Exact Results in Susy QFT from Six Dimensions

Daniel L. Jafferis

Harvard University

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Clay Cordova, D.J.
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work in progress

- Motivations – supersymmetric partition functions and quantum field theories from 6 dimensions.
- 6d supersymmetric backgrounds.
- Reduction to 5d super-Yang-Mills.
- Emergent gauge symmetry from supersymmetry, Toda theory from CS on an interval.

Exact results from SUSY

- Operators: chiral ring, F-terms
- States: index of BPS 1-particle states
- Partition functions: Nekrasov instanton partition function, superconformal index, sphere partition functions

Localization

- Supersymmetric path integrals are invariant under deformation by Q variations.
- Can (sometimes) use this to reduce to an integral over supersymmetric configurations.
- Also implies agreement between UV and IR calculations (ie. one can use a Lagrangian or 6d description).

Supersymmetry in curved space

- How can one determine curvature couplings such that susy is preserved?

- Couple to off-shell supergravity, putting the theory in the geometry of interest, and taking M_{Pl} to infinity. Certain background fields in addition to the metric must be turned on to preserve supersymmetry.

Supersymmetric background fields

- Twisting involves turning on background R-gauge fields.

- Much more general possibilities – off-shell backgrounds including all fields of the gravity multiplet. One may give up reflection positivity.

Spheres

- Any conformal field theory can be put on the sphere – it is conformal to flat space.
- Characterizes the number of degrees of freedom of the CFT – it gives rg inqualities.
- (Sometimes) localization implies that the CFT answer equals that from a susy but nonconformal background.

Squashings

- Often exist supersymmetric squashed sphere backgrounds and supersymmetric mass deformations.
- Deritatives w.r.t. to those parameters give certain n-point functions in flat space of operators in the stress tensor and current multiplets.

Localizing the path integral

- In Euclidean path integrals, the meaning of supersymmetry is that the expectation values of Q(..) vanish.
- This can be used to show that the full partition function localizes to an integral over Q-fixed configurations. There is a 1-loop determinant from integrating out the other modes. [Witten]

$S_{loc} = \{Q, V\}, \quad [Q^2, V] = 0$ $\qquad Z(t) = \int \prod d\Phi \; e^{-S - tS_{loc}}$

$$\frac{d}{dt} Z = -\int \prod d\Phi e^{-S - tS_{loc}} \{Q, V\} = 0$$

Reality property

- One needs to find a V such that δ V is bounded from below. The standard choice is

$$V = \sum \psi(\delta\psi)^*$$

- Works if $[(Q^2)^*, Q] = 0$.

- One can do the Q^2 collective integral first, then the Q collective Grassmann integral. It vanishes away from fixed points.

(2,0) theory in six dimensions

- Labeled by ADE "gauge group", su(N) version describes the dynamics of multiple M5 branes.
- No marginal or relevant deformations, but believed to be a local CFT.
- Intrinsically strongly coupled, no known Lagrangian description.

(2,0) theory on a circle

- Compactifying on a circle flows in the IR to 5d $\mathcal{N}$=2 YM with gauge group G.
- In 5d, Yang-Mills theory is IR free, and strongly coupled in the UV.
- Instanton-solitons are identified as the KK modes.
- At least for susy quantities – higher order operators in whatever is the exact 5d theory are plausibly Q-exact.

Compactifications

- On Riemann surfaces to 4d N=2 theories
- On 3-manifolds to 3d N=2 theories
- On 4-manifolds to 2d (0,2) theories
- A new window on strongly coupled SCFTs in lower dimensions, some which lack Lagrangians.

Curious correspondences

- Observables such as the sphere partition function of the resulting (6−d) dimensional SCFT are equal to the partition function of a particular theory on M_d.

- These 2d Toda and 3d noncompact CS theories are not supersymmetric, don't look like standard gauge theories…

[Alday Gaiotto Tachikawa. Terashima Yamazaki, Dimofte Gaiotto Gukov]

4d Gaiotto theories

- Take 2 dimensions to be a small 2-manifold.

- To preserve some supersymmetry, one may take the normal directions to be in the cotangent bundle: $R^{3,1} \times T^{*}\Sigma_2 \times R^2$ is the 11d geometry.

- The IR 4d N=2 CFT only depends on the complex structure of Σ_2

Twisting on M_2 for 4d $\mathcal{N}$=2

- One may preserve 8 supercharges upon compactification on a general 2-manifold by compensating the spin connection with a background R-gauge field.

$$\partial_\mu \epsilon^m + \frac{1}{4}\omega_\mu^{bc}\Gamma_{bc}\epsilon^m - \frac{1}{2}V_{\mu n}^m \epsilon^n = 0$$

$$SO(5)_R \times SO(5,1) \supset SO(3)_R \times SO(2)_R \times SO(2) \times SO(3,1)$$

- $SO(2)_R \times SO(2)$ is broken to the diagonal, resulting in 4d N=2 supersymmetry for a general metric on Σ_2.

$\mathcal{T}_{\mathfrak{g}}[\Sigma_2]$

- Provides a new perspective on 4d QFTs.

- Can be constructed out of certain strongly interacting building blocks that lack Lagrangian descriptions, coupled by 4d gauge theories. Associated to pants decomposition of the Riemann surface.

S^4 partition function

- Computable from a supersymmetric Lagrangian using localization. Preserves nonconformal OSp(2|4) supersymmetry.

- Results in an integral over the Coulomb branch of the square of the Nekrasov instanton partition function.

[Pestun]

$$Z = \int_{\text{Coulomb}} da \; e^{-\frac{8\pi^2 r^2}{g_{YM}^2} S_{\text{classical}}(a)} Z_{\text{1-loop}}(a) \; \left| Z_{\text{instanton}}(ia, \tfrac{1}{r}, \tfrac{1}{r}) \right|^2$$

- Instantons are localized at the poles of the sphere, and are given exactly by Nekrasov's partition function in the Omega background in R^4.
- The epsilon parameters are set by the radius.
- One might also consider squashing the sphere.

Alday-Gaiotto-Tachikawa

- Nekrasov partition functions of 4d N=2 QFT in the Omega background are identified with Toda conformal blocks.
- The S^4 partition function is the Toda partition function. This becomes a correlation function when the Riemann surface has punctures.
- The epsilon parameters become the Toda parameter, b.

Liouville and Toda theory

- Liouville action is

$$S = \frac{1}{4\pi}\int d^2x\sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (b+b^{-1})R\phi + 4\pi e^{2b\phi})$$

- Toda is a ADE generalization of this. It has currents which form a W-algebra.

N M5 branes on a 3-manifold

- Take 3 dimensions to be a small 3-manifold.
- To preserve some supersymmetry, one may take the normal directions to be in the cotangent bundle: $R^{2,1} \times T^*M_3 \times R^2$ is the 11d geometry. $SO(3)_R \times SO(3)$ is broken to the diagonal, resulting in 3d N=2 supersymmetry.
- The IR 3d N=2 CFT is independent of the metric on M_3, and has no flavor symmetries for compact hyperbolic manifolds.

Twisting on M_3 for 3d $\mathcal{N}$=2

- One may preserve 4 supercharges upon compactification on a general 3-manifold by compensating the spin connection with a background R-gauge field.

$$\partial_\mu \epsilon^m + \frac{1}{4}\omega_\mu^{bc}\Gamma_{bc}\epsilon^m - \frac{1}{2}V_{\mu n}^m \epsilon^n = 0$$

$$SO(5)_R \times SO(5,1) \supset SO(3)_R \times SO(2)_R \times SO(3) \times SO(2,1)$$

$$(4,4) \to (2,+1,2,2) \oplus (2,-1,2,2)$$

- $SO(3)_R \times SO(3)$ is broken to the diagonal, resulting in 3d N=2 supersymmetry.

$\mathcal{T}_{\mathfrak{g}}[M_3]$

- Provides a new perspective on 3d SCFTs.

- Abelian CSM Lagrangians have been constructed for some classes of examples, using tetrahedral decomposition.

  [Dimofte Gaiotto Gukov, …]

- Harder for compact 3-manifolds.

3-sphere partition function

- Characterizes the number of degrees of freedom of the CFT.
- Computable from a supersymmetric Lagrangian using localization. Preserves nonconformal $SU(2|1) \times SU(2)$ supersymmetry.

[Kapustin Willett Yaakov, D.J., Hama Hosomichi Lee]

Superconformal symmetries on S^3

- The conformal group in 3d is USp(4) = SO(3,2).
- In Euclidean signature, one has the real form USp(2,2) = SO(4,1).
- On S^3, the USp(2) × USp(2) = SO(4) subgroup acts as rotations of the sphere.
- The $N = 2$ superconformal group is OSp(2|4).
- The R-symmetry is SO(2) = U(1).

Supersymmetry on the sphere

- The sphere possesses homogeneous Killing spinors, $\nabla_\mu \epsilon = \pm \frac{i}{2} \gamma_\mu \epsilon$. The associated generators square to isometries.

- It corresponds to keeping Q and S while throwing away $\bar{Q}$ and $\bar{S}$ of the superconformal algebra.

- Closely related to the 4d superconformal index on $S^3 \times R$.

OSp(2|2) × SU(2)

- The OSp(2|2) subgroup of OSp(2|4) does *not* contain any conformal transformations. The bosonic generators are the *R*-symmetry and $SU(2)_L$ isometries.

$$\{Q^i_A, Q^j_B\} = \delta^{ij} J_{AB} + i\epsilon_{AB}\epsilon_{ij} R$$

- Parity exchanges the two SU(2)s and is broken by this choice.

$$\delta = \frac{1}{\sqrt{2}}(Q^1_1 + iQ^2_1),\ \tilde{\delta} = \frac{1}{\sqrt{2}}(Q^1_2 - iQ^2_2)$$

Localization of CSM theories

$$S_{CS}^{\mathcal{N}=2} = \frac{k}{4\pi} \int (A \wedge dA + \frac{2}{3} A^3 - \bar{\chi}\chi + 2D\sigma)$$

Coupled to chiral multiplets with R charge q.

$$Z = \int \prod_{\text{Cartan}} \frac{d\sigma}{2\pi} \;\; \exp\left[\frac{i}{4\pi} k \text{tr} \sigma^2\right] \; \text{Det}_{\text{Ad}} \left(\sinh \frac{b\sigma}{2} \sinh \frac{\sigma}{2b}\right)$$

$$\times \; \prod_{\substack{\text{chirals} \\ \text{in rep } R_i}} \text{Det}_{R_i} \left(s_b\left(\frac{i}{2}(b + b^{-1})(1 - q_i) - \sigma\right)\right)$$

$$s_b(x) = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + \frac{b+b^{-1}}{2} - ix}{mb + nb^{-1} + \frac{b+b^{-1}}{2} + ix}$$

3d-3d conjecture

- Terashima Yamazaki, Dimofte Gaiotto Gukov conjectured that the squashed S^3 partition function of the N M5 on M_3 theory is given by a noncompact CS partition function on M_3 with level determined by the squashing.

- How can this bosonic theory with emergent gauge symmetry arise from reduction of a susy gauge theory?

Complex Chern-Simons theory

$$S = \frac{q}{8\pi}\int \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \frac{\tilde{q}}{8\pi}\int \mathrm{Tr}\left(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} + \tfrac{2}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\right)$$
$$= \frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA + \tfrac{2}{3}A^3 - X\wedge d_A X\right) + \frac{u}{2\pi}\int \mathrm{Tr}\left(\tfrac{1}{3}X^3 - X\wedge F_A\right)$$

- The Chern-Simons levels are $q = k + iu$ and $\tilde{q} = k - iu$
- Noncompact gauge symmetry $\mathcal{A} \to \mathcal{A} + d_{\mathcal{A}}\, g$, for $g \in \mathfrak{g}_{\mathbb{C}}$

[Witten]

Reality properties

- Can’t regulate with YM – wrong sign kinetic term. Subtle to define theory non-perturbatively.
- k is an integer so that e^{iS} is invariant under large gauge transformations.
- u is either real or pure imaginary to obtain a unitary theory. Path integral is oscillatory for real u.

Superconformal indices

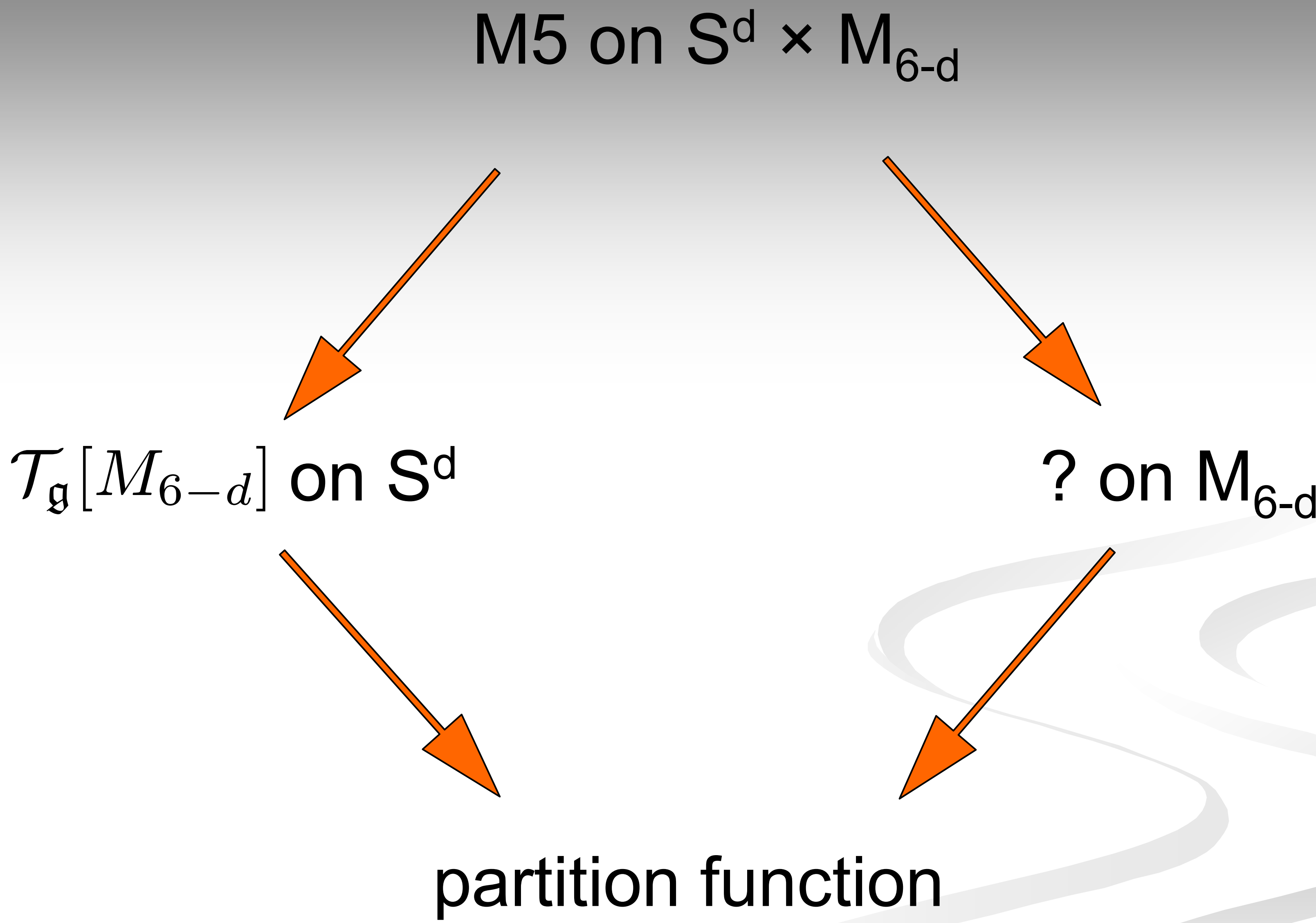
- There are similar correspondences for the 3d and 4d superconformal index of these theories.

[Dimofte Gaiotto Gukov, Gadde Rastelli Razamat Yan]

- For $S^2 \times S^1$, it was shown by Yagi and Sungjay Lee, Yamazaki using similar ideas that one obtains complex Chern-Simons with $k = 0$ and u pure imaginary.

Direct approach

- Find the full 6d background S $\times$ M that preserves supersymmetry.
- Partition function is independent of the size of M due to supersymmetry.
- Reduce on S to find the theory on M.


M5 on $S^d \times M_{6-d}$
$\mathcal{T}_{\mathfrak{g}}[M_{6-d}]$ on S^d
? on M_{6-d}
partition function

A surprising feature

- The proposed theories are not supersymmetric, and not standard gauge theories!

5d intermediary

- Hard to proceed directly from six dimensions, since there is no Lagrangian of the (2,0) theory to reduce on the squashed sphere. After all, this is why the $\mathcal{T}_{\mathfrak{g}}[M]$ theories are interesting.

- Thus one wants to first find a circle isometry to obtain 5d YM in some background – then one can simply derive the theory on M.

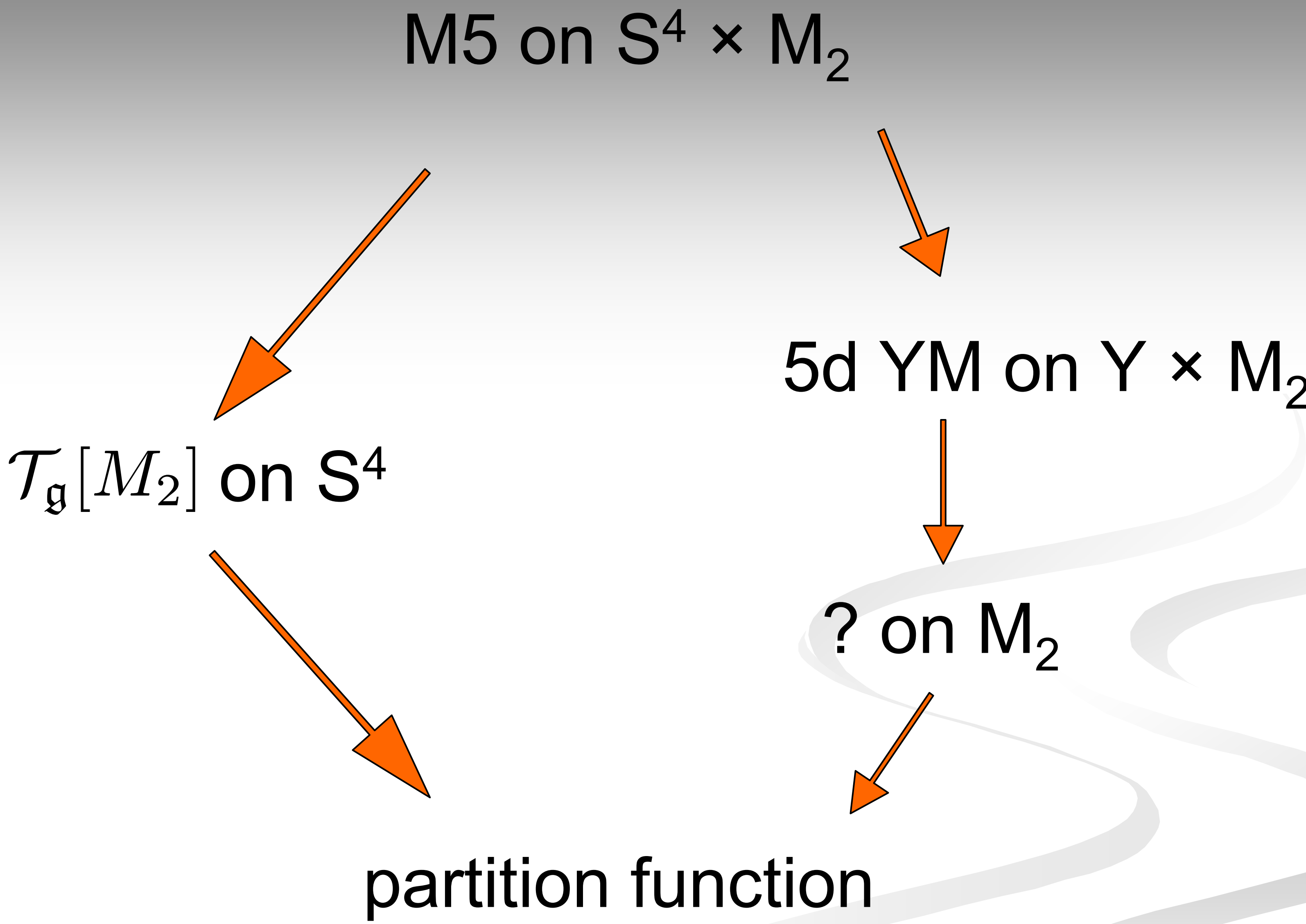
(2,0) theory on a circle

- It has been suggested that 5d maximally supersymmetric Yang-Mills *defines* the 6d (2,0) theory, along the lines of asymptotic safety.

  [Douglas, Lambert Papageorgakis Schmidt-Sommerfeld]

- On our context, the circle *is* small.

- Moreover higher order terms are presumably Q-exact. Would require the absence of certain corrections, due to new massive fields.


M5 on $S^4 \times M_2$
5d YM on $Y \times M_2$
$\mathcal{T}_{\mathfrak{g}}[M_2]$ on S^4
? on M_2
partition function

Turning the crank

- Need to find the full 6d background – it doesn't factorize.
- Pick a circle to reduce on supersymmetrically.
- For AGT, it shrinks somewhere, so b.c.s are required.
- Find the effective 3d or 2d theory – this is algorithmic.

3d-3d

- The full 6d background of $S^3_\ell \times M_3$ preserves 4 supersymmetries.
- The partition function doesn't depend on the overall size due to conformal invariance, and lack of anomalies in the 3-3 split.
- Metric deformations of M_3 are Q-exact.

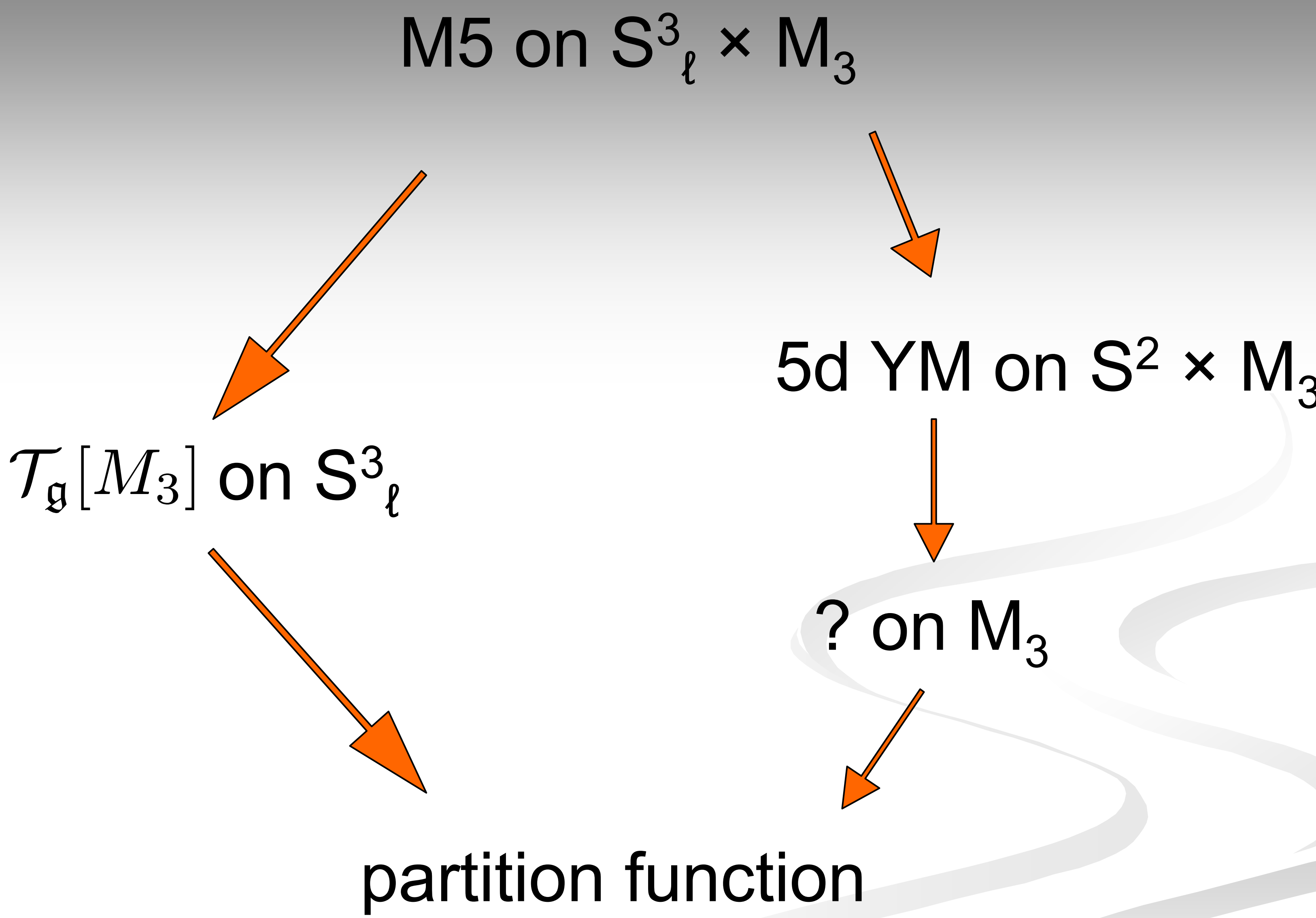
Hopf reduction: 5d Yang-Mills

- One can reduce from S^3 to S^2 along the Hopf fiber.

$$ds^2 = \frac{r^2}{4}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) + r^2\left(d\psi + \cos^2(\theta/2)d\phi\right)^2$$

- This results in 5d YM on S^2 with graviphoton flux and other background fields.

- Nonrenormalizable theory with $\frac{1}{g_{\mathrm{YM}}^2} = \frac{1}{r}$, but here we actually want small *r*.


M5 on $S^3_\ell \times M_3$
5d YM on $S^2 \times M_3$
$\mathcal{T}_{\mathfrak{g}}[M_3]$ on S^3_ℓ
? on M_3
partition function

Background couplings

- One needs to find the complete 6d background in which supersymmetry is preserved. This involves twisting on the 3-manifold, but something else on the squashed sphere.

- Then reduce to five dimensional background, coupled to the dynamical maximally susy Yang-Mills Lagrangian theory.

Supersymmetry in curved space

- How can one determine curvature couplings such that susy is preserved?

- Couple to off-shell supergravity, putting the theory in the geometry of interest, and taking M_{Pl} to infinity. Certain background fields in addition to the metric must be turned on to preserve supersymmetry.

5d maximal supergravity

- Only interested here in the gravitino and dilatino supersymmetry variations, whose vanishing gives the conditions for preserving rigid susy.
- Can be obtained by reduction of 6d off-shell conformal sugra [Bergshoeff Sezgin van Proeyen]. Here the R symmetries are both SO(5).
- Generalizes 5d N=1 supergravity of [Kugo Ohashi].

6d →5d, off-shell fields

Field	Interpretation	$sp(4)$	w
$\underline{e}_\mu^a$	Metric	**1**	-1
$\underline{V}_\mu^{mn}$	R Gauge Field	**10**	0
$\underline{T}_{\mu\nu\rho}^{mn}$	Auxiliary 3-form	**5**	-2
$\underline{D}^{mn,rs}$	Auxiliary scalar	**14**	2

Metric reduces to 5d as metric, graviphoton and dilaton. $\underline{e}_\mu^a = \begin{pmatrix} e_\mu^a & e_\mu^5 = \alpha^{-1} C_\mu \\ e_z^a = 0 & e_z^5 = \alpha^{-1} \end{pmatrix}$

$$\underline{V}_a^{mn} \to V_a^{mn}, a \neq 5, \ \underline{V}_5^{mn} \equiv S^{mn}, \ \underline{T}_{abc}^{mn} \to \underline{T}_{ab5}^{mn} \equiv T_{ab}^{mn}, \ \underline{D}^{mn,rs} \to D^{mn,rs}$$

5d $\mathcal{N}$=2 off-shell SUGRA

- A taste of the supersymmetry variations:

$$\delta\psi_a^m = \mathcal{D}_a\epsilon^m + \frac{i}{2\alpha}\left[G_{ab}\Omega^{mn} - \alpha S^{mn}\eta_{ab}\right]\Gamma^b\epsilon_n + \frac{i}{8\alpha}\left[G^{bc}\Omega^{mn} - 4\alpha\left(T^{mn}\right)^{bc}\right]\Gamma_{abc}\epsilon_n,$$

$$\delta\chi_r^{mn} = \left[T_{ab}^{mn}T_{cdrs} - \frac{1}{\alpha}T_{ab}^{mn}G_{cd}\Omega_{rs} + \frac{1}{12}\left(\mathcal{D}^e S_r^{[m}\delta_s^{n]} + \mathcal{D}_f T^{mnfe}\Omega_{rs}\right)\varepsilon_{eabcd}\right]\Gamma^{abcd}\epsilon^s$$
$$+\left[\frac{5}{2\alpha}T_{ab}^{mn}G^a{}_c\Omega_{rs} - 4T_{ab}^{mn}T^a{}_{crs} + 2T_{bc}^{mn}S_{rs} - S_p^{[m}T_{bc}^{n]p}\Omega_{rs} - R_{bcr}{}^{[m}\delta_s^{n]}\right.$$
$$\left.+\frac{1}{2}\mathcal{D}_a T_{de}^{mn}\Omega_{rs}\varepsilon^{ade}{}_{bc}\right]\Gamma^{bc}\epsilon^s + \left[\frac{1}{\alpha}T_{ab}^{mn}G^{ab}\Omega_{rs} - 2T_{ab}^{mn}T_{rs}^{ab} - \frac{4}{15}D_{rs}^{mn}\right]\epsilon^s - (\text{traces})$$

5d Yang-Mills action

$$
\begin{aligned}
S_A &= \frac{1}{8\pi^2}\int \mathrm{Tr}\Big(\alpha F\wedge *F + C\wedge F\wedge F\Big)\,,\\
S_\varphi &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\mathcal{D}_a\varphi^{mn}\mathcal{D}^a\varphi_{mn} - 4\varphi^{mn}F_{ab}T^{ab}_{mn} - \varphi^{mn}(M_\varphi)_{mn}^{rs}\varphi_{rs}\Big)\,,\\
S_\rho &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\rho_{m\gamma}i\mathcal{D}^\gamma_\beta\rho^{m\beta} + \rho_{m\gamma}(M_\rho)^{mn\gamma}{}_\beta\rho^\beta_n\Big)\,.\\
S_{int} &= \frac{1}{32\pi^2}\int d^5x\sqrt{|g|}\;\alpha\mathrm{Tr}\Big(\rho_{m\alpha}[\varphi^{mn},\rho^\alpha_n] - \frac{1}{4}[\varphi_{mn},\varphi^{nr}][\varphi_{rs},\varphi^{sm}] - \frac{2}{3}S_{mn}\varphi^{mr}[\varphi^{ns},\varphi_{rs}]\Big)\,.
\end{aligned}
$$

- Note the CS term, the F-scalar mixing, and the cubic scalar potential induced by background fields.

Round sphere case

- Note that $H_3 \times S^3$ with equal radii is conformally flat.
- Thus the (2,0) theory can be put canonically on this space.
- $SO(3)_R$ twisting on H_3 leads to $S^3 \times R^3$.

Background sugra on $S^2 \times R^3$

- For general squashing, all background fields are involved. In 5d, there is graviphoton flux on S^2.

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + \left(\tfrac{r\ell}{2}\right)^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right), \; C = \cos^2(\theta/2) d\phi, \; \alpha = 1/r$$

$$T_{\hat{A}BC} = t\varepsilon_{\hat{a}bc}, \; V_{A\hat{B}\hat{C}} = v\varepsilon_{a\hat{b}\hat{c}}, \; S_{\hat{A}\hat{B}} = s\varepsilon_{\hat{x}\hat{y}}, \; D_{\hat{A}\hat{B}} = d\left(\delta_{\hat{a}\hat{b}} - \tfrac{3}{2}\delta_{\hat{x}\hat{y}}\right)$$

$$t = s = -\frac{\sqrt{1-\ell^2}}{2r\ell^2}, \; v = -\frac{i}{2r\ell^2}, \; d = \frac{3}{2r^2\ell^2}\left(1 + \frac{1}{\ell^2}\right)$$

Reduction on S^2

- In the limit of a small sphere, the light fields that survive the dimensional reduction are S^2 constant modes of the gauge field (along the 3-manifold) and the scalars, and particular modes of the fermions that transform in the 2 of the SU(2) rotations of the sphere.

- The fermionic action is not diagonalized by the mass basis, so one must include some massive modes and integrate them out.

Gauge sector

- In 5d, the graviphoton induces a Chern-Simons term.

$$S_A = \frac{1}{8\pi^2} \int_{\mathbb{R}^3 \times S^2} \left(\alpha \ \mathrm{Tr}(F \wedge *F) + G \wedge CS(A)\right)$$

- S^2 is simply connected, one just reduces to 3d.

$$S_A = \frac{r\ell^2}{8\pi} \int_{\mathbb{R}^3} \mathrm{Tr}(F \wedge *F) + \frac{1}{4\pi} \int_{\mathbb{R}^3} CS(A)$$

- The 3d YM term disappears in the r $\longrightarrow$ 0 limit.

Scalar action

- The 5 scalars decompose into (3,1) + (1,2) under R-symmetry breaking SO(5) to SO(3) × SO(2).
- The 5d sugra induced masses and those from the R gauge field covariant derivative cancel.

$$S_X = \frac{r\ell^2}{8\pi}\int d^3x \mathrm{Tr}\left(\nabla_a X_b \nabla_a X_b\right) + \frac{1}{4\pi}\int d^3x\, i\varepsilon_{abc}\mathrm{Tr}\left(X_a\nabla_b X_c - i\sqrt{1-\ell^2}X_a F_{bc}\right)$$

$$S_Y = \frac{r\ell^2}{8\pi}\int d^3x \mathrm{Tr}\left(\nabla_a Y_z \nabla_a Y_z\right)$$

Fermion action

- Expanding the twisted fermions in terms of scalars and 1-forms in M_3, a doublet of modes on S^2, and keeping track of the $SO(2)_R$ index:

$$\rho^m = \varepsilon^{\alpha\hat{\alpha}}\lambda^{\sigma\hat{\sigma}} + (\gamma^a)^{\alpha\hat{\alpha}}\,\xi_a^{\sigma\hat{\sigma}}$$

$$S_{ferm} = \frac{1}{32\pi^2}\int d^3x\ \mathrm{Tr}\Bigg[\left(\ \xi_a^{i\hat{\imath}}\varepsilon_{ij}B_{\hat{\imath}\hat{\jmath}} - e\widetilde{\xi}_a^{i\hat{\imath}}B_{ij}\varepsilon_{\hat{\imath}\hat{\jmath}}\right) i\nabla_a\lambda^{j\hat{\jmath}}$$
$$- \frac{i}{r\ell}\left(\ \xi_a^{i\hat{\imath}}\xi_a^{j\hat{\jmath}} - \widetilde{\xi}_a^{i\hat{\imath}}\widetilde{\xi}_a^{j\hat{\jmath}}\right)\varepsilon_{ij}B_{\hat{\imath}\hat{\jmath}} - \frac{4i}{r\ell^2(1+\ell)}\widetilde{\lambda}^{i\hat{\imath}}\widetilde{\lambda}^{j\hat{\jmath}}\varepsilon_{ij}B_{\hat{\imath}\hat{\jmath}}\Bigg]$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e = \sqrt{\tfrac{1-\ell}{1+\ell}}$$

Non-abelian interactions

- These come from the standard quartic terms and Yukawa couplings, and the background induced cubic scalar potential.

$$S_{pot} = \frac{r\ell^2}{8\pi} \int d^3x \, \mathrm{Tr} \left(\frac{1}{2}[X_a, X_b][X_a, X_b] + [X_a, Y_z][X_a, Y_z] + \frac{1}{2}[Y_z, Y_w][Y_z, Y_w] \right)$$

$$+ \frac{i\sqrt{1-\ell^2}}{12\pi} \int d^3x \; i\varepsilon_{abc} \mathrm{Tr} \left(X_a[X_b, X_c] \right)$$

$$S_{yuk} = \frac{1}{32\pi^2} \int d^3x \, \mathrm{Tr} \left(\widetilde{\xi}_a^{i\hat{i}} [X_a, \lambda^{j\hat{j}}] B_{ij} \varepsilon_{\hat{i}\hat{j}} - e \xi_a^{i\hat{i}} [X_a, \lambda^{j\hat{j}}] \varepsilon_{ij} B_{\hat{i}\hat{j}} + \left(\frac{2}{1+\ell} \right) \widetilde{\lambda}^{i\hat{i}} [Y_z, \lambda^{j\hat{j}}] B_{ij} \kappa^z_{\hat{i}\hat{j}} \right)$$

Complex connection

- By changing the contour of integration of X, it is natural to define a complex 1-form,

$\mathcal{A}$ = A + i X.

- The action looks almost invariant under a new symmetry, $A \to A - [X, g]$, $X \to X + dg + [A, g]$

except for the term $(D^A_\mu X^\mu)^2$

Obtaining ghosts

- Integrating out the fermions whose mass diverges in the small sphere limit leads to a second order action for the 4 massless fermions.

$$S_\lambda = \frac{ir\ell^2}{64\pi^2(1+\ell)} \int d^3x \; \mathrm{Tr} \Big(\nabla_a \lambda^{i\hat{i}} \nabla_a \lambda^{j\hat{j}} \varepsilon_{ij} B_{\hat{i}\hat{j}} + [X_a, \lambda^{i\hat{i}}][X_a, \lambda^{j\hat{j}}] \varepsilon_{ij} B_{\hat{i}\hat{j}} - \tfrac{1}{2}[Y_z, \lambda^{i\hat{i}}][Y_w, \lambda^{j\hat{j}}] \left(\delta^{zw} \varepsilon_{ij} B_{\hat{i}\hat{j}} + i\varepsilon^{zw} \varepsilon_{ij} \varepsilon_{\hat{i}\hat{j}} \right) \Big)$$

- Non-linear ghost action is Q-equivalent to a quadratic action.

Faddeev-Popov

- The Faddeex-Popov determinant for fixing the noncompact part of the gauge symmetry with the gauge fixing term $(D_\mu^A X^\mu)^2$ is precisely the fermionic determinant!

$$\delta\left(D_\mu X^\mu\right) = D_A^2 g + \left(\mathrm{ad}_X\right)^2 g$$

- There are 4 rather than 2 fermions, and the doubling is exactly cancelled by the 1-loop determinant of the Y scalars.

Complex Chern-Simons

- Therefore,

$$Z_{S^3_\ell}[T_{\mathfrak{g}}(M_3)] = Z_{M_3}[CS_{\mathfrak{g}_C}(1, \sqrt{1-\ell^2})]$$

- Note that both branches of unitary reality conditions for the level u appear.

M5 on $S^4 \times M_2$

$\mathcal{T}_{\mathfrak{g}}[M_2]$ on S^4

? on M_2

partition function

Background couplings

- One needs to find the complete 6d background in which supersymmetry is preserved. This involves twisting on the 2-manifold, but something else on the 4-sphere.

- Then reduce to five dimensional background, coupled to the dynamical maximally susy Yang-Mills Lagrangian theory.

Round sphere case

- Note that $H_2 \times S^4$ with equal radii is conformally flat.
- Thus the (2,0) theory can be put canonically on this space.
- $SO(2)_R$ twisting on H_2 leads to $S^4 \times R^2$.
- The flat space 2d theory is twisted!

Hopf reduction

- S^4 looks like S^3 fibered over an interval, and shrinking at the 2 ends.

- We pick the reduction circle to be the Hopf fiber of the S^3. It has two fixed points on S^4, and is compatible with Pestun's localizing supersymmetry.

5d background

- Given by the Riemann surface $\times S^2$ fibered over an interval, and shrinking at the ends.
- The dilaton varies and goes to 0 at the ends.
- The sphere carries one unit of graviphoton flux.
- Other backgrounds fields are needed for susy.

Weyl rescaling

- The 6d theory is Weyl invariant, so the partition function is independent of local Weyl rescalings of the background fields (up to overall anomaly factors).
- Use this to obtain $S^3 \times (\Sigma_2$ warped over $R^1)$ in 6d.
- In 5d now the dilaton constant, and the metric looks like $S^2 \times (\Sigma_2$ warped over $R^1)$.
- R-gauge equivalent to twisting on the 3d part.

Back to 3d-3d

- Here the 3-manifold is the Riemann surface warped over an interval. Only the b.c.s matter since the SL(N,C) Chern-Simons theory is topological.

SU(2) × U(1) squashed S^4

- One can take the squashed 3-sphere times arbitrary 3-manifold background that we found before, specialize to a Riemann surface warped over an interval, and Weyl transform back.

- Results in a supersymmetric squashed 4-sphere background for 4d N=2 theories that preserves SU(2) × U(1), not just U(1) × U(1).

[Nosaka Terashima]

Boundary conditions

- The M-theory circle shrinks at the boundaries, so one expects a Nahm pole type boundary condition, of D4 branes ending on D6 branes.

- In the abelian case, the b.c. is determined by requiring the 6d fields to be smooth.

$$D_w X_i = \epsilon_{ijk}[X_j, X_k]$$

- The scalars have a Nahm pole,

  $X_i \sim T_i / w$ sitting in an SU(2) subalgebra,

$$[T_i, T_j] = \epsilon_{ijk} T_k$$

- The gauge field is adjusted to make the complex connection flat.

$$\mathcal{A} = i T_3 \, \frac{dw}{w} + T_+ \, \frac{dx_+}{w} + \ldots$$

- The subleading terms contain the dynamical Toda fields. They come from modes of the 5d scalars localized at the poles of the 4 sphere.

CS boundary conditions

- After the Weyl rescaling and reduction to three dimensions, these become a pole type boundary conditions for the complex connection.

- It is exactly the Brown-Henneaux b.c. for euclidean AdS under the identification of SL(2,C) CS with 3d gravity.

$$A = \omega \sim \frac{1}{2}\begin{pmatrix} 0 & -rd\bar{z} \\ rdz & 0 \end{pmatrix} \qquad X = e \sim \frac{1}{2}\begin{pmatrix} dr/r & rd\bar{z} \\ -rdz & dr/r \end{pmatrix}$$

3d gravity and Toda

- Chern-Simons theory on a space with boundary results in a chiral WZW model on the boundary.

[Witten]

- Here there are two boundaries, so one ends up with the full CFT. [Elitzer Moore Schwimmer Seiberg]

- The particular boundary conditions here set some of the currents to 0, truncating SL(N,C) WZW to the Toda theory.

[Balog Feher Forgacs O'Raifeartaigh Wipf;
Coussaert Henneaux van Driel]

Summary

- Gave a direct derivation of the Toda theory and SL(N,C) CS theory from reduction of the (2,0) theory on a 4-sphere and 3-sphere, rsp.
- These theories are already twisted in flat space.
- Fermions turned into Faddeev-Popov ghosts of an emergent gauge symmetry.
- Nahm b.c.s turned into Brown-Henneaux b.c.s in SL(N,C) CS that resulted in Toda theory.