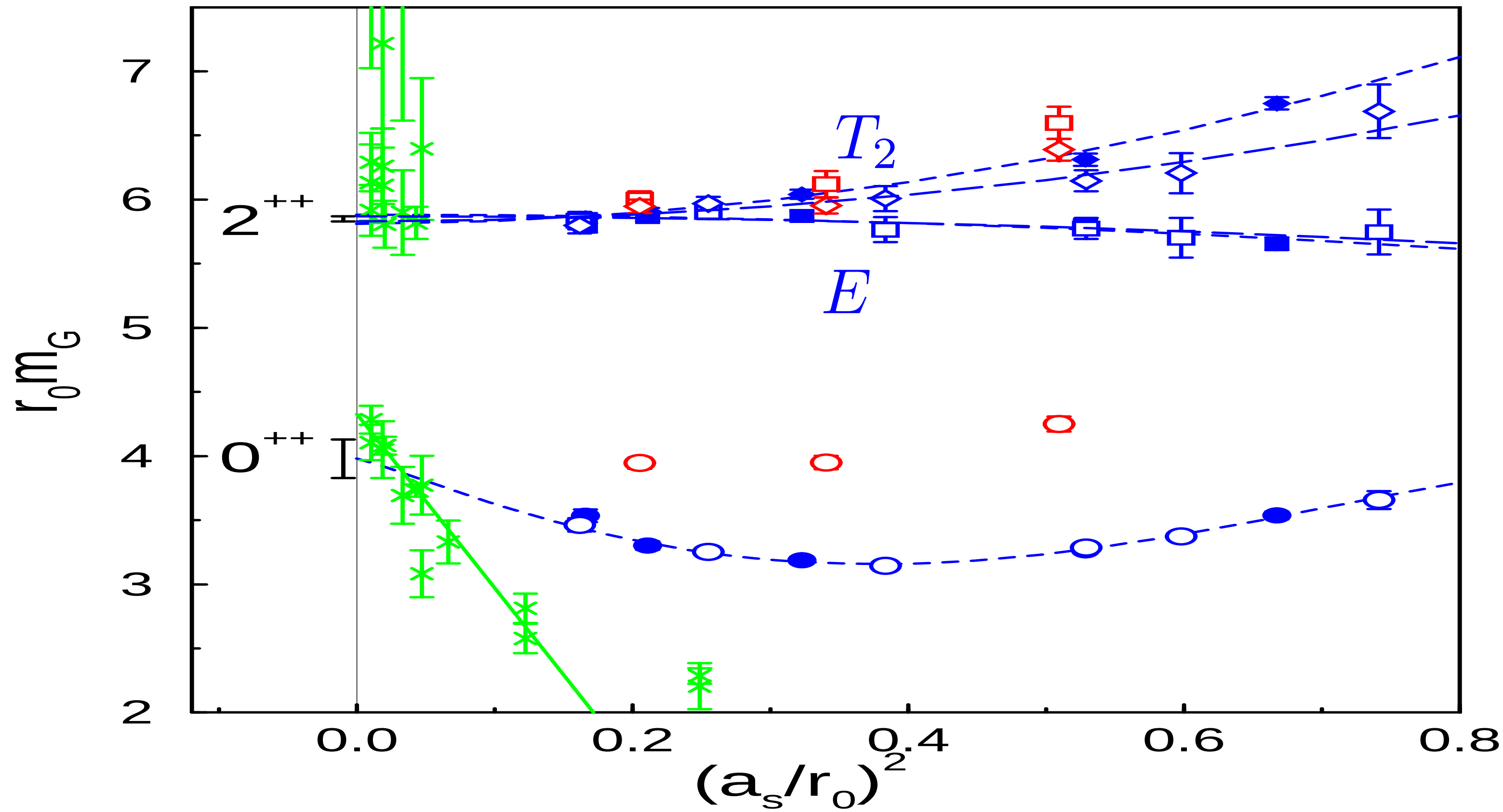
a) **generic $\mathrm{O}(a^2)$ artifacts in pure Yang–Mills' theory**,
since no gauge invariant scalar dim 5 operator,
but $\mathrm{O}(a)$ **effects with pure Wilson fermions**

b) possible to construct $\mathrm{O}(a^2)$**–improved lattice actions** for YM,
$\mathrm{O}(a)$-improved Wilson fermions.

Also need improved composite operators....

whether these are useful in simulations is a practical question

Scaling of lowest glueballs (Peardon & Morningstar)



Data from **Improved action** (anisotropic lattice)

Wilson action and New action

indications of UNIVERSALITY

Leading finite size effects given by infinite volume physics

known from Quantum Mechanics (emphasized by Parisi)

Extended to framework of QFT by Lüscher (1986)

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Stable particle masses $(m = M(\infty))$: $M(L) - m =$
$$-\frac{3}{16\pi m^2 L}\left\{\lambda^2 \mathrm{e}^{-\frac{\sqrt{3}}{2}mL} + \frac{m}{\pi}\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-\sqrt{m^2+y^2}L}F(iy) + \ldots\right\}$$

λ: 3 point coupling; $F(\nu)$ forward scattering amplitude, $\nu = pq/m$

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Lowest energy of 2 particles in periodic box size L:

$$W = 2m - \frac{4\pi a_0}{mL^3}\left\{1 + c_1\frac{a_0}{L} + c_2\left(\frac{a_0}{L}\right)^2\right\} + \mathrm{O}(L^{-6})$$

s-wave scattering length: $a_0 = \lim_{p\to 0}\frac{1}{2ip}\left(\mathrm{e}^{2i\delta_0(p)} - 1\right)$

(Huang& Yang 1957) $c_1 = -2.83.., c_2 = 6.37...$

$\rightarrow$ **extract S-matrix data from spectra measured in the Euclidean framework; no analytic continuation required**

e.g. **s-wave phase shift**: (Lüscher (1991)) from

2-particle energies: $W = 2\sqrt{m^2 + \mathbf{k}^2}\,,\ \ \mathbf{k} \neq \frac{2\pi}{L}\mathbf{n}\,,\ \ \mathbf{n} \in \mathbb{Z}^3$

$$k \cot \delta_0(k) = \frac{2}{\sqrt{\pi}L} Z_{00}(1; q^2)\,,\ \ k = |\mathbf{k}|\,,\ \ q = \frac{kL}{2\pi}$$

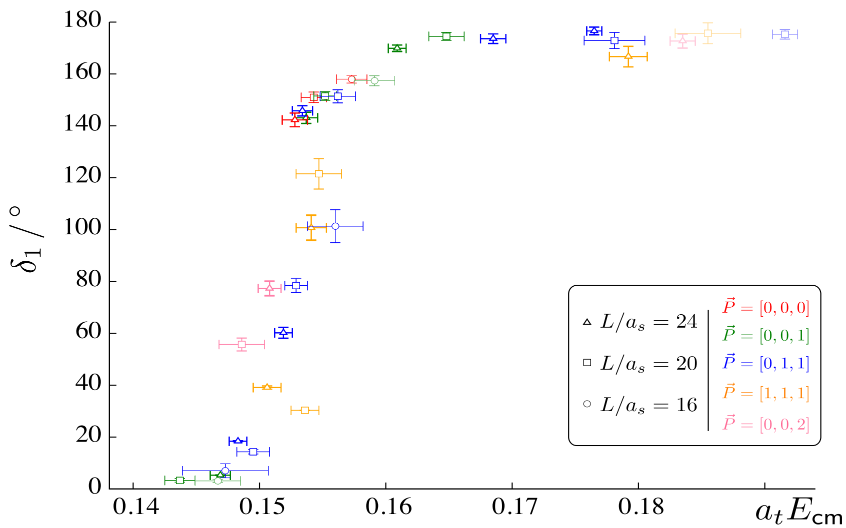
generalized zeta-function: $Z_{0,0}(s, q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(\mathbf{n}^2 - q^2\right)^{-s}$

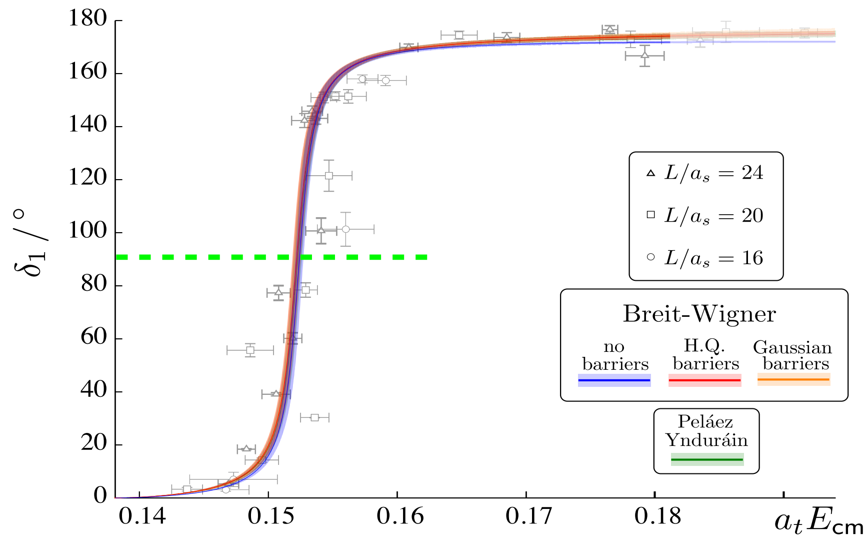
lattice results for $\pi\pi$ $I = 0, 2$ phase shifts...

extensions to other channels, moving frames, resonances...

Dudek-Edwards-Thomas, PRD87(2013)034505

$\delta_1(E_{\mathrm{cm}})$





2-flavor anisotropic clover fermion

$$a_s \sim 0.12\,\mathrm{fm}$$

$$m_\pi \sim 400\,\mathrm{MeV}$$

FLAG Working Group

S. Aoki, Y. Aoki, C. Bernard, T. Blum, G. Colangelo, M. Della Morte, S. Dürr, A.X. El Khadra, H. Fukaya, R. Horsley. A. Jüttner, T. Kaneko, J. Laiho, L. Lellouch, H. Leutwyler, V. Lubicz, E. Longhi, S. Necco, T. Onogi, C. Pena, C.T. Sachrajda, S.R. Sharpe, S. Simula, R. Sommer, R.S. Van de Water, A. Vladikas, U. Wenger, H. Wittig

USEFUL AIM: **Answer the question "What is currently the best lattice value for a particular quantity in a way that is readily accessible to non-experts"**

e.g. running coupling, light quark masses, $f_k/f_\pi, B_K$,.....

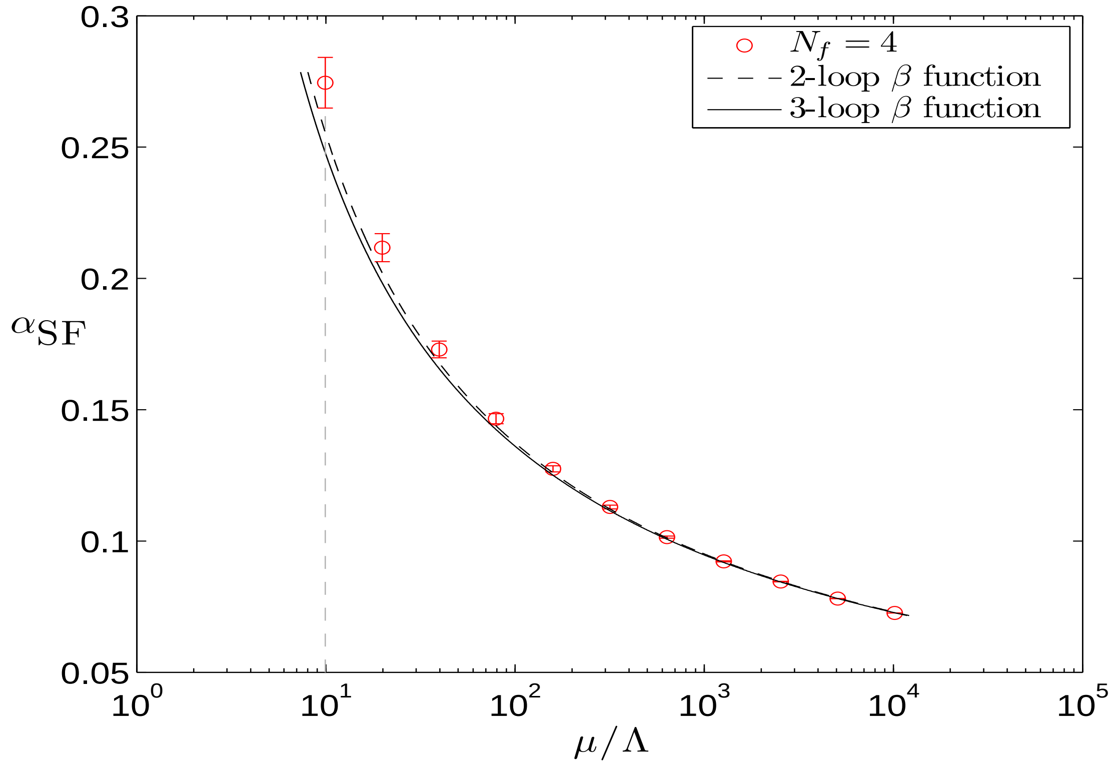
Eur. Phys. J C71 (2011) 1695; arXiv:1310:8555 e.g. **Running coupling** - Many **non-perturbative definitions**:

- from static potential at short distances (UKQCD, Necco&Sommer)
- from lattice observables at $\mu \sim a^{-1}$ (HPQCD)
- from gluon vertices (Boucaud et al, ETMC)
- from Schrödinger Functional (SF) (finite volume, Alpha Collab.)
- from gradient flow

Advantages and disadvantages reviewed in FLAG report

Only the SF method cleanly separates lattice artifacts

Nonperturbatively defined running coupling with $N_\mathrm{f} = 4$



ALPHA Collaboration

2010

- Measure the coupling over a wide range of energies
- Observe domain where PT behavior sets in!
- use PT at HE to relate to $\overline{\mathrm{MS}}$ scheme

for $N_\mathrm{f} = 2$ obtain $\Lambda^{(2)}_{\overline{\mathrm{MS}}} = 245(16)(16)\mathrm{MeV}$ (using $r_0 = 0.5\mathrm{fm}$)

FLAG rate lattice results according to **"quality criteria"**

- ★ "when the systematic error has been [...] convincingly shown to be under control"
- ○ "when a reasonable attempt [...] has been made although this could be improved"
- □ "when no or a clearly unsatisfactory attempt [...] has been made"

Compilation and global estimates: FLAG WG

Schrödinger functional:

Collaboration	Ref.	N_f	publication status	renormalization scale	perturbative behaviour	continuum extrapolation	scale	$\Lambda_{\overline{\rm MS}}$[MeV]	$r_0\Lambda_{\overline{\rm MS}}$
ALPHA 10A	[484]	4	A	★	★	★	only running of α_s in Fig. 4		
Perez 10	[485]	4	P	★	★	○	only step scaling function in Fig. 4		
PACS-CS 09A	[486]	2+1	A	★	★	○	m_ρ	$371(13)(8)(^{+0}_{-27})^{\#}$	$0.888(30)(18)(^{+0}_{-65})^{\dagger}$
			A	★	★	○	m_ρ	$345(59)^{\#\#}$	$0.824(141)^{\dagger}$
ALPHA 12 *	[59]	2	A	★	★	★	$f_{\rm K}$	310(20)	0.789(52)
ALPHA 04	[487]	2	A	■	★	★	$r_0{}^{\S}$	$245(16)(16)^{\S}$	$0.62(2)(2)^{\S}$
ALPHA 01A	[488]	2	A	★	★	★	only running of α_s in Fig. 5		
CP-PACS 04 $^{\&}$	[482]	0	A	★	★	○	only tables of $g^2_{\rm SF}$		
ALPHA 98 ††	[489]	0	A	★	★	○	$r_0 = 0.5$fm	238(19)	0.602(48)
Lüscher 93	[479]	0	A	★	★	○	$r_0 = 0.5$fm	233(23)	$0.590(60)^{\S\S}$

$\alpha_{\overline{\mathrm{MS}}}$ determined from non-perturbative measurement of an observable at scale μ and computing perturbative series:

$$\mathcal{O}(\mu) = c_1 \alpha_{\overline{\mathrm{MS}}}(\mu) + c_2 \alpha_{\overline{\mathrm{MS}}}(\mu)^2 + \ldots$$

Define: $\alpha_{\mathrm{eff}} \equiv \mathcal{O}(\mu)/c_1$

Meaning of FLAG symbols:

★ Renormalization scale: $\alpha_{\mathrm{eff}} < 0.2$ throughout

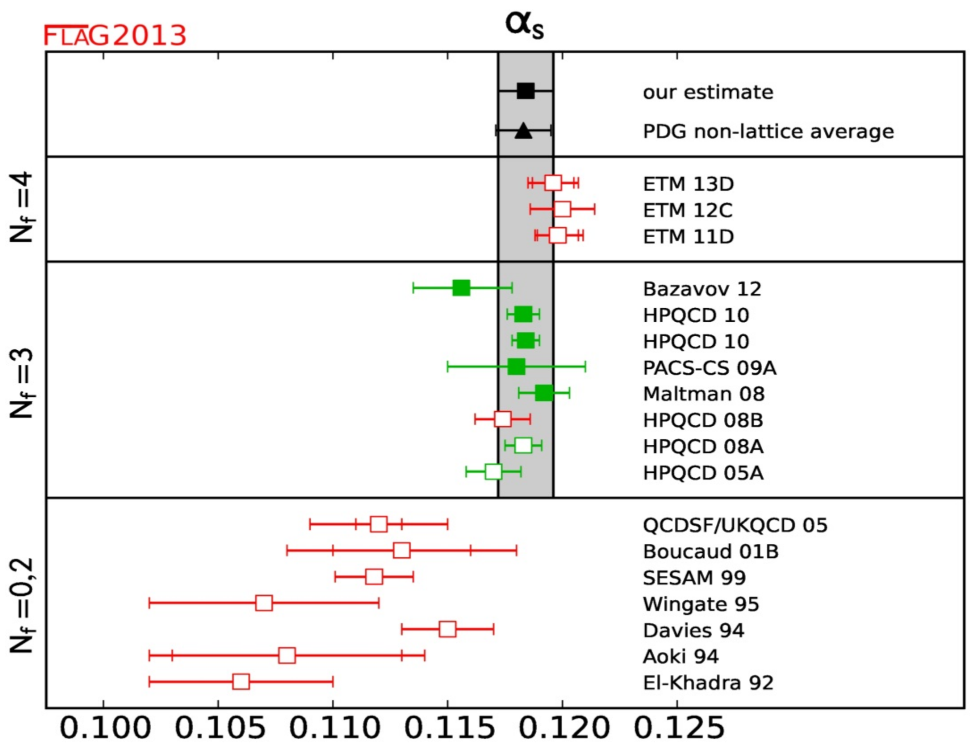
★ Perturbative behaviour: verified over a factor 2 in α_{eff}; ....

★ Continuum extrapolation: at least 3 lattice spacings
with $\mu a < 0.5$ for $\alpha_{\mathrm{eff}} \leq 0.3$

○ Renormalization scale: $\alpha_{\mathrm{eff}} < 0.4$; at least one with $\alpha_{\mathrm{eff}} < 0.25$

....

Compilation and global estimates: FLAG WG



FLAG lattice estimate:

$$\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.1184 \pm 0.0012$$

PDG non-lattice estimate

$$\alpha^{(5)}_{\overline{\rm MS}}(m_Z) = 0.01183 \pm 0.0012$$

Muon anomalous magnetic moment $a_\mu = \frac{1}{2}(g-2)_\mu$

possible sign of new physics?

	estimate $\times 10^{11}$	
experiment E821 − SM	$261(78)$	Hagiwara et al
	$287(80)$	Davier et al
present experimental error	± 63	
present SM error	± 49	
expected error Fermilab E989	± 16	
expected improved SM error	± 35	
electroweak contribution	$154(1)$	
HL$\times$ L (“Glasgow concensus”)	$105(26)$	

lattice determinations of HVP under way since ~ 2003

Hadronic vacuum polarization (HVP) contribution from **Euclidean correlation function** (Blum, 2003; Roskies, Leon, Remiddi, 1990)

$$a_\mu^{(HVP)} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2\, I(Q^2)\,, \quad I(Q^2) = f(Q^2, m_\mu^2)\,\left[\Pi(0) - \Pi(Q^2)\right]$$

$$\int \mathrm{d}^4 Q\, \mathrm{e}^{iQx} \langle J_\mu(x) J_\nu(0)\rangle = \left(Q^2 \delta_{\mu\nu} - Q_\mu Q_\nu\right) \Pi(Q^2)$$

Problem: $I(Q^2)$ peaked at $Q^2 \sim m_\mu^2/4$

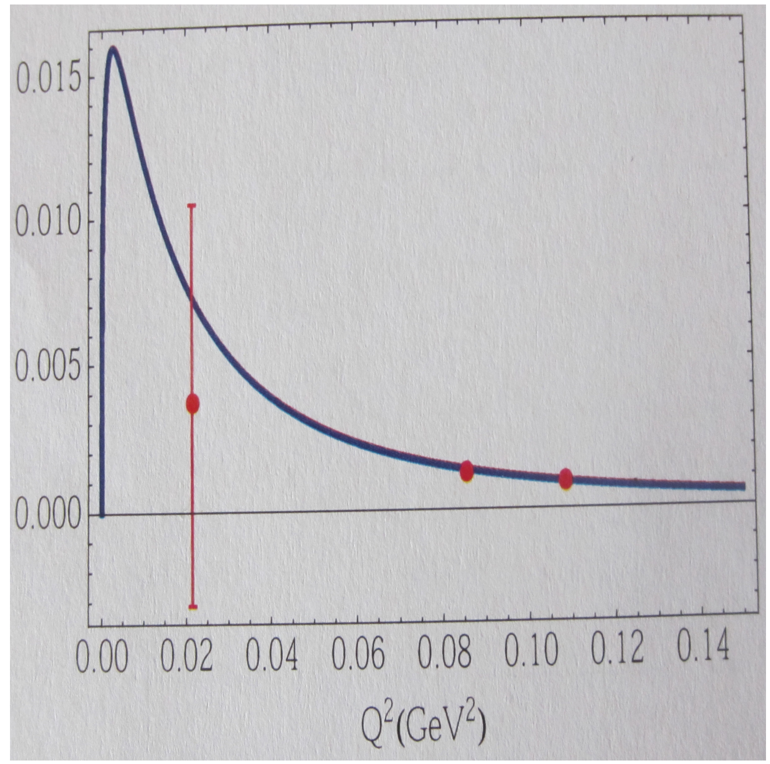
periodic bc $2\pi/L \simeq m_\mu/2 \to L = 25\mathrm{fm}$

then $a \sim 0.06\mathrm{fm} \to L/a \sim 400$

$\to$ used twisted pbc

and/or do clever fits of $\Pi(Q^2)$

$I(Q^2)$



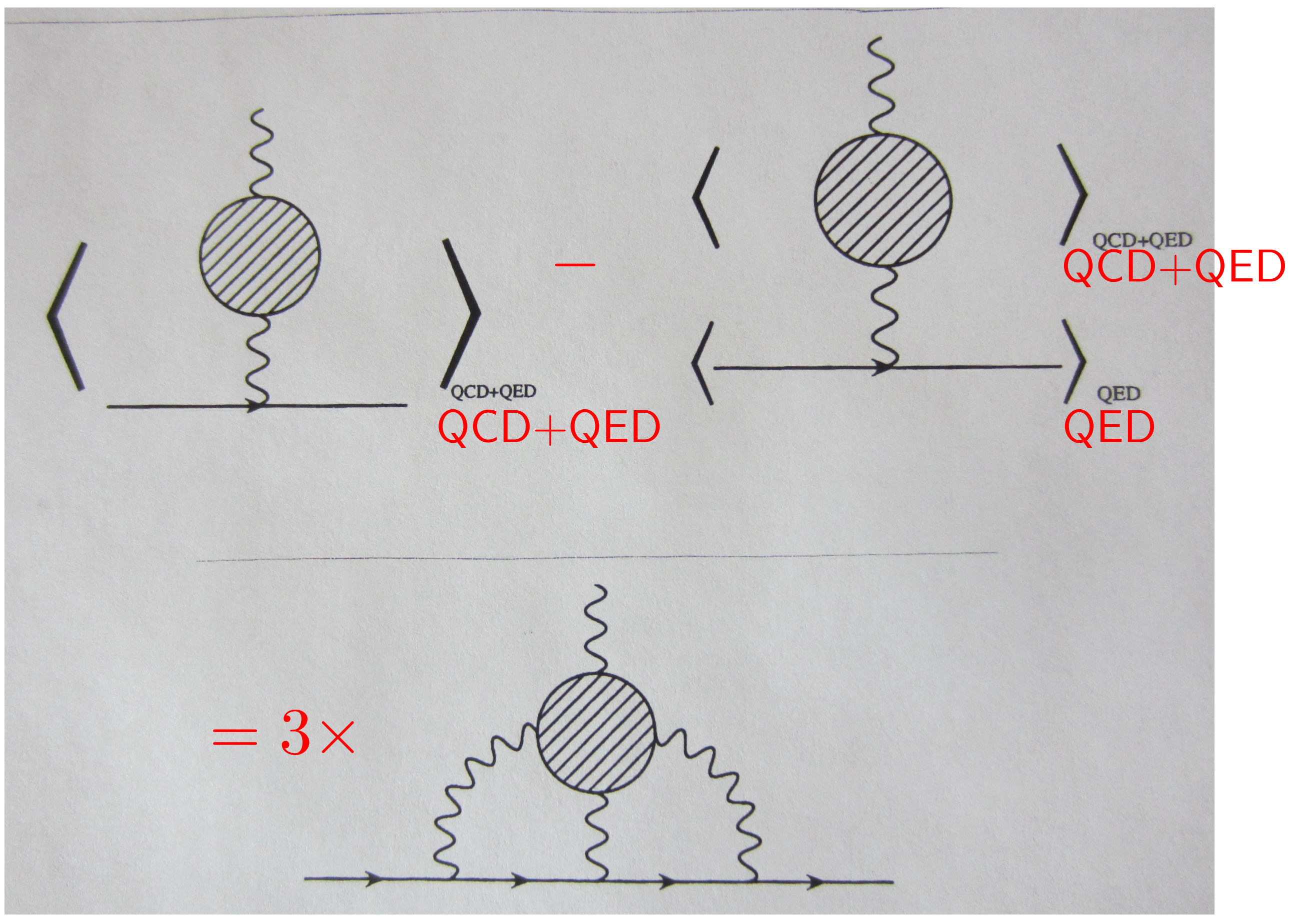
Present errors on lattice determinations of HVP at $3-5\%$ level

cf from e^+e^- error $\sim 0.6\%$

Hadronic light by light scattering contribution

Not possible to obtain from scattering data and dispersion relations

Lattice computation: Blum, Chowdhury, Hayakawa, Izubuchi, (arXiv.1407.2923)



SUMMARY

- ⋆ much algorithmic progress in the last decade
- ⋆ serious dynamical quark simulations of QCD with $m_\pi \sim 140\mathrm{MeV}$ under way
- ⋆ the (effort to) control the various systematic errors is essential for the quality of a lattice experiment
- ⋆ indication of chiral logs
- ⋆ lattice + χPT gives reasonable agreement with experiment
- ⋆ indicates that QCD could well be the theory of the strong interactions, having a mass gap, confinement and asymptotic freedom at HE

Many further lattice studies of non-perturbative phenomena

QCD studies (not mentioned so far) under way:

e.g. QCD at finite temperature, nuclear physics,

QCD at finite density (problem: action is complex)

Standard Model studies: inclusion of electromagnetism

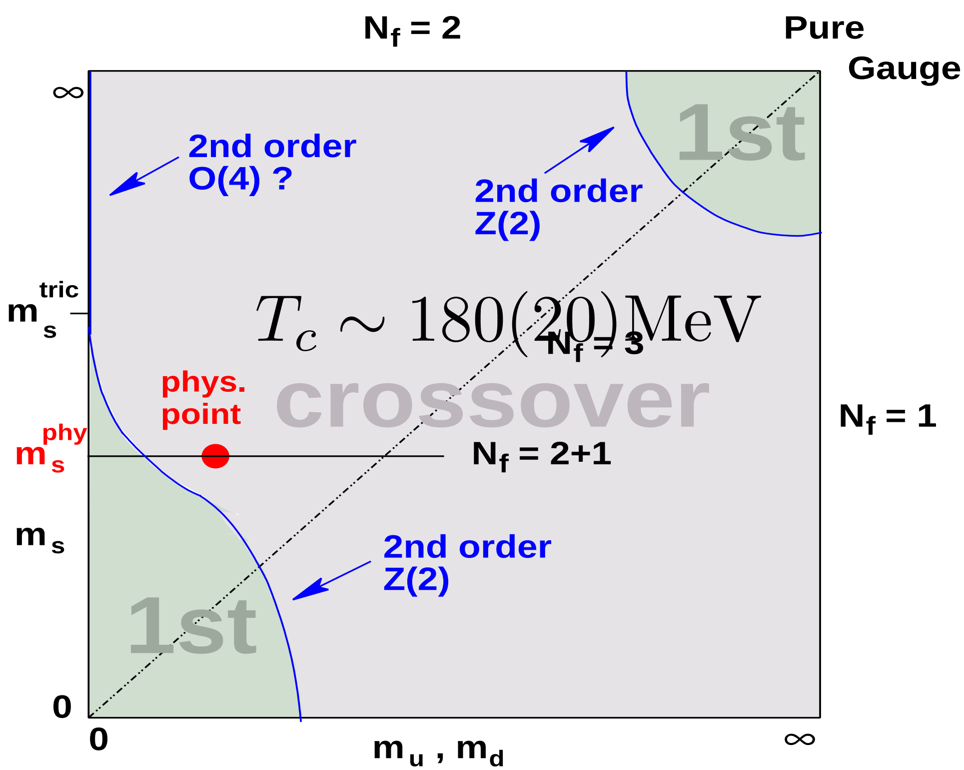
e.g. electromagnetic mass differences: BMW collab. arXiv1406.4088

triviality of ϕ^4_4, QED_4 - rigorous proof still lacking

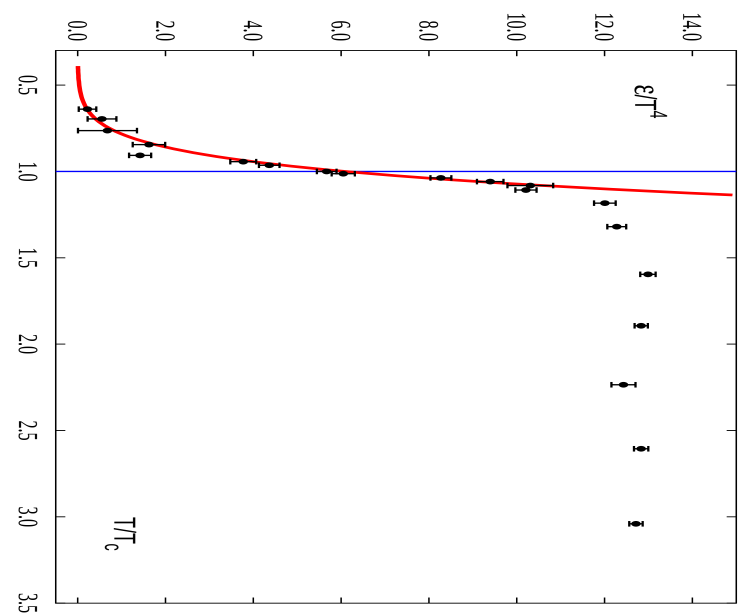
Beyond Standard Model studies

Supersymmetry, walking technicolor, ...

Finite Temperature



Heller LAT06



MESON DISTRIBUTION AMPLITUDES

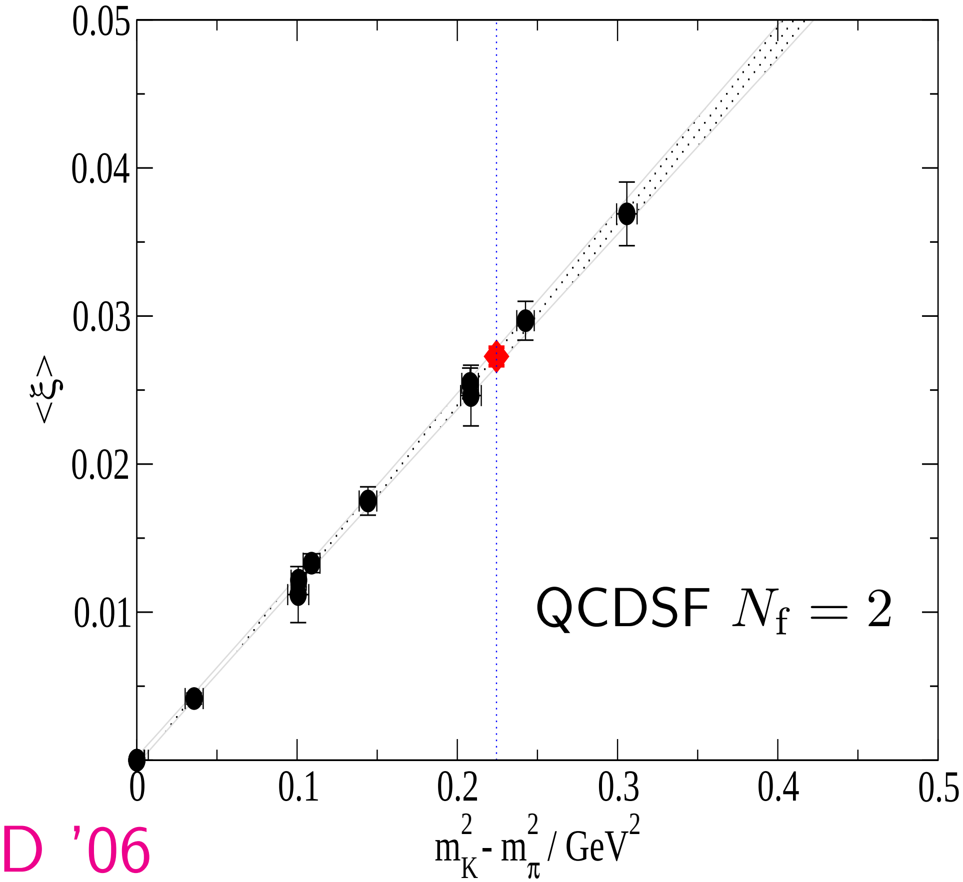
$$\langle 0|\bar{q}(z)\gamma_\rho\gamma_5 P\exp\left[i\int_{-z}^{z} A(x)\cdot \mathrm{d}x\right] s(-z)|K(p)\rangle_{z^2=0}$$

$$= f_K i p_\rho \int_{-1}^{1} \mathrm{d}\xi\, \mathrm{e}^{i\xi p\cdot z}\phi_K(\xi,\mu)$$

Moments: $\langle \xi^n\rangle_K(\mu) = \int_{-1}^{1} \mathrm{d}\xi\, \xi^n \phi_K(\xi,\mu)$

expressed as matrix elements of **local operators**

$$\langle \xi\rangle_K(\mu) f_K p_\rho p_\nu = \langle 0|\bar{q}(0)\gamma_\rho\gamma_5 \overleftrightarrow{D}_\nu s(0)|K(p)\rangle$$



$\langle \xi \rangle_K^{\overline{\mathrm{MS}}}(2\mathrm{GeV})$

$\sim 0.030(12)$ Sum rules

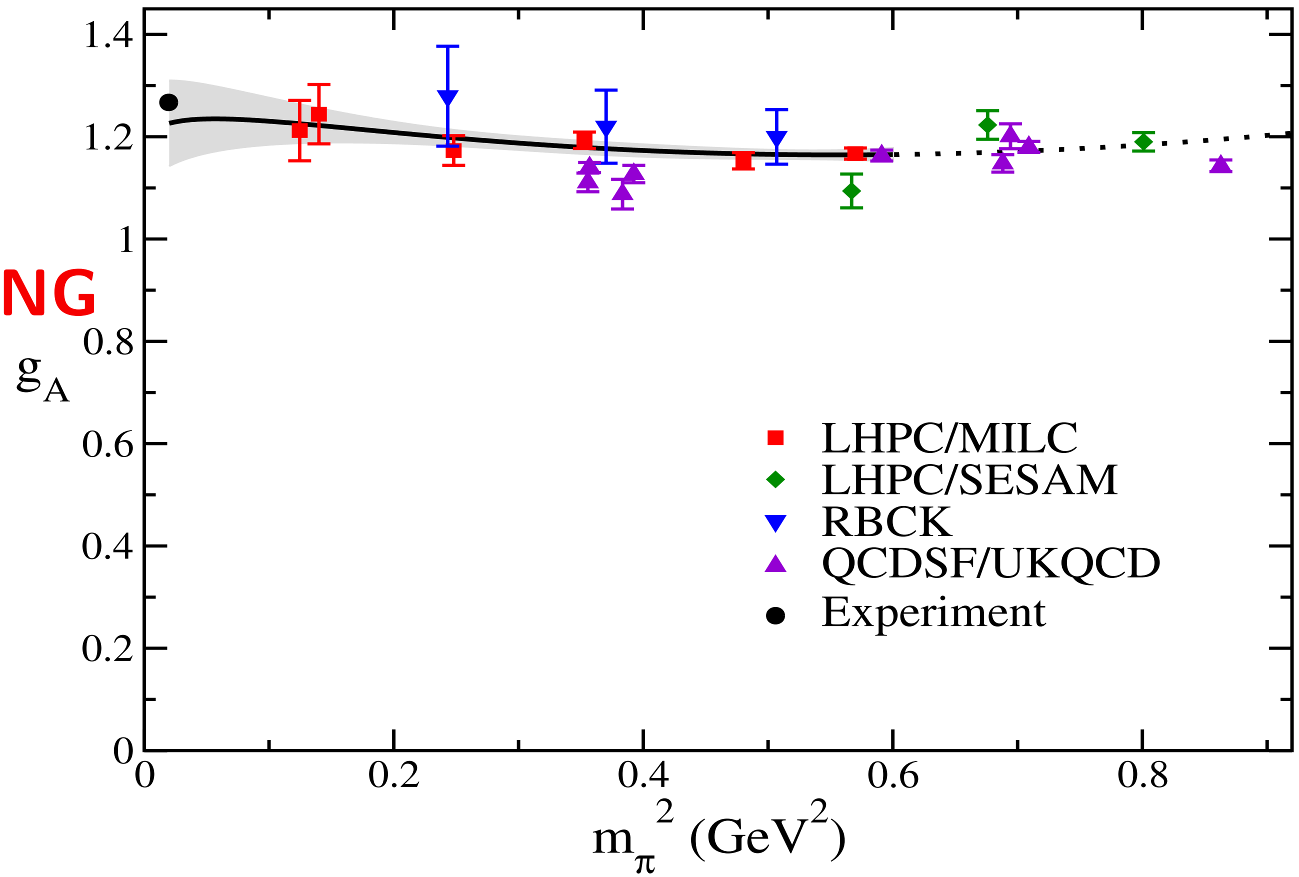
$0.032(03)$ RBC/UKQCD '06

$0.027(02)$ QCDSF '06

QCDSF results for 2nd moment:

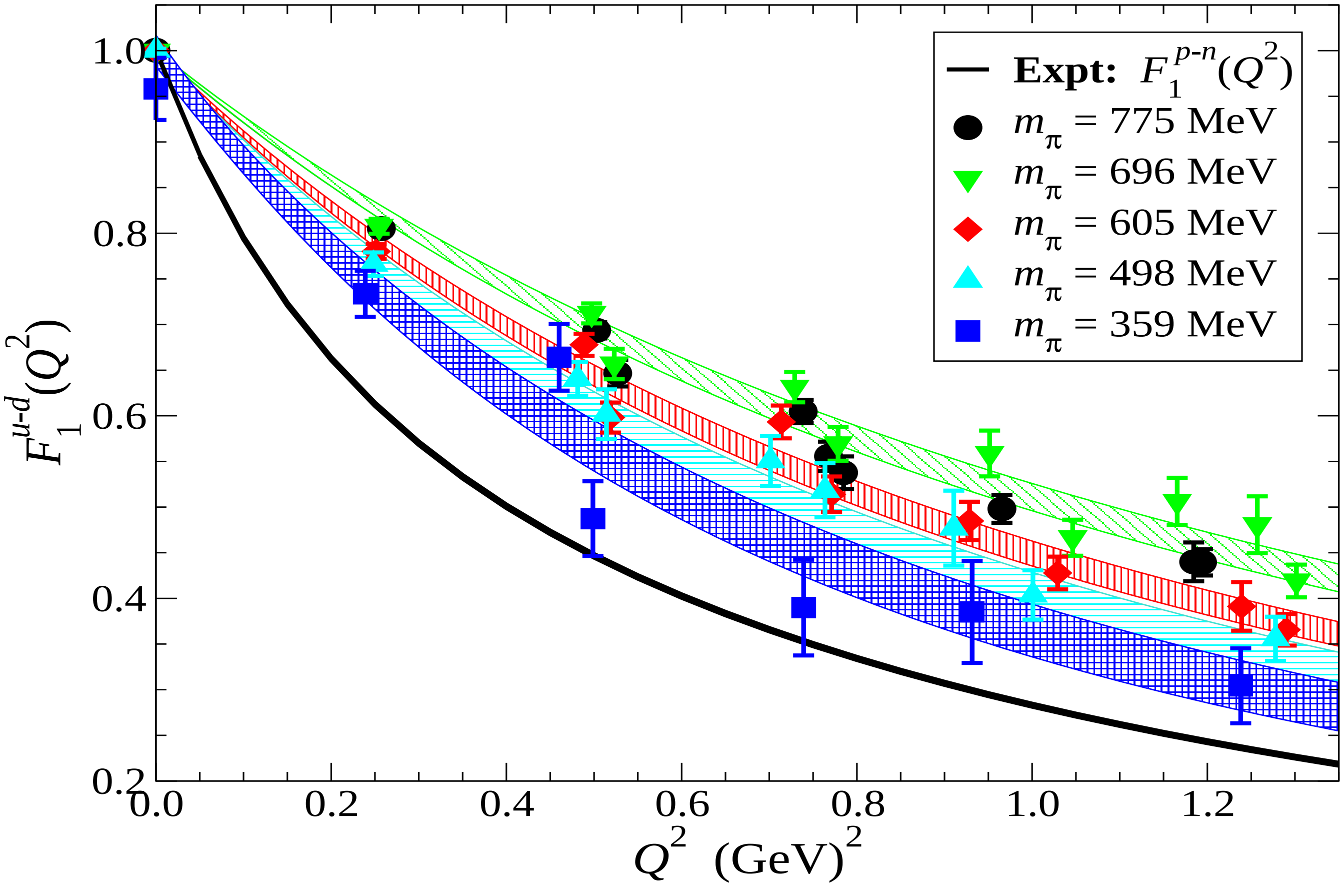
$$\langle \xi^2 \rangle_K^{\overline{\mathrm{MS}}}(2\mathrm{GeV}) = 0.26(2), \ \langle \xi^2 \rangle_\pi^{\overline{\mathrm{MS}}}(2\mathrm{GeV}) = 0.27(4)$$

NUCLEON AXIAL COUPLING



- consistent results among groups
- finite volume effects
- weak dependence on m_π
- LAT06: $g_A(m_\pi = 140\text{MeV}) = 1.23(8)$; cf exp. $1.2695(29)$
- dynamical 2+1 at smaller m_π under way (RBC/UKQCD/QCDSF)

ISOVECTOR F_1 FORM FACTOR



(LHPC data '06)

m_π still large, but approach experiment as m_π decreases

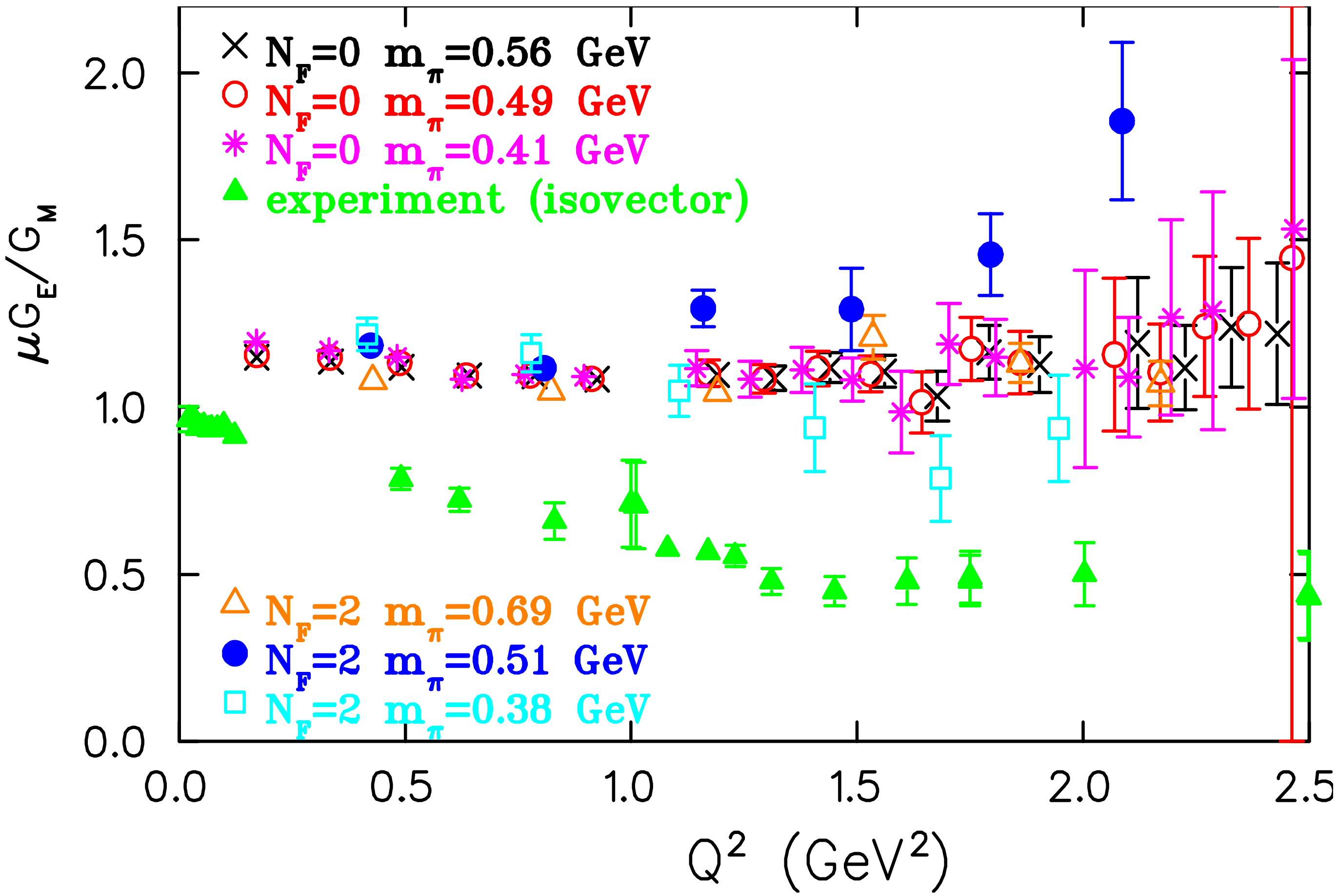
Difficulty: momenta quantized in units $2\pi/L$ for periodic bc

e.g. $L = 24a$ with $a = 0.1\text{fm}$ gives $2\pi/L \sim 0.52\text{GeV}$

(Bedaque, Sachrajda et al): using twisted pbc $p_i = (2\pi n + \theta_i)/L$

Ratio of electric and magnetic isovector form factors

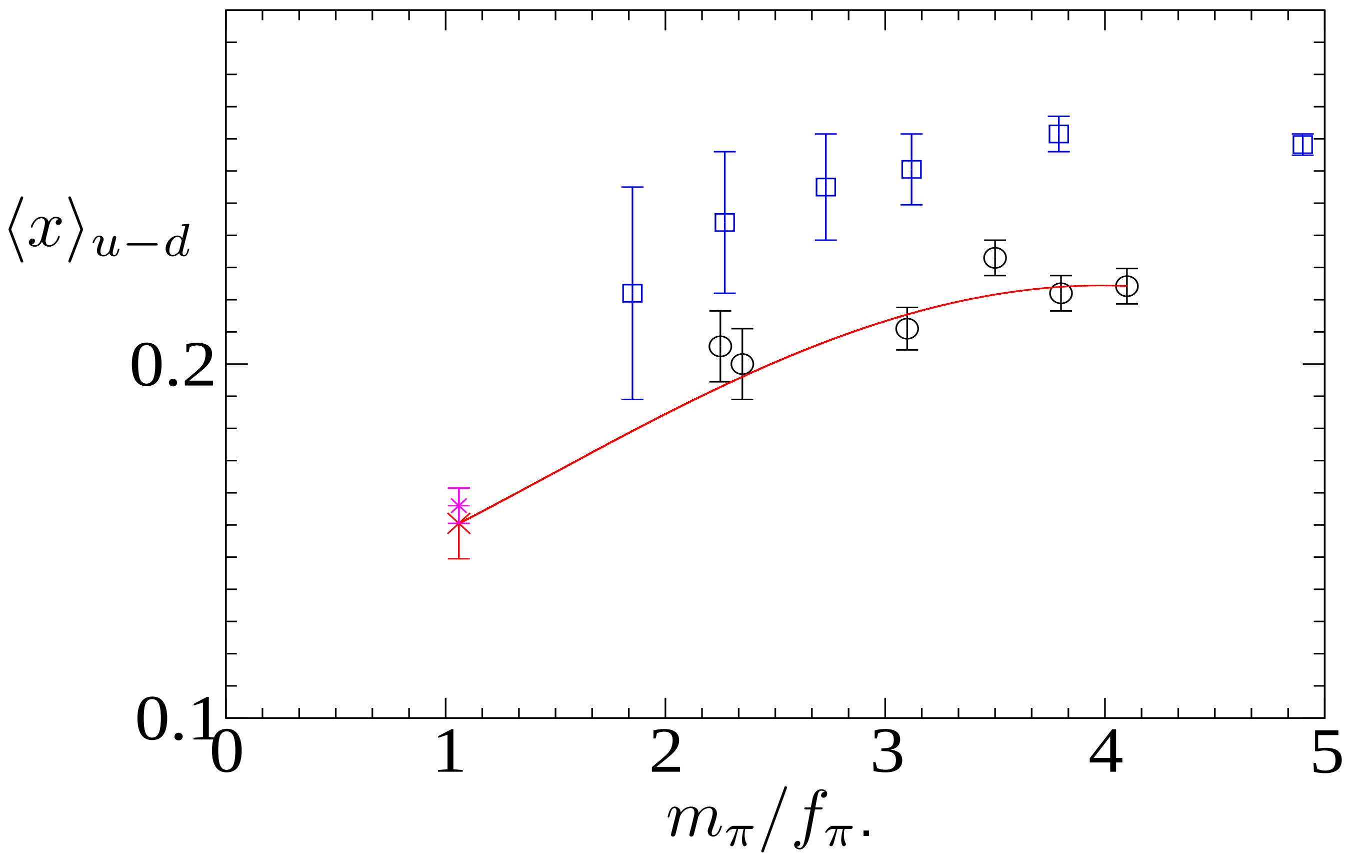
Alexandrou et al '06 data



Present disagreement with JLab experimental data

Lattice artifacts??

MOMENTS OF STRUCTURE FUNCTIONS



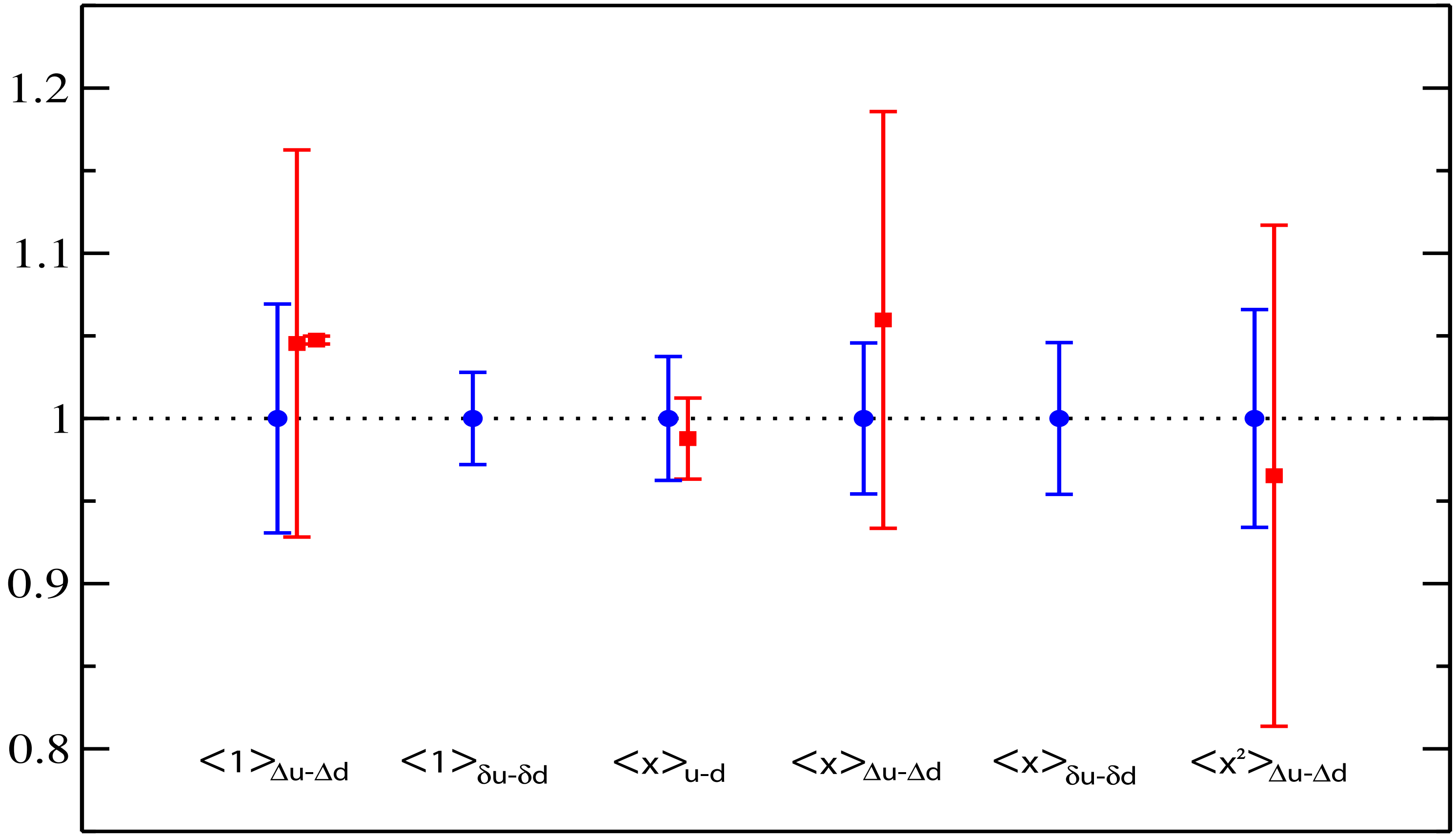
QCDSF: quenched overlap

LHPC: 2+1 dynam.

large χ**PT logs**: $\langle x \rangle_{u-d} = C\left[1 - r^2 (A \ln r^2 + B) + \ldots\right]$

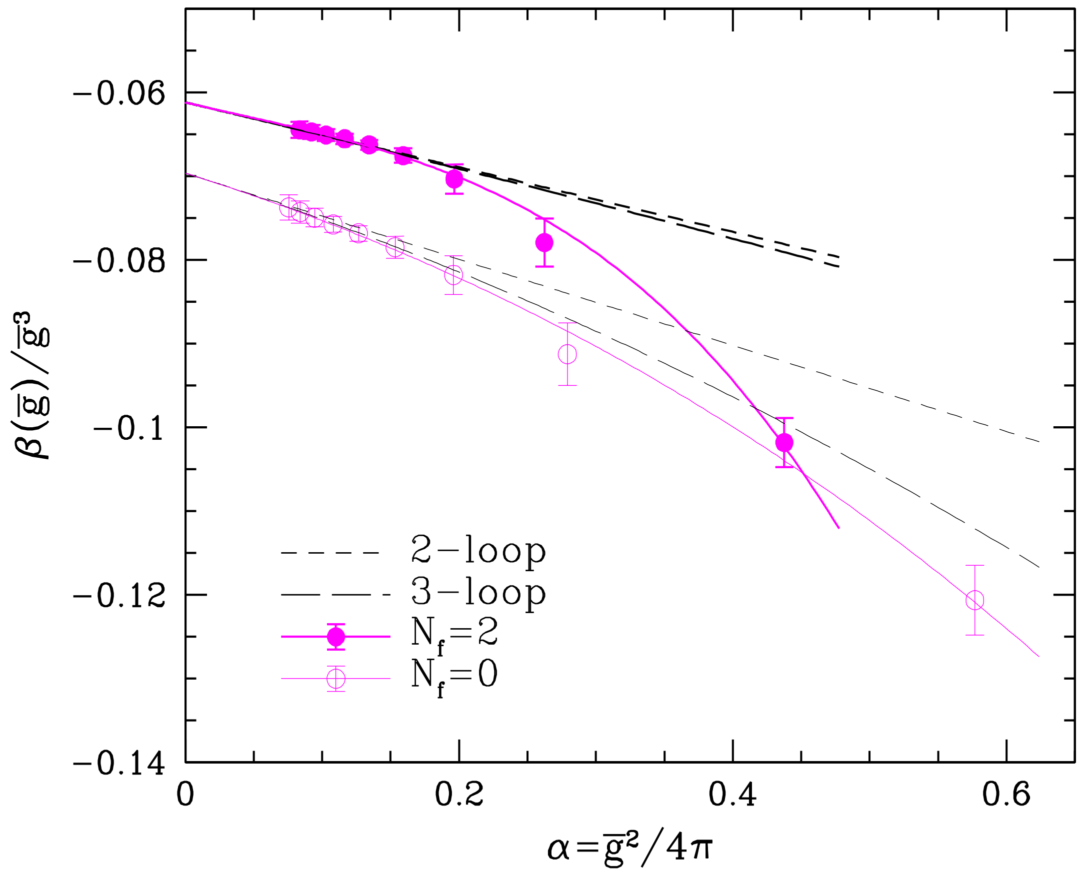
$r = m_\pi / (4\pi f_\pi)$; $A = 6g_A^2 + 2 \sim 11$

Ratios of Moments: Lattice/DIS



Chiral logs not yet seen in dynamical simulations

But if fit to χPT get good agreement with experiment



See sea quark effects

cf Lepage et al use **rooted staggered fermions**

a) At given g_0: measure e.g. a charmonium level splitting $a\Delta$

b) set Δ to experimental value $\rightarrow a(g_0)$

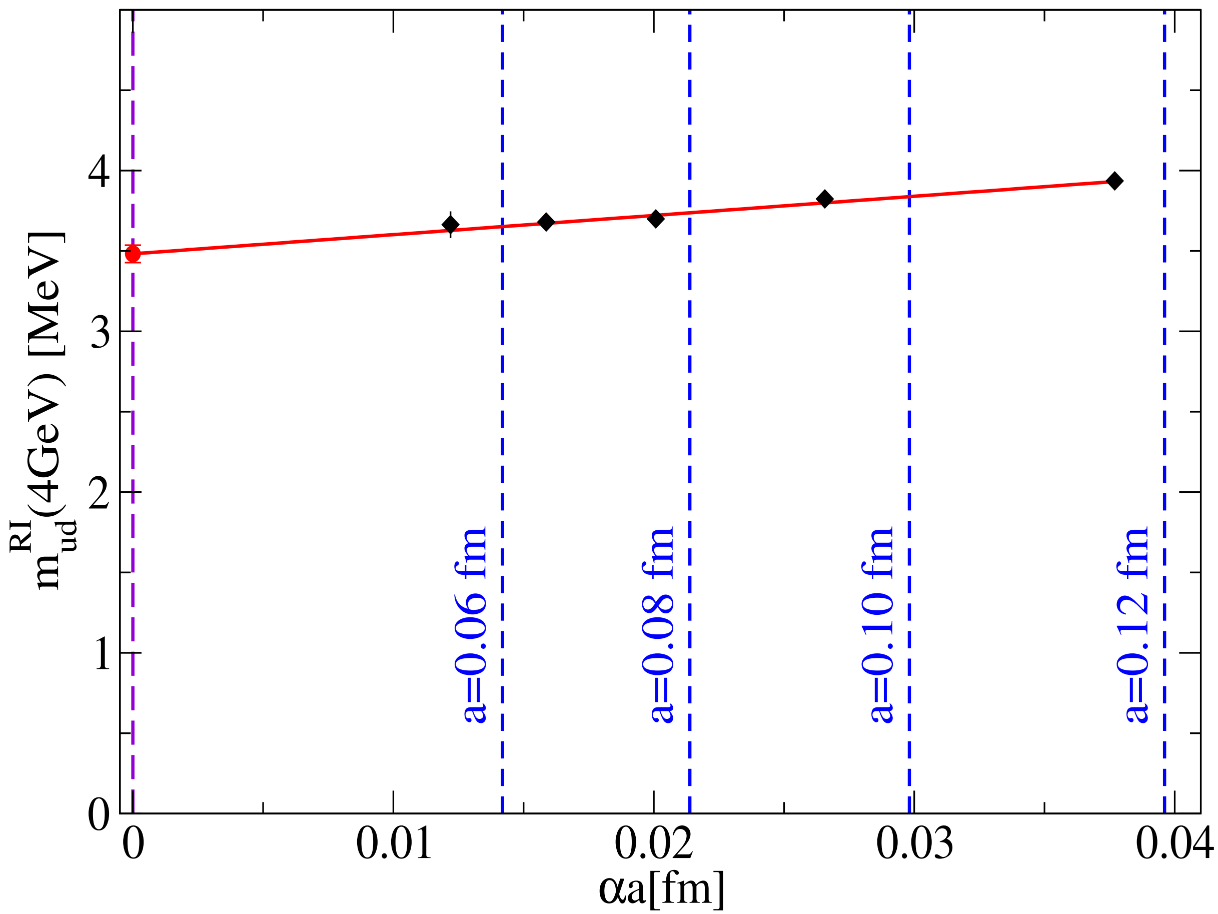
c) use $g_{\overline{\mathrm{MS}}}(a^{-1}) = g_0 + c_1 g_0^3 + \ldots$ as input for PT RG evolution

Lattice artifacts

"conventinal wisdom": The continuum limit is approached as $\sim (aM)^p \ln^q(Ma)$ with $p = 1$ or 2

theoretical framework: Symanzik's effective action

ud-quark mass
BMW (2010)



4. Finite-volume method to weak matrix elements

Weak transition matrix elements from finite volume correlation functions

Laurent Lellouch (Annecy, LAPTH) , Martin Luscher (CERN)

Commun.Math.Phys. 219 (2001) 31-44

$K \to \pi\pi$ decay amplitude

$$T(K \to \pi\pi) = \langle \pi\ p_1, \pi\ p_2\,\mathrm{out}|\mathcal{L}_{\mathrm{w}}(0)|K\ p\rangle = A e^{i\delta_0}$$

decay rate in infinite volume

$$\Gamma = \frac{k_\pi}{16\pi m_K^2}|A|^2, \qquad k_\pi \equiv \frac{1}{2}\sqrt{m_K^2 - 4m_\pi^2}$$

infinite volume

$$|A|^2 = 8\pi \left\{ q\frac{\partial\phi}{\partial q} + k\frac{\partial\delta_0}{\partial k} \right\}_{k=k_\pi} \left(\frac{m_K}{k_\pi}\right)^3 |M|^2$$

finite volume

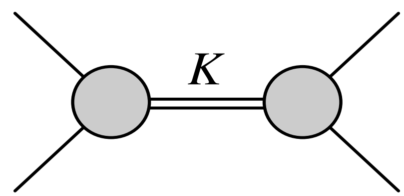
finite volume matrix element

$$M = \langle \pi\pi|H_{\mathrm{w}}|K\rangle,$$

$$W = 2\sqrt{m_\pi^2 + k^2},$$

$$n\pi - \delta_0(k) = \phi(q), \qquad q \equiv \frac{kL}{2\pi},$$

$\pi\pi$ scattering phase shift in finite vplume



Some results

$K \to (\pi\pi)_{I=2}$ decay amplitude

T. Blum *et al.*, PRL108(2012)141061
T. B.um *et al.*, PRD86(2012)074513

Lattice

$$\boxed{\begin{aligned}\mathrm{Re}A_2 &= 1.381(46)_{\mathrm{stat}}(258)_{\mathrm{syst}}10^{-8}\ \mathrm{GeV},\\ \mathrm{Im}A_2 &= -6.54(46)_{\mathrm{stat}}(120)_{\mathrm{syst}}10^{-13}\ \mathrm{GeV}.\end{aligned}}$$

$$a^{-1} = 1.364\ \mathrm{GeV},\ m_\pi = 142\ \mathrm{MeV},\ m_K = 506\ \mathrm{MeV}$$

$$W_{2\pi} = 486\ \mathrm{MeV}$$

LL formula

$$\mathcal{A}_i = \left[\frac{\sqrt{2^{n_{\mathrm{tw}}}}}{2\pi q_\pi}\sqrt{\frac{\partial\phi}{\partial q_\pi}+\frac{\partial\delta}{\partial q_\pi}}\right]\frac{2}{\sqrt{2^{n_{\mathrm{tw}}}}}L^{3/2}\sqrt{m_K}E_{\pi\pi}\mathcal{M}_i,$$

Experiment

$$\mathrm{Re}A_2 = 1.479(4)\times 10^{-8}\mathrm{GeV}$$

K^+ decays

$\Delta I = 1/2$ rule

P. Boyle *et al.*, PRL110(2013)152001

Lattice

$$\boxed{\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = \begin{cases} 9.1(2.1) & \text{for } m_K = 878\ \mathrm{MeV},\quad m_\pi = 422\ \mathrm{MeV}\\ 12.0(1.7) & \text{for } m_K = 662\ \mathrm{MeV},\quad m_\pi = 329\ \mathrm{MeV}.\end{cases}}$$

$$a^{-1} = 1.73\ \mathrm{GeV}$$

Experiment

$$\frac{\mathrm{Re}\ A_0}{\mathrm{Re}\ A_2} = 22.45(6)$$