Adiabatic Fluids

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Hydrodynamics as an effective field theory

- Hydrodynamics describes low-energy, near-equilibrium behaviour fluctuations of an equilibrium density matrix on scales large compared to the characteristic mean free path.

- Organize data into conserved currents: $T^{\mu\nu},\ J^{\mu}$

- Dynamics: conservation laws for the currents (up to anomalies)

$$\nabla_\mu T^{\mu\nu} = 0\,, \qquad \nabla_\mu J^{\mu} = 0$$

- Summarize hydrodynamic data as constitutive relations for the currents in terms of operators built from the hydrodynamical variables

$$\begin{aligned} T_{\alpha\beta} &= \varepsilon\, u_\alpha\, u_\beta + P\, P_{\alpha\beta} + 2\, q_{(\alpha}\, u_{\beta)} + \Pi_{\alpha\beta} \\ J^\alpha &= q\, u^\alpha + \nu^\alpha \end{aligned}$$

Transport coefficients

- The task of a hydrodynamicist is to determine the constitutive relations determining the conserved currents in terms of the hydrodynamic variables

$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} - \zeta\,\theta\,P^{\mu\nu} + \cdots$$

- The basic organizing principle is the same as in any effective field theory. One may imagine that one is working with an effective current algebra.

- The specific values of the transport coefficients themselves is determined by microscopic details of the underlying quantum system.

- It is worth recording a remapping of hydrodynamic variables:

$$\boldsymbol{\beta}^\mu \equiv \frac{u^\mu}{T}\,, \qquad \Lambda_{\boldsymbol{\beta}} \equiv \frac{\mu}{T} - \frac{u^\sigma}{T} A_\sigma$$

Constraints on hydrodynamics

- Constitutive relations obtained in a gradient expansion with transport coefficients/thermodynamic response parameters determined by microscopics.

$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} - \zeta\,\theta\,P^{\mu\nu} + \cdots$$

- The transport data are constrained macroscopically by demanding the second law of thermodynamics hold locally, e.g., $\eta, \zeta \geq 0$

$$\exists\; J_S^\mu \quad \to \quad \nabla_\alpha J_S^\alpha \geq 0$$

- The main surprise is that not all constraints are inequalities; there are non-trivial equality constraints:
  - anomaly induced transport is completely fixed

    Son Surowka '09

    Jensen Loganayagam Yarom '13
  - non-trivial relations for neutral fluid at 2∂ order (5 relations among 15 a-priori independent transport coefficients)

    Bhattacharyya '12

Constraints on hydrodynamics

- Hydrodynamic transport can be classified into three categories
  - Hydrostatic or thermodynamic response: fixed by equilibrium
  - Genuine hydrodynamic transport
  - Berry transport: undetermined by any form of entropy analysis
- Hydrostatic data can be understood by time-independent configurations of the fluid in the presence of non-trivial background sources.
- Can equivalently be encoded in a generating function, the ``equilibrium partition function'' which is a functional of stationary background sources.

$$\mathcal{K} \equiv \{K^\mu, \Lambda_K\}\,, \quad g_{\mu\nu}\, K^\mu\, K^\nu \leq 0 \;\longrightarrow\; \delta_{\mathcal{K}} g_{\mu\nu} = \delta_{\mathcal{K}} A_\mu = 0$$

Banerjee et. al. '12 Jensen et. al. '12

An autonomous theory of hydrodynamics?

- ✦ Are the constraints exhaustive?
  - ✴ gradient expansion is systematic but not derived from usual principles for effective field theories
- ✦ First principles understanding of entropy current?
- ✦ Would ideally like to have an effective action for deriving the dynamics.
  - ✴ dissipation introduces some difficulties.
  - ✴ require dynamics to be equivalent to current conservation.

➡ There exists a class of non-dissipative actions which seem to capture interesting aspects of hydrodynamical constraints. Not a-priori guaranteed!

➡ Benchmarking: anomaly induced transport.

Non-dissipative fluids: Definition

- Requirements of an effective action for NDF
  - Dynamical eom = conservation equations

$$\delta S_{eff} = 0 \qquad \Longrightarrow \qquad \nabla_\mu T^{\mu\nu} = 0$$

  - Lack of dissipation $\Longrightarrow$ conserved entropy current $\nabla_\alpha J^\alpha_S = 0$

- Ideal fluids clearly comprise one such system. The surprise is that there are non-trivial non-linear examples which seem to suggest some interesting constraints on hydrodynamic transport.

- Formalism is quite old: Taub '54, Carter '73

- Modern presentation: Dubovsky, Hui, Nicolis, Son '11

- Systematic analysis: Bhattacharyya, Bhattacharya, MR '12 & Haehl, MR '13

Lagrangian fields & symmetries

- The fundamental fields for NDF are taken to be Lagrangian variables which are labels for the fluid elements: $\phi_I\,, I = 1, \cdots, d-1$
- NB: view fluid as a space filling D-brane.
- Field reparameterization invariance: require arbitrary volume preserving diffeomorphisms in configuration space $\text{Sdiff}(\mathcal{M}_\phi)$

$$\phi^I \to \xi^I(\phi)\ , \qquad \text{Jacobian}(\xi, \phi) = 1$$

- The diffeo invariance in configuration space guarantees that Euler-Lagrange equations are identical to energy momentum conservation.

$$\delta_\phi S_{eff} = 0 \iff \nabla_\mu T^{\mu\nu} = 0$$

Entropy current

- Volume preserving symmetry $\Longrightarrow$ conserved entropy current

$$J_{\rm S}^{\beta} = \frac{1}{(d-1)!}\, \epsilon^{\beta\alpha_1\ldots\alpha_{d-1}}\, \epsilon_{I_1\ldots I_{d-1}} \prod_{j=1}^{d-1} \partial_{\alpha_j}\phi^{I_j} \qquad\qquad \nabla_\alpha J_S^\alpha = 0$$

- Interpret this current as being the entropy current to all orders by passing to the *entropy frame*

$$J_{\rm S}^{\alpha} = s\, u^{\alpha} \qquad\qquad s = \sqrt{-\, g_{\alpha\beta}\, J_{\rm S}^{\alpha}\, J_{\rm S}^{\beta}}$$

- Operator dimensions as appropriate for a phase field: $[d\phi] = 0$

- Intuitively expect that all dissipative transport coefficients will be vanishing in the theory; borne out by explicit analysis.

- The effective action should be viewed as the Legendre transform of an off-equilibrium Gibbs potential.

Neutral fluids: 0∂ and 1∂

- Zeroth order action reproducing ideal fluid behaviour

$$S_0 \propto \int d^d x \, \sqrt{-g} \, f(s)$$

$$T^{\mu\nu} = (s\, f'(s) - f(s))\, g^{\mu\nu} + s\, f'(s)\, u^\mu\, u^\nu$$

- Basically the action is the energy density as a function of entropy density.

- 1∂ corrections: No corrections for parity-even fluid dynamics since only available term is a total derivative

$$S_1 \propto \int d^d x \, \sqrt{-g} \, J_s^\alpha \, \nabla_\alpha f_1(s) = \int d^d x \, \sqrt{-g} \, \nabla_\alpha \left(f_1(s)\, J_s^\alpha \right)$$

- Parity-odd terms are of course interesting and non-vanishing in the presence of anomalies.

Adiabatic Fluids

Definition: An adiabatic fluid is one where off-shell entropy production is compensated for by energy-momentum and charge flow.

$$\nabla_\mu J_S^\mu + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu} - \mathrm{T}_H^{\mu\perp}\right) + (\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda) \cdot \left(D_\nu J^\nu - \mathrm{J}_H^\perp\right) = 0\,.$$

$$\boldsymbol{\beta}^\mu \equiv \frac{u^\mu}{T}\,, \qquad \Lambda_{\boldsymbol{\beta}} \equiv \frac{\mu}{T} - \frac{u^\sigma}{T} A_\sigma$$

- ✳ Locally an off-shell version of the Clausius relation hold.
- ✳ On-shell such fluids are non-dissipative, but the advantage of going off-shell is a certain linearity (adiabatic fluids can be superposed).
- ✳ Implementing second law off-shell is equivalent to requiring the l.h.s of AE to be positive definite using Lagrange multipliers.
- ✳ We are just focussing on the marginal situation to maximize control.

Adiabatic Fluids: A classification

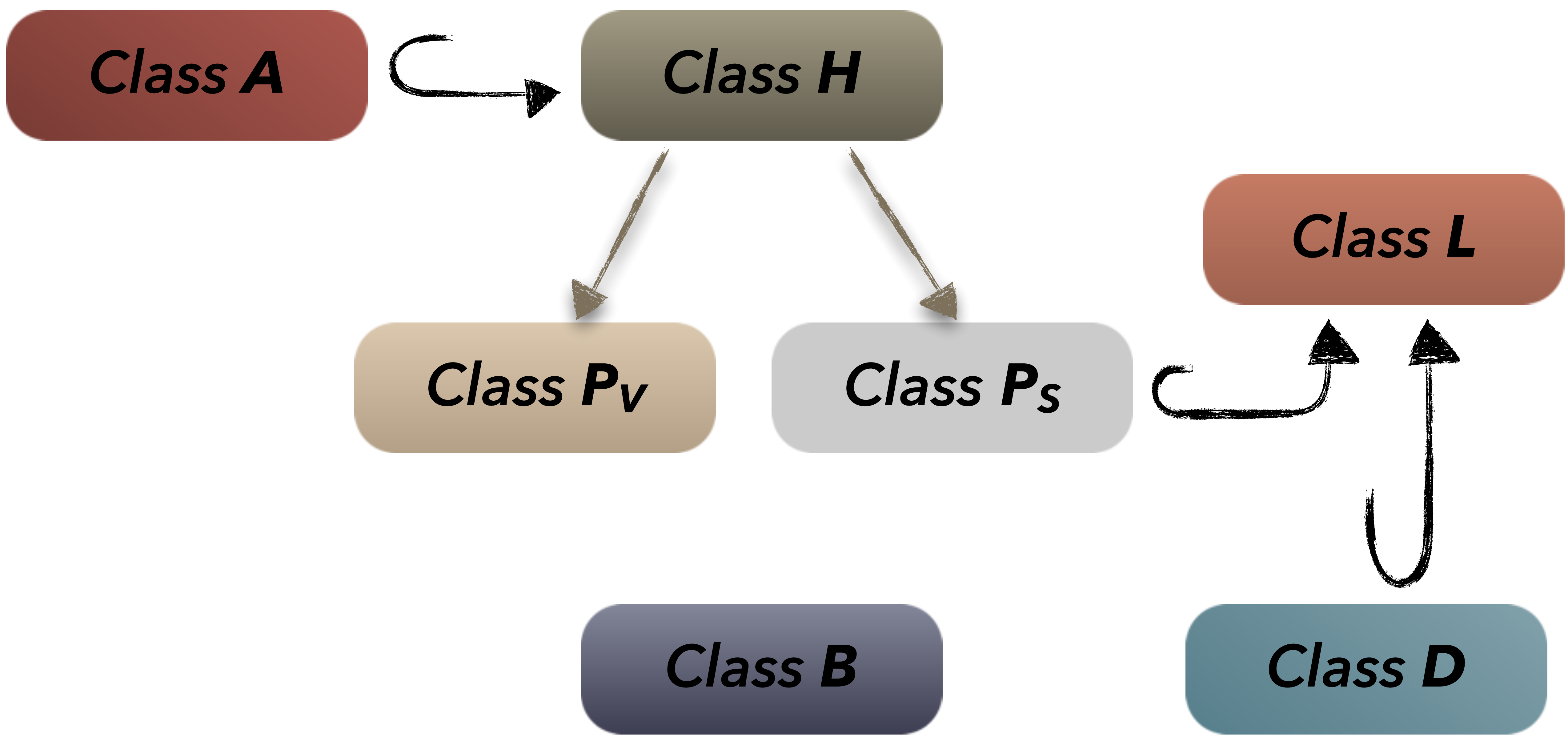
- The off-shell formalism is quite powerful. One can classify the solutions to AE into various classes & understand the origins of various constraints.
  - Class H: Hydrostatic configurations (subclasses P_V and P_S)

    Obtained by identifying hydro fields with background Killing fields
  - Class L: Lagrangian solutions

    Local Lagrangians functions of $\boldsymbol{\Psi} \equiv \{g_{\mu\nu}, A_\mu, \boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$
  - Class D: Non-dissipative fluids
  - Class A: Anomaly induced transport
  - Class B: Berry curvature terms

Adiabatic Fluids: A classification

A schematic set of connections between the various classes is as follows:



Class L Adiabatic fluids

- Consider diffeomorphism and gauge invariant scalar Lagrangian densities which are functionals of hydrodynamic fields $\boldsymbol{\Psi} \equiv \{g_{\mu\nu}, A_\mu, \boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$

$$S_{\text{hydro}} = \int d^d x \sqrt{-g}\; \mathcal{L}\left[\boldsymbol{\Psi}\right]$$

- The basic variational principle of this theory defines currents:

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}\;\mathcal{L}\right) - \nabla_\mu(\not{\delta}\Theta_{\text{PS}})^\mu \\ &\qquad = \frac{1}{2}\, T^{\mu\nu}\;\delta g_{\mu\nu} + J^\mu \cdot \delta A_\mu + T\, V_\sigma\; \delta\boldsymbol{\beta}^\sigma + T\,\zeta \cdot \left(\delta\Lambda_{\boldsymbol{\beta}} + A_\sigma\, \delta\boldsymbol{\beta}^\sigma\right) \end{aligned}$$

- Entropy density is defined as in thermodynamics

$$s \equiv \left(\frac{1}{\sqrt{-g}}\,\frac{\delta}{\delta T}\,\int \sqrt{-g}\;\mathcal{L}\left[\boldsymbol{\Psi}\right]\right)\Bigg|_{\{u^\sigma,\,\mu,\,g_{\alpha\beta},\,A_\alpha\}=\text{fixed}} \qquad\qquad J^\mu_S = s\, u^\mu$$

Class L Adiabatic fluids

Now diffeomorphism and gauge symmetries of the Lagrangian imply a set of Bianchi identities:

$$\nabla_\nu T^{\mu\nu} = J_\nu \cdot F^{\mu\nu} + \frac{g^{\mu\nu}}{\sqrt{-g}} \delta_{\mathfrak{B}} \left(\sqrt{-g}\; T V_\nu\right) + g^{\mu\nu}\, T\, \zeta \cdot \delta_{\mathfrak{B}} A_\nu$$

$$D_\sigma J^\sigma = \frac{1}{\sqrt{-g}} \delta_{\mathfrak{B}} \left(\sqrt{-g}\; T\zeta\right)$$

Together with the identity and an off-shell Euler relation

$$\nabla_\sigma J^\sigma_S = \nabla_\sigma (T\, s\, \boldsymbol{\beta}^\sigma) = \frac{1}{\sqrt{-g}} \delta_{\mathfrak{B}} \left(\sqrt{-g}\; T s\right)$$

$$T\, s + \mu \cdot \zeta + u^\sigma\, V_\sigma = 0$$

one ends up with the non-anomalous adiabaticity equation

$$\nabla_\mu J^\mu_S + \boldsymbol{\beta}_\mu \left(\nabla_\nu T^{\mu\nu} - J_\nu \cdot F^{\mu\nu}\right) + \left(\Lambda_{\boldsymbol{\beta}} + \boldsymbol{\beta}^\lambda A_\lambda\right) \cdot D_\nu J^\nu = 0$$

Dynamics in Class L

- The dynamics in Class L is supposed to reduce to the conservation of energy-momentum and charge currents.
- Naive variation with respect to $\{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\}$ does not respect this requirement, since it would lead to vanishing of the adiabatic heat/charge currents.
- Constrained variational principle: vary the hydrodynamic fields along a family related by Lie transport.

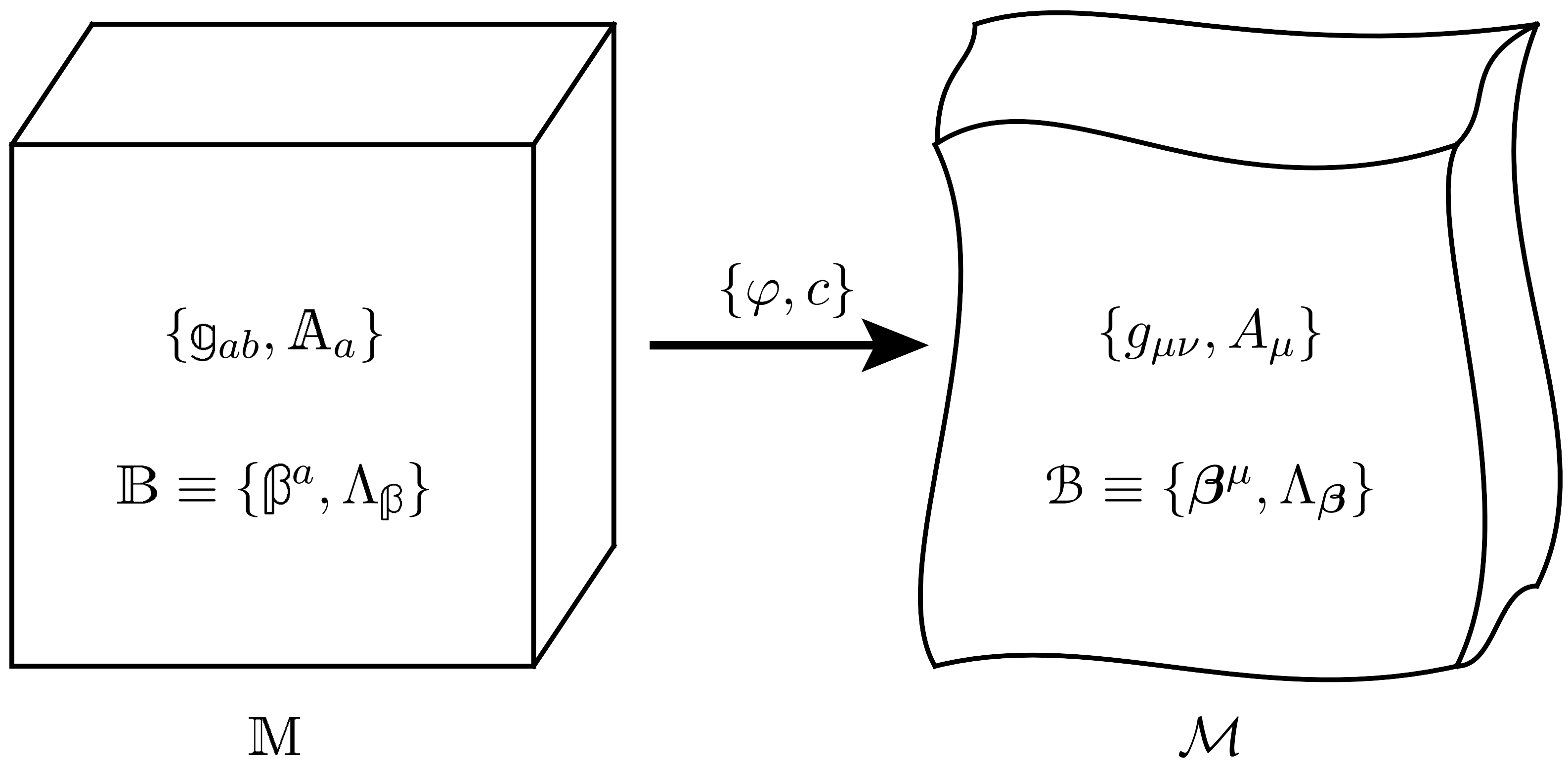
$$\mathbb{\delta} : \quad \mathbb{\delta}\boldsymbol{\beta}^\mu = \delta_\chi \boldsymbol{\beta}^\mu \,, \qquad \mathbb{\delta}\Lambda_{\boldsymbol{\beta}} = \delta_\chi \Lambda_{\boldsymbol{\beta}} \,, \qquad \mathbb{\delta} g_{\mu\nu} = \mathbb{\delta} A_\mu = 0$$

- This variation leads to equations of motion which when combined with the Bianchi identities leads to conservation

$$\frac{1}{\sqrt{-g}} \delta_{\mathscr{B}} \left(\sqrt{-g}\; T\, V_\mu\right) + T\,\zeta \cdot \delta_{\mathscr{B}} A_\mu \simeq 0$$
$$\frac{1}{\sqrt{-g}} \delta_{\mathscr{B}} \left(\sqrt{-g}\; T \zeta\right) \simeq 0$$

$$\xRightarrow{+\text{ Bianchi}} \qquad \nabla_\nu T^{\mu\nu} \simeq 0 \,, \qquad D_\nu J^\nu \simeq 0$$

Reference fields for Class L



The constrained variational principle can be alternately phrased as fixing a reference configuration and varying along the pull-back maps by diffeos and gauge transformations.

$$e^\mu_a \, \partial_\nu \varphi^a = \delta^\mu_\nu \,, \qquad e^\mu_a \, \partial_\mu \varphi^b = \delta^a_b \,.$$

$$\boldsymbol{\beta}^\mu = e^\mu_a(x) \; \beta^a[\varphi(x)]$$

$$\Lambda_{\boldsymbol{\beta}} = c(x) \; \Lambda_\beta[\varphi(x)] \; c^{-1}(x) + \boldsymbol{\beta}^\sigma(x) \, \partial_\sigma c(x) \; c^{-1}(x)$$

Reference fields for Class L

The pull-back maps come with some residual gauge symmetries, which one can view as diffeos and gauge transformations on the reference space.

$$\{\Lambda_{\mathbb{B}},\ c\} \sim \{f^{-1}\,\Lambda_{\mathbb{B}}\, f - f^{-1}\,\boldsymbol{\beta}^\sigma \partial_\sigma f,\ c\, f\}$$

$$\begin{aligned} \varphi^a &\mapsto f^a(\varphi) \\ \Lambda_{\mathbb{B}}[\varphi^a] &\mapsto \Lambda_{\mathbb{B}}[f^a(\varphi)] \\ \mathbb{B}^a[\varphi^b] &\mapsto \frac{\partial f^a}{\partial \varphi^c}\, \mathbb{B}^c[f^b(\varphi)] \end{aligned}$$

This allows us to pass over completely to the reference manifold

$$\int \sqrt{-g}\ \mathcal{L}(g, A, K, \Lambda_{\boldsymbol{\beta}}) = \int_{\mathbb{M}} \sqrt{-\mathfrak{g}}\ \mathcal{L}\left(\mathfrak{g}, \mathbb{A}, \mathbb{B}, \Lambda_{\mathbb{B}}\right) \equiv \int_{\mathbb{M}} \sqrt{-\mathfrak{g}}\ \mathbb{L}\left[\Psi\right]$$

One advantage of doing so will be clear when we consider Schwinger-Keldysh functionals for general non-equilibrium situations (also for anomalous transport in Class L).

Gauge fixing in Class L

Fields on the reference manifold can be gauge fixed to ``static gauge"

$$\Lambda_{\beta} = 0, \qquad \beta^{a=0} = 1 \qquad \text{and} \qquad \beta^{a=I} = 0 \quad \text{for } I \in \{1, \dots, d-1\}$$

There are still some residual gauge symmetries in this gauge which

$$\begin{aligned} \varphi^J &\mapsto h^J(\varphi^I)\,, \qquad \det\left(\frac{\partial h^J}{\partial \varphi^I}\right) \neq 0 \\ \varphi^0 &\mapsto \varphi^0 + g(\varphi^I) \\ c &\mapsto c\, f(\varphi^I) \end{aligned}$$

Reminiscent of symmetries described earlier in the context of non-dissipative fluids. We still have details to fill in to make precise contact.

Embedding Class H (P_S) into Class L

- The hydrostatic limit of a hydrodynamic system is obtained by subjecting the fluid to an arbitrary time-independent background sources.

$$\mathcal{K} \equiv \{K^\mu, \Lambda_K\}\,, \quad g_{\mu\nu}\, K^\mu\, K^\nu \leq 0 \;\longrightarrow\; \delta_{\mathcal{K}} g_{\mu\nu} = \delta_{\mathcal{K}} A_\mu = 0 \qquad\qquad \{\boldsymbol{\beta}^\mu, \Lambda_{\boldsymbol{\beta}}\} \;=\; \{K^\mu, \Lambda_K\}$$

Banerjee et. al. '12 Jensen et. al. '12

- The hydrostatic partition function is the Lagrangian evaluated on the hydrostatic configuration and integrated over a truncation of the physical manifold to a unit affine interval along the Killing field

$$W_{\text{Hydrostatic}} = \int_{\Sigma_E \times I_K} d^d x \sqrt{-g}\; \mathcal{L}_{\text{Hydrostatic}} \;+ \text{Boundary contributions}$$

- Makes completely clear that hydrodynamic entropy current is a Noether charge (cf., Iyer-Wald constructions for stationary black hole entropy).

Bhattacharyya '14

Embedding Class D into Class L

- Non-dissipative fluid effective actions use entropy as a fundamental variable.

$$\{\boldsymbol{\beta}^\sigma, \Lambda_{\boldsymbol{\beta}}\} \longrightarrow \{\mathbf{S}_{\alpha_1\ldots\alpha_{d-1}}, (\Lambda_{\mathbf{S}})_{\alpha_1\ldots\alpha_d}\}$$

$$^{\varepsilon}\mathbf{S}^\sigma \equiv \frac{1}{(d-1)!}\,\varepsilon^{\sigma\alpha_1\ldots\alpha_{d-1}}\,\mathbf{S}_{\alpha_1\ldots\alpha_{d-1}} = T\,s\,\boldsymbol{\beta}^\sigma = s\,u^\sigma$$

$$^{\varepsilon}\Lambda_{\mathbf{S}} \equiv \frac{1}{d!}\,\varepsilon^{\alpha_1\ldots\alpha_d}\,(\Lambda_{\mathbf{S}})_{\alpha_1\ldots\alpha_d} = T\,s\,\Lambda_{\boldsymbol{\beta}} = s\,(\mu - u^\alpha\,A_\alpha)$$

- Legendre transformation of the Lagrangian density gives the desired non-dissipative effective action

$$\int\sqrt{-g}\;\mathcal{L}_S\left[\boldsymbol{\Psi}_S\right] \equiv \int\sqrt{-g}\;\left(\mathcal{L}\left[\boldsymbol{\Psi}\right] - T\,s\right)\Big|_{\{\boldsymbol{\beta},\Lambda_{\boldsymbol{\beta}}\}\mapsto\{\mathbf{S},\boldsymbol{\Lambda}_{\mathbf{S}}\}}$$

Embedding Class D into Class L

Introducing reference entropic fields and going to the ``static gauge"

$$\mathbb{S}_{123\cdots(d-1)} = 1 \,, \qquad \mathbb{S}_{0I_1I_2\cdots I_{(d-2)}} = 0 \,, \qquad (\Lambda_{\mathbb{S}})_{0123\ldots(d-1)} = 0$$

one ends up finding residual gauge freedom, which coincides with the volume-preserving diffeomorphisms used in the earlier discussions.

volume-preserving spatial diffeos: $\phi^J \mapsto h^J(\phi^I) \,, \quad \det\left(\frac{\partial h^J}{\partial \phi^I}\right) = 1$

thermal shift symmetry: $\phi^0 \mapsto g(\phi^I, \phi^0) \,, \quad \frac{\partial g}{\partial \phi^0} \neq 0$

chemical shift symmetry: $\mathsf{c} \mapsto \mathsf{c}\, f(\phi^I)$

Temporal diffeomorphism field is non-dynamical: implies that component of energy-momentum conservation is traded for entropy conservation.

Some curiosities

- Lagrangian constructions for neutral fluids up to 2∂ and charged parity odd fluids in 3d to 1∂ have been explicitly carried out.
- As expected there are additional constraints on transport than those seen from the second law.
- Very curiously, some of these constraints are realized as *universal relations* amongst transport data in holography.

$$T^{\mu\nu}_{(2),\mathcal{W}} = \tau \left(u^\alpha \nabla_\alpha \sigma^{\mu\nu} + \frac{\Theta}{d-1} \sigma^{\mu\nu} \right) + \kappa \, C^{\mu\alpha\nu\beta} \, u_\alpha \, u_\beta + \lambda_1 \, \sigma^{\langle\mu\alpha} \, \sigma_\alpha{}^{\nu\rangle} + \lambda_2 \, \sigma^{\langle\mu\alpha} \, \omega_\alpha{}^{\nu\rangle} + \lambda_3 \, \omega^{\langle\mu\alpha} \, \omega_\alpha{}^{\nu\rangle}$$

$$\tau = \lambda_1 - \frac{1}{2} \lambda_2 \, , \qquad \lambda_1 = \kappa$$

Haack, Yarom '08

- Are these constraints hinting at a geometric origin of Class L fluids?

Class B adiabatic fluids

- This class of constitutive relations solves adiabaticity trivially. Non-equilibrium, non-dissipative data!

$$(T^{\mu\nu})_B \equiv -\frac{1}{4}\left(\tilde{\eta}^{(\mu\nu)(\alpha\beta)} - \tilde{\eta}^{(\alpha\beta)(\mu\nu)}\right) T\delta_{\mathbb{B}} g_{\alpha\beta} + \tilde{\Xi}^{(\mu\nu)\alpha} \cdot T\delta_{\mathbb{B}} A_\alpha$$

$$(J^\alpha)_B \equiv -\frac{1}{2}\tilde{\Xi}^{(\mu\nu)\alpha} T\delta_{\mathbb{B}} g_{\mu\nu} - \tilde{\sigma}^{[\alpha\beta]} \cdot T\delta_{\mathbb{B}} A_\beta$$

- The entropy current is canonical (given just by projections of energy-momentum and charge currents)

Hall Transport in 3 dimensions

$$(T^{\mu\nu})_B = -\tilde{\eta}_H\, u_\rho\, \left(\varepsilon^{\rho\mu\alpha}\, \sigma^\nu_\alpha + \varepsilon^{\rho\nu\alpha}\, \sigma^\mu_\alpha\right)$$

$$(J^\alpha)_B = \tilde{\sigma}_H \cdot u_\rho\, \varepsilon^{\rho\alpha\beta}\left[E_\beta - T\, D_\beta\left(\frac{\mu}{T}\right)\right]$$

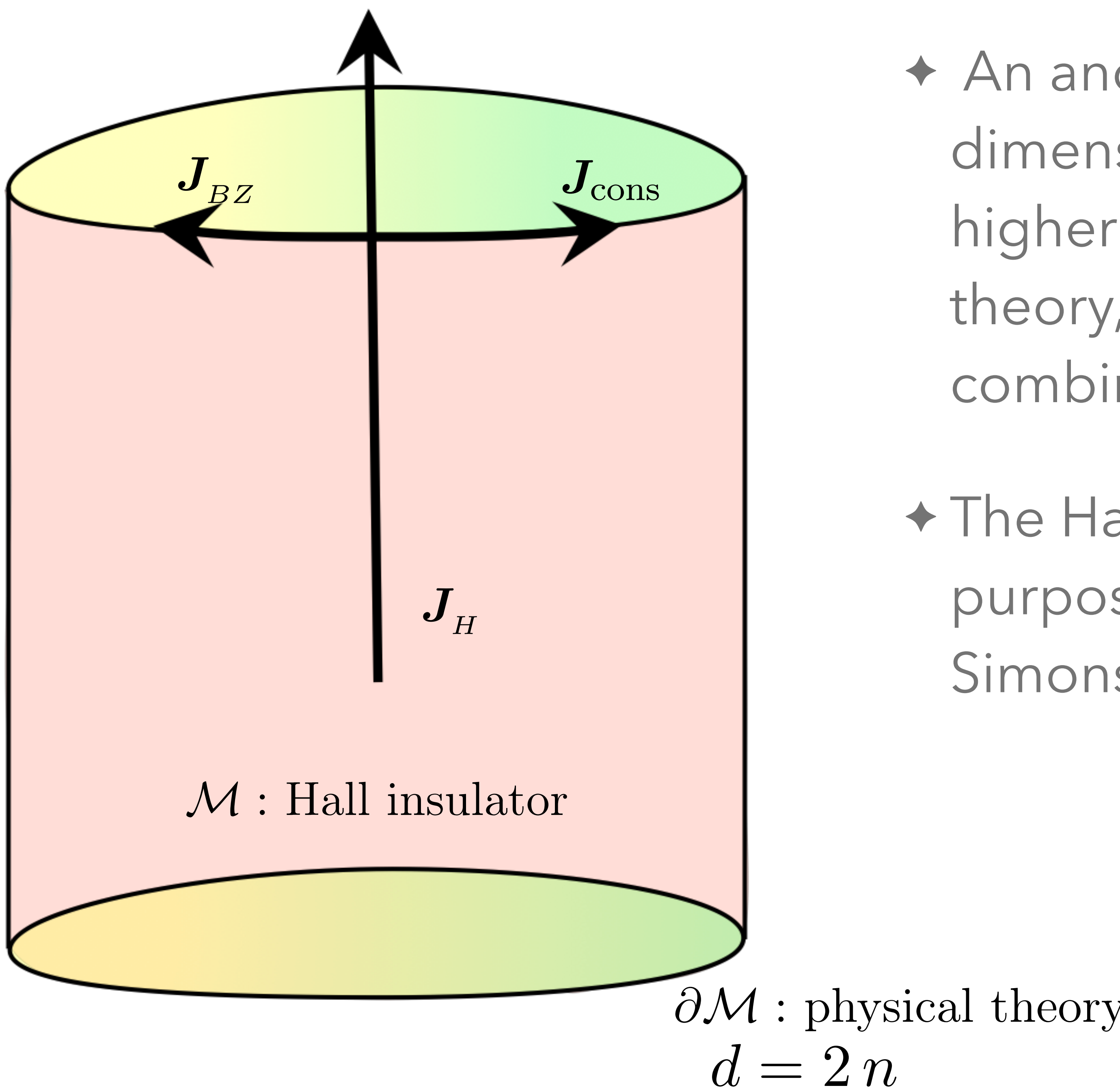
Neutral fluids in arbitrary dimensions

$$(T^{\mu\nu})_B = -\lambda_\sigma\left(\Theta\,\sigma^{\mu\nu} - \sigma^2\, P^{\mu\nu}\right) - \lambda_\omega\left(\omega^{\mu\alpha}\sigma^\nu_\alpha + \omega^{\nu\alpha}\sigma^\mu_\alpha\right)$$

Class A: Anomalies as a litmus test

- ✦ Anomalies provide an interesting window into the structure of quantum field theories and can be used to understand constraints on admissible dynamics.
- ✦ Since quantum anomalies leave behind indelible signatures in transport phenomena they can be used to learn some useful constraints on hydrodynamics.
- ✦ Constraints on anomalous transport have been derived from entropy analysis, generating functions and fluid/gravity & all of these approaches agree on the constitutive relations. Jensen, Loganayagam, Yarom '12-'13
- ✦ Since we know that anomalous transport is adiabatic we can use it as a benchmark for hydrodynamic effective actions.
- ✦ Punchline: Anomalous transport can indeed be recovered from a Class L Lagrangian but with some surprising twists.

Global Anomalies: Anomaly inflow



- An anomalous theory in d = 2n dimensions can be coupled to a higher dimensional topological theory, so as to render the combined system anomaly free.
- The Hall insulator for present purposes will just be a Chern-Simons theory:

$$\boldsymbol{I}_{2n+1}^{CS} = c_{_A}\,\boldsymbol{A} \wedge \boldsymbol{F}^n$$

Currents & Ward identities

- The inflow picture makes clear the various currents in the game.

$$
\begin{aligned}
J^{a}_{H} &= \frac{(n+1)\ c_{A}}{2^{n}}\ \epsilon^{a p_1 p_2 \cdots p_{2n-3} p_{2n-2}}\ F_{p_1 p_2} \cdots F_{p_{2n-1} p_{2n}} \\
J^{\perp}_{H} &= \frac{(n+1)\ c_{A}}{2^{n}} \epsilon^{\alpha_1 \beta_1 \cdots \alpha_n \beta_n}\ F_{\alpha_1 \beta_1} \cdots F_{\alpha_n \beta_n}\ , \\
J^{\alpha}_{BZ} &= \frac{n\ c_{A}}{2^{n-1}}\ \epsilon^{\alpha\beta\delta_1\delta_2 \cdots \delta_{2n-3}\delta_{2n-2}}\ A_{\beta}\ F_{\delta_1 \delta_2} \cdots F_{\delta_{2n-3}\delta_{2n-2}}
\end{aligned}
$$

- The covariant current:

$$J^{\gamma}_{\text{cov}} = J^{\gamma}_{\text{cons}} + J^{\gamma}_{BZ}$$

- Ward identities:

$$\nabla_{\alpha} T^{\alpha\beta} = F^{\beta\alpha}\left(J_{\text{cov}}\right)_{\alpha}\ , \qquad \nabla_{\alpha} J^{\alpha}_{\text{cov}} = J^{\perp}_{H}$$

Anomalous constitutive relations I

- Adiabaticity equation for anomalous transport:

$$(\nabla_\alpha + \mathfrak{a}_\alpha)\, q^\alpha_{anom} - J^\alpha_{anom}\, E_\alpha = T\, \nabla^\alpha J^\alpha_{\mathrm{S},\, anom} + \mu \left(\nabla_\alpha J^\alpha_{anom} - J^\perp_{H} \right)$$

- This equation can be solved explicitly to obtain the anomalous currents.

- For abelian anomalies in four dimensions the currents are determined to be

$$J^\mu_{\mathrm{S},anom} = 0$$

$$q^\alpha_{anom} = -4\, c_A\, \mu^3\, \omega^\alpha - 3\, c_A\, \mu^2\, B^\alpha\,, \qquad \Pi^{\alpha\beta}_{anom} = 0\,,$$

$$J^\alpha_{anom} = -6\, c_A\, \mu^2\, \omega^\alpha - 6\, c_A\, \mu\, B^\alpha\,,$$

Loganayagam '11

Anomalous constitutive relations II

- We can translate the solution of adiabaticity equation to Landau frame using ideal fluid eoms to recover more familiar expressions: $C = -6\, c_A$

$$
\begin{aligned}
T^{\alpha\beta}_{(\mathrm{Landau})} &= (\varepsilon + P) u^\alpha u^\beta + P g^{\alpha\beta} + \dots\,, \\
J^{\alpha}_{(\mathrm{Landau})} &= \rho\, u^\alpha + C\, \mu^2 \left(1 - \frac{2}{3}\frac{\rho\,\mu}{\varepsilon + P}\right) \omega^\alpha + C\mu \left(1 - \frac{1}{2}\frac{\rho\,\mu}{\varepsilon + P}\right) B^\alpha + \dots\,, \\
J^{\alpha}_{\mathrm{s}\,(\mathrm{Landau})} &= s\, u^\alpha - C\, \frac{s}{\varepsilon + P} \left(\frac{2}{3}\, \mu^3\, \omega^\alpha + \frac{1}{2}\, \mu^2\, B^\alpha\right) + \dots\,,
\end{aligned}
$$

Son Surowka '09

- Similar discussion applies to other dimensions and non-abelian currents.

The anomalous effective action

Solution to the adiabaticity equation for anomalous transport derived from an effective action:

$$S_{anom} = \int_{\mathcal{M}_{d+1}} \sqrt{-g_{d+1}}\, \mathcal{L}_{anom} = \int_{\mathcal{M}_{d+1}} \boldsymbol{V}_{\boldsymbol{\mathcal{P}}}[\boldsymbol{A}, \hat{\boldsymbol{A}}] = \int_{\mathcal{M}_{d+1}} \frac{\boldsymbol{u}}{2\boldsymbol{\omega}} \wedge \Big(\boldsymbol{\mathcal{P}}[\boldsymbol{F}] - \widehat{\boldsymbol{\mathcal{P}}}[\hat{\boldsymbol{F}}]\Big)$$

This functional depends on the *hydrodynamic shadow gauge fields*

$$\hat{\boldsymbol{A}} = \boldsymbol{A} + \mu\, \boldsymbol{u}$$

For the cognoscenti: the effective action is a transgression form which interpolates between the physical and the shadow connection.

$$\boldsymbol{A}_t = t\, \boldsymbol{A}_{t=1} + (1-t)\, \boldsymbol{A}_{t=0} \qquad\qquad \boldsymbol{A}_{t=0} = \hat{\boldsymbol{A}} \text{ and } \boldsymbol{A}_{t=1} = \boldsymbol{A}$$

To implement the inflow, all fields are taken to live in an auxiliary higher dimensional spacetime via $\boldsymbol{\Psi}_{d+1} = \{g_{mn}, A_m, \boldsymbol{\beta}^m, \Lambda_{\boldsymbol{\beta}}\}\,, \qquad \boldsymbol{\beta}^\perp = 0$

Anomalous constitutive relations

From the usual variational principle we pick up the anomalous constitutive relations

$$T^{\alpha\beta}_{anom} = q^{\alpha}_{\boldsymbol{\mathcal{P}}} u^{\beta} + q^{\beta}_{\boldsymbol{\mathcal{P}}} u^{\alpha} \,, \qquad J^{\alpha}_{anom} = J^{\alpha}_{\boldsymbol{\mathcal{P}}} \,, \qquad J^{\alpha}_{S,\,anom} = 0$$

$$\begin{aligned} \star \boldsymbol{J}_{\boldsymbol{\mathcal{P}}} &\equiv \int_0^1 dt \left[\left(\frac{\partial^2 \boldsymbol{\mathcal{P}}}{\partial \boldsymbol{F} \partial \boldsymbol{F}} \right)_t \cdot \frac{d\boldsymbol{A}_t}{dt} \right] \\ &= \frac{\boldsymbol{u}}{2\boldsymbol{\omega}} \wedge \left\{ \frac{\partial \boldsymbol{\mathcal{P}}}{\partial \boldsymbol{F}} - \frac{\partial \widehat{\boldsymbol{\mathcal{P}}}}{\partial \hat{\boldsymbol{F}}} \right\} \end{aligned}$$

$$\begin{aligned} \star \boldsymbol{q}_{\boldsymbol{\mathcal{P}}} &= \int_0^1 ds \int_0^s dt \left[\mu \cdot \left(\frac{\partial^2 \boldsymbol{\mathcal{P}}}{\partial \boldsymbol{F} \partial \boldsymbol{F}} \right)_t \cdot \frac{d\boldsymbol{A}_t}{dt} \right] \\ &= -\frac{\boldsymbol{u}}{(2\boldsymbol{\omega})^2} \wedge \left\{ \boldsymbol{\mathcal{P}} - \widehat{\boldsymbol{\mathcal{P}}} - \left(\boldsymbol{F} - \hat{\boldsymbol{F}} \right) \cdot \frac{\partial \widehat{\boldsymbol{\mathcal{P}}}}{\partial \hat{\boldsymbol{F}}} \right\} \end{aligned}$$

These reduce to the currents we described earlier for abelian flavour anomalies upon explicit evaluation, but the form above completes the story for arbitrary flavour symmetries.

Mixed anomalies

- Mixed anomalies (flavour+Lorentz) also work but we need to incorporate a spin chemical potential, shadow spin connection and deform the entropy current.

$$\Omega^{\mu}{}_{\nu} = \frac{1}{2}\,T\left(D_\nu \boldsymbol{\beta}^\mu - D^\mu \boldsymbol{\beta}_\nu\right) \qquad \hat{\boldsymbol{\Gamma}}^{\mu}{}_{\nu} = \boldsymbol{\Gamma}^{\mu}{}_{\nu} + \Omega^{\mu}{}_{\nu}\,\boldsymbol{u}$$

$$\int_{\mathcal{M}_{d+1}} \sqrt{-g_{d+1}}\,\mathcal{L}_{anom} = \int_{\mathcal{M}_{d+1}} \boldsymbol{V}_{\boldsymbol{\mathcal{P}}}[\boldsymbol{A},\boldsymbol{\Gamma},\hat{\boldsymbol{A}},\hat{\boldsymbol{\Gamma}}] = \int_{\mathcal{M}_{d+1}} \frac{\boldsymbol{u}}{2\boldsymbol{\omega}} \wedge \left(\boldsymbol{\mathcal{P}}[\boldsymbol{F},\boldsymbol{R}] - \widehat{\boldsymbol{\mathcal{P}}}[\hat{\boldsymbol{F}},\hat{\boldsymbol{R}}]\right)$$

- The currents can be derived again by the standard variational principle, but the entropy current is non-canonical.

$$\begin{aligned}
T^{\alpha\beta}_{anom} &= q^{\alpha}_{\boldsymbol{\mathcal{P}}} u^\beta + q^{\beta}_{\boldsymbol{\mathcal{P}}} u^\alpha + \frac{1}{2}\,D_\rho\left(\Sigma^{\alpha[\beta\rho]}_{\boldsymbol{\mathcal{P}}} + \Sigma^{\beta[\alpha\rho]}_{\boldsymbol{\mathcal{P}}} - \Sigma^{\rho(\alpha\beta)}_{\boldsymbol{\mathcal{P}}}\right) - \frac{1}{2}\hat{\Sigma}^{\perp(\alpha\beta)}_{H} \\
J^{\alpha}_{anom} &= J^{\alpha}_{\boldsymbol{\mathcal{P}}}\,, \\
J^{\alpha}_{S,\,anom} &= -\frac{1}{2}\boldsymbol{\beta}_\sigma \hat{\Sigma}^{\perp[\alpha\sigma]}_{H}\,.
\end{aligned}$$

Anomalous Ward identities

The Ward identities from diffeomorphism and gauge invariance (after using the bulk Bianchi identities) are

$$
\begin{aligned}
&D_\beta \left(T^{\alpha\beta}_{non-anom} + T^{\alpha\beta}_{anom}\right) \\
&\qquad \simeq (J^\sigma_{non-anom} + J^\sigma_{anom}) \cdot F^\alpha{}_\sigma + \frac{1}{2} D_\gamma \left(\Sigma_H^{\perp[\alpha\gamma]} - \hat{\Sigma}_H^{\perp[\alpha\gamma]}\right) - \left(\mu \cdot \hat{\mathrm{J}}^\perp_H + \frac{1}{2}\Omega^\nu{}_\mu \hat{\Sigma}_H{}^{\perp\mu}{}_\nu\right) u^\alpha \,, \\
&D_\sigma (J^\sigma_{non-anom} + J^\sigma_{anom}) \simeq \mathrm{J}^\perp_H - \hat{\mathrm{J}}^\perp_H \,.
\end{aligned}
$$

These are not the correct conservation equations.

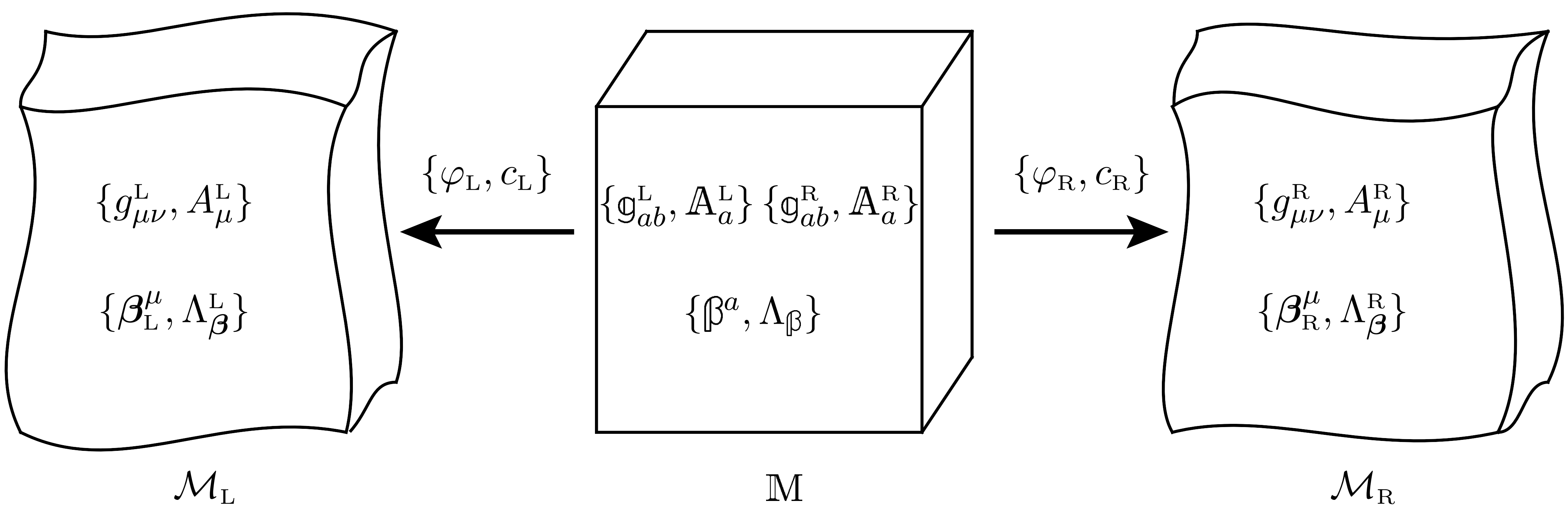
The shadow terms on the r.h.s suggest that the inflow has excess contributions. This is manifest in the transgression language at the level of the action.

A Schwinger-Keldysh functional for anomalies

- A convenient way to capture the Ward identities is to think about the problem in the thermofield doubled theory.

$$\begin{aligned} S_{tot} &\equiv S_{tot}[\mathbf{\Psi}_{\rm L}, \mathbf{\Psi}_{\rm R}] \\ &= S_{eff}\,[\mathbf{\Psi}_{\rm R}] - S_{eff}\,[\mathbf{\Psi}_{\rm L}] + S_{cross}[\mathbf{\Psi}_{\rm R}, \mathbf{\Psi}_{\rm L}] \end{aligned}$$

influence functional



Doubled cure for Ward identities

- The conserved currents in hydrodynamics are defined by variation with respect to the ``difference sources".

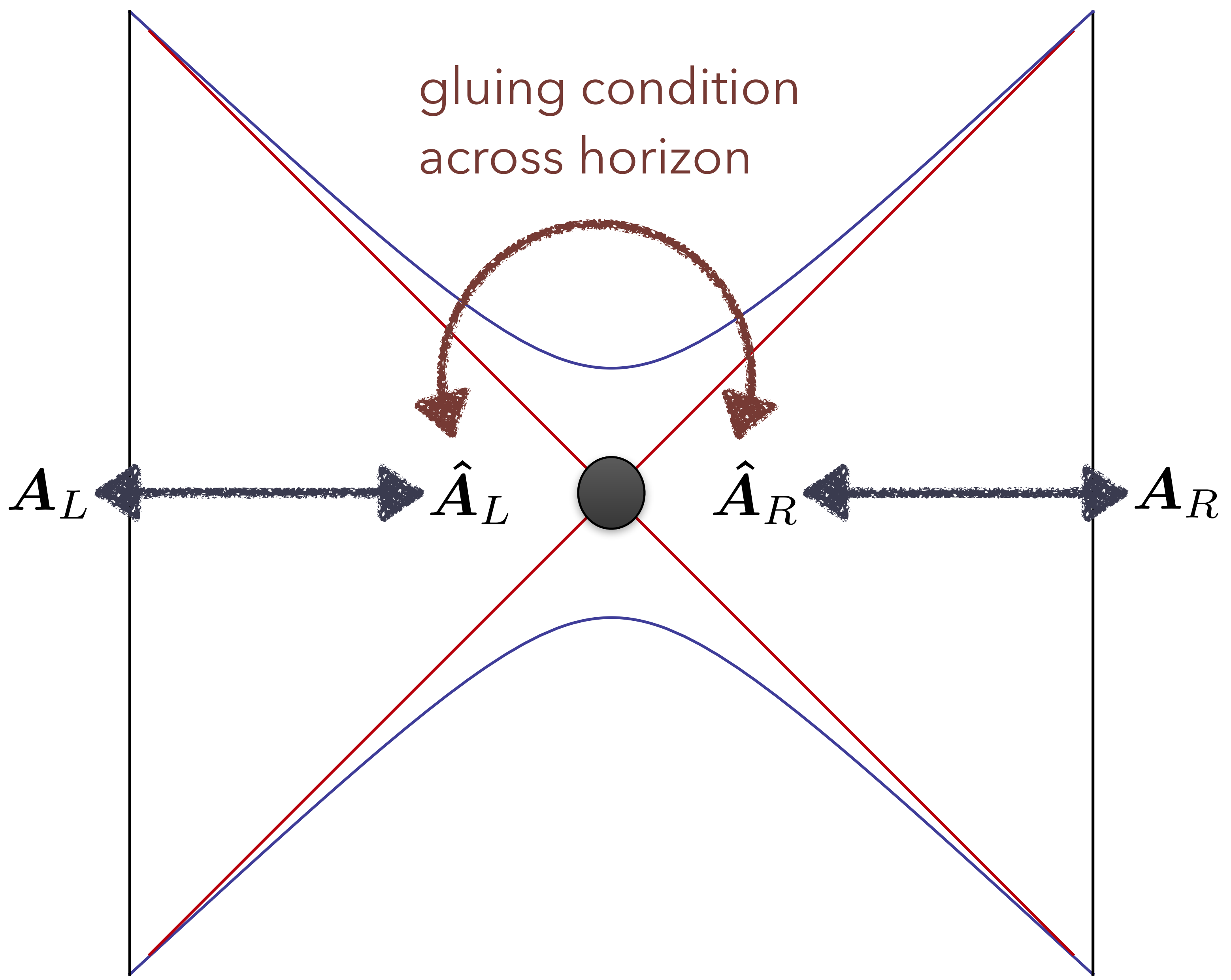
$$T^{ab}_{hydro} = \frac{2}{\sqrt{-\mathfrak{g}}}\left(\frac{\delta S_{tot}}{\delta \mathfrak{g}^{\rm R}_{ab}[\varphi_{\rm R}]} - \frac{\delta S_{tot}}{\delta \mathfrak{g}^{\rm L}_{ab}[\varphi_{\rm L}]}\right)\Bigg|_{\substack{\varphi^a_{\rm R}(x)\,=\,\varphi^a_{\rm L}(x)\,\equiv\,\varphi^a(x)\\ c_{\rm R}(x)\,=\,c_{\rm L}(x)\,\equiv\,c(x)}}$$

$$J^{a}_{hydro} = \frac{1}{\sqrt{-\mathfrak{g}}}\left(\frac{\delta S_{tot}}{\delta \mathbb{A}^{\rm R}_{a}[\varphi_{\rm R}]} - \frac{\delta S_{tot}}{\delta \mathbb{A}^{\rm L}_{a}[\phi_{\rm L}]}\right)\Bigg|_{\substack{\varphi^a_{\rm R}(x)\,=\,\varphi^a_{\rm L}(x)\,\equiv\,\varphi^a(x)\\ c_{\rm R}(x)\,=\,c_{\rm L}(x)\,\equiv\,c(x)}}$$

- There is a unique influence functional which respects the symmetries of the construction and provides the correct currents + Ward identities:

$$S_{IF} = \int_{\mathcal{M}_{d+1}} \boldsymbol{V_{\mathcal{P}}}[\hat{\boldsymbol{A}}_{\rm R}, \hat{\boldsymbol{\Gamma}}_{\rm R}; \hat{\boldsymbol{A}}_{\rm L}, \hat{\boldsymbol{\Gamma}}_{\rm L}] \equiv \int_{\mathcal{M}_{d+1}} \boldsymbol{V_{\mathcal{P}}}\left(\hat{\boldsymbol{A}}[\Psi_{\rm R}], \hat{\boldsymbol{\Gamma}}[\Psi_{\rm R}]; \hat{\boldsymbol{A}}[\Psi_{\rm L}], \hat{\boldsymbol{\Gamma}}[\Psi_{\rm L}]\right)$$

A gravitational picture for SK construction



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Summary & Open Questions

- We have now a reasonable theory of adiabatic fluids which encompasses many known constraints on hydrodynamic transport.
- The off-shell formulation makes transparent various symmetries in previous investigations and makes clear the general structure of transport.
- Ideally, we would like to embed Class B and Class P_V into Class L have a unified Lagrangian derivation of all known constraints.
- There appear to be tantalizing connections to holography which should be fleshed out.
- Can one deform adiabatic fluids to incorporate dissipation?