Large-Spin Expansions of Giant Magnons

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Introduction and Motivation

$$\left\{ \text{Type IIB String Theory in } \mathrm{AdS}_5 \times \mathrm{S}^5 \right\} \longleftrightarrow \left\{ \mathcal{N} = 4,\ SU(N) \text{ Super Yang-Mills Theory in 4d} \right\}$$

Maldacena, 1998

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  Step 1: Compute scaling dimensions Δ of all gauge-invariant operators of $\mathcal{N} = 4$ SYM.

  Step 2: Identify their dual string states.

  Step 3: Compute the energies E of the dual string states.

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- Before comparing spectra, we have to develop techniques to compute them (in appropriate form)!

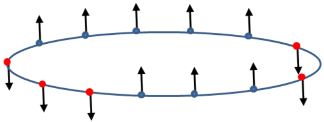
Section 2

Infinite-Size Magnons

Magnons in $\mathcal{N} = 4$ SYM

- The $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM consists of the single-trace operators $\hat{O}^L = \text{Tr}\left[\mathcal{Z}^{L-M}\mathcal{W}^M\right]$ (the 6 real scalars Φ_i of $\mathcal{N} = 4$ SYM have been combined into 3 complex scalars $\mathcal{Z}$, $\mathcal{W}$, $\mathcal{Y}$).
- Owing to the cyclic property of the trace, these can be written as a closed spin-chain, e.g.

$\text{Tr}\left[\mathcal{Z}^5\mathcal{W}^2\mathcal{Z}^3\mathcal{W}^3\right] \longleftrightarrow$



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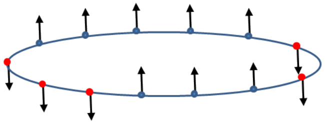
$\text{Tr}\left[\mathcal{Z}^5\mathcal{W}^2\mathcal{Z}^3\mathcal{W}^3\right] \longleftrightarrow$  $= |\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow>$

- The $\mathfrak{su}(2)$ one-loop dilatation operator of $\mathcal{N}=4$ SYM has the form of an integrable spin chain:

$$\mathcal{D} = L\cdot\mathbb{I} + \frac{\lambda}{8\pi^2}\hat{\text{H}} + \sum_{n=2}^{\infty}\lambda^n\mathcal{D}_n\,, \quad \hat{\text{H}} = \sum_{j=1}^{L}\left(\mathbb{I}_{j,j+1} - \mathbb{P}_{j,j+1}\right) = 2\sum_{j=1}^{L}\left(\frac{1}{4} - \boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_{j+1}\right) \qquad \left(\lambda = g_{\text{YM}}^2 N_c\right)$$

Minahan-Zarembo, 2002

Bethe Ansatz

- The Heisenberg $\text{XXX}_{1/2}$ ferromagnetic (quantum) spin chain Hamiltonian $\hat{H}$ can be diagonalized by BA:

$$\text{Tr}\left[\mathcal{Z}^J\mathcal{W}^M\right] \sim |x_1, x_2, \ldots, x_M\rangle = |\uparrow\ldots\uparrow\underset{x_1}{\downarrow}\uparrow\ldots\uparrow\underset{x_2}{\downarrow}\uparrow\ldots\uparrow\underset{x_M}{\downarrow}\uparrow\ldots\uparrow>$$

$$\Delta = J + M + \frac{\lambda}{2\pi^2}\sum_j \sin^2\frac{p_j}{2} + \mathcal{O}\left(\lambda^2\right), \qquad \sum_j p_j = \frac{2\pi k}{L}, \; k \in \mathbb{Z}, \quad j = 1, 2, \ldots, M.$$

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- An all-loop, asymptotic BA for $\mathfrak{su}(2)$ has been proposed...

$$\Delta = J + M + \frac{\lambda}{8\pi^2}\sum_{j=1}^{M} E\left(p_j\right), \; E\left(p_j\right) = \frac{8\pi^2}{\lambda}\left[\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}} - 1\right] = 4\sin^2\frac{p_j}{2} - \frac{\lambda}{\pi^2}\sin^4\frac{p_j}{2} + \mathcal{O}\left(\lambda^2\right)$$

Beisert-Dippel-Staudacher (BDS), 2004

One-Magnon States

- Let us consider $M = 1$ magnon states:

$$\widehat{\mathcal{O}} = \sum_m e^{imp} \left|\ldots \mathcal{Z}\mathcal{Z}\mathcal{W}(m)\mathcal{Z}\mathcal{Z}\ldots\right\rangle$$

- In infinite size, their all-loop dispersion relation becomes:

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}\,, \quad \text{all } \lambda,\ J \to \infty$$

This relation follows by extending the corresponding symmetry algebra $\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2) \subset \mathfrak{psu}(2,2|4)$.

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- We may obtain its weak and strong coupling limits as follows:

$$\Delta - J = 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} - \frac{\lambda^2}{8\pi^4}\sin^4\frac{p}{2} + \frac{\lambda^3}{16\pi^6}\sin^6\frac{p}{2} - \ldots\,, \qquad \lambda \to 0 \text{ (weak coupling)}$$

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2} + 0 + \frac{\pi}{2\sqrt{\lambda}}\csc\frac{p}{2} - \frac{\pi^3}{8\lambda^{3/2}}\csc^3\frac{p}{2} + \ldots\,, \qquad \lambda \to \infty \text{ (strong coupling)}$$

Hofman-Maldacena (HM) Giant Magnons

- Giant Magnons (GMs) are open, single-spin strings that rotate in $\mathbb{R} \times S^2 \subset AdS_5 \times S^5$:

$$\left\{t = \tau, \rho = \overline{\theta} = \overline{\phi}_1 = \overline{\phi}_2 = 0\right\} \times \left\{\theta = \theta\left(\sigma - v\tau\right), \phi = \tau + \varphi\left(\sigma - v\tau\right), \theta_1 = \phi_1 = \phi_2 = 0\right\}$$

Hofman-Maldacena, 2006

HM Giant Magnon ($v = 0.9$)

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- GMs extend between parallels $0 \leq z \leq 1 - v^2$. Their angular extent is equal to their momentum $p = \Delta\phi$.

- Their conserved energy E and spin J diverge $\to \infty$. Their difference remains constant:
$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin\frac{p}{2}, \quad J, \sqrt{\lambda} = \frac{R^2}{\alpha'} \to \infty$$
The leading term of the all-loop (BDS) dispersion relation is recovered.

- It can be proved that the GM s-matrix matches the one for magnons obtained from the gauge theory side.

- In AdS/CFT Duality, GMs are the string theory duals of $\mathcal{N} = 4$ SYM theory magnon excitations: $|\ldots \uparrow\uparrow\downarrow\uparrow\uparrow \ldots > \leftrightarrow |\ldots \mathcal{Z}\mathcal{Z}\mathcal{W}\mathcal{Z}\mathcal{Z} \ldots >$.

Gubser-Klebanov-Polyakov (GKP) String

- GMs are open strings which are incompatible with the spectrum of a IIB string theory possessing only closed strings.
- Their dual gauge theory magnons have non-vanishing momentum p which is incompatible with the cyclicity of the trace.
- To get physical configurations, superpose more than one magnon/GMs to form closed strings and operators of vanishing total momentum.

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- To get physical configurations, superpose more than one magnon/GMs to form closed strings and operators of vanishing total momentum.
- Superposing two GMs with $v = 0$, $p = \pi$ and angular momenta equal to $J/2$, a closed folded string rotating on S^2 is obtained. This is the infinite-volume Gubser-Klebanov-Polayakov (GKP) string on S^2:

$$\left\{t = \tau, \rho = \bar{\theta} = \bar{\phi}_1 = \bar{\phi}_2 = 0\right\} \times \left\{\theta = \theta\left(\sigma\right), \phi = \tau, \theta_1 = \phi_1 = \phi_2 = 0\right\}$$

Gubser-Klebanov-Polyakov, 2002

- The GKP dispersion relation is:

$$E - J = \frac{2\sqrt{\lambda}}{\pi}, \quad J,\, \lambda \to \infty$$

- It is dual to the 2-magnon $\mathcal{N} = 4$ SYM operator Tr $\left[\mathcal{W}\,\mathcal{Z}^m\,\mathcal{W}\,\mathcal{Z}^{J-m}\right] + \ldots$

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Section 3

Finite-Size Giant Magnons

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- We therefore want to establish the validity of AdS/CFT for finite system sizes, $L = J + M$.
- In the following we will examine what happens on the string theory side of the correspondence by studying the dispersion relation of finite-size GMs.

Finite-Size Giant Magnons

- Finite-size GMs are described in terms of a linear and an angular velocity v and ω:

$$\left\{t=\tau,\rho=\bar{\theta}=\bar{\phi}_1=\bar{\phi}_2=0\right\}\times\left\{\theta=\theta\left(\sigma-v\omega\tau\right),\phi=\omega\tau+\varphi\left(\tau,\sigma\right),\theta_1=\phi_1=\phi_2=0\right\}$$

- They possess two basic regions, the Elementary $0\leq|v|<1/\omega<1$ and the Doubled $0\leq|v|<1<1/\omega$.
- The Elementary Region is Stable while the Doubled is Unstable.
- As before, GMs have 3 conserved charges, Energy E, Spin J and momentum/angular extent $\Delta\varphi=p$.
- Elementary GMs do not touch the equator of S^2, but extend between parallels $1-1/\omega^2<z<1-v^2$.
- For $\omega\to1$ the endpoints of GMs touch the equator and their 2 regions merge to one as their size $J\to\infty$.
- Putting together two GMs with $v=0$, maximum momentum $p=\Delta\varphi=\pi$ and angular momentum equal to $J/2$ we obtain the Gubser-Klebanov-Polyakov (GKP) string on S^2, dual to the 2-magnon state $\mathrm{Tr}\left[\mathcal{Z}^J\mathcal{W}^2\right]$.

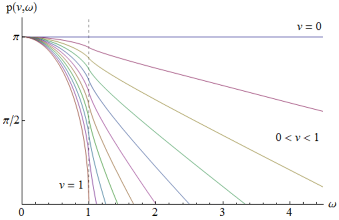
Finite-Size GM & GKP String: Elementary Region, $0 \leq |v| < 1/\omega < 1$

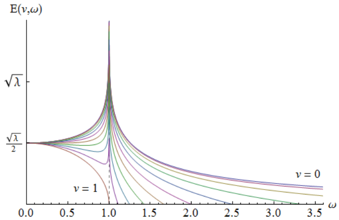
GM ($v = 0.5$ & $\omega = 1.1$)

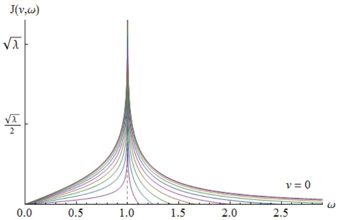
GKP String ($v = 0$ & $\omega = 1.5$)

GM momentum, energy and spin

- We may plot the momentum, energy and spin of the giant magnon as functions of its angular velocity ω:







Dispersion Relation

- The general dispersion relation of finite-size GMs is:

$$E - J = \epsilon_\infty + \underbrace{\sqrt{\lambda}\,\delta\epsilon_{\text{cl}} + \delta\epsilon_{\text{1-loop}} + \frac{1}{\sqrt{\lambda}}\delta\epsilon_{\text{2-loop}} + \ldots,}_{\text{finite-size corrections}} \qquad \lambda \to \infty$$

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- The first few terms of the classical finite-size corrections $\delta\epsilon_{\text{cl}}$ have been known ($\mathcal{R} \equiv 2\pi J \csc p/2/\sqrt{\lambda}$):

$$\delta\epsilon_{\text{cl}} = -\frac{4}{\pi}\sin^3\frac{p}{2}\,e^{-2-\mathcal{R}}\left\{1 + \left[2\mathcal{R}^2\cos^2\frac{p}{2} + 2\,(3\cos p + 2)\,\mathcal{R} + (6\cos p + 7)\right]e^{-2-\mathcal{R}} + \ldots\right\}.$$

Arutyunov-Frolov-Zamaklar, 2006

- The leading term of $\delta\epsilon_{\text{cl}}$ has also been obtained by finite gap methods and the Lüscher formulae.

Minahan-Ohlsson Sax, 2008 & Heller-Janik-Łukowski, 2007-2008

Classical Finite-Size Corrections

- The structure of classical finite-size corrections $\delta\epsilon_{\text{cl}}$ is the following (no negative powers, $\mathfrak{b}_1 = \mathfrak{c}_1 = 0$):

$$\delta\epsilon_{\text{cl}} = \frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \left[\mathfrak{a}_n\left(p\right) \mathcal{J}^{2n-2} + \mathfrak{b}_n\left(p\right) \mathcal{J}^{2n-3} + \mathfrak{c}_n\left(p\right) \mathcal{J}^{2n-4} + \ldots\right] e^{-n(2+\mathcal{R})}, \quad \mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}}$$

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- The first few leading terms are known:

$$\delta\epsilon_{\text{cl}}\Big|_{\text{leading}} = -\frac{4}{\pi}\sin^3\frac{p}{2}\,e^{-L_{\text{eff}}}\Big[1+2\,L_{\text{eff}}^2\cos^2\frac{p}{2}\,e^{-L_{\text{eff}}} + 8\,L_{\text{eff}}^4\cos^4\frac{p}{2}\,e^{-2L_{\text{eff}}} + \frac{128}{3}L_{\text{eff}}^6\cos^6\frac{p}{2}\,e^{-3L_{\text{eff}}} + \frac{800}{3}L_{\text{eff}}^8\cos^8\frac{p}{2}\,e^{-4L_{\text{eff}}} + \frac{9216}{5}L_{\text{eff}}^{10}\cos^{10}\frac{p}{2}\,e^{-5L_{\text{eff}}} + \ldots\Big].$$

Klose-McLoughlin, 2008

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- In the following we will obtain closed-form expressions for the leading, subleading

Classical Finite-Size Corrections

- The structure of classical finite-size corrections $\delta\epsilon_{\mathrm{cl}}$ is the following (no negative powers, $\mathfrak{b}_1 = \mathfrak{c}_1 = 0$):

$$\delta\epsilon_{\mathrm{cl}} = \frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \Big[\underbrace{\mathfrak{a}_n(p)\,\mathcal{J}^{2n-2}}_{\text{leading}} + \underbrace{\mathfrak{b}_n(p)\,\mathcal{J}^{2n-3}}_{\text{subleading}} + \underbrace{\mathfrak{c}_n(p)\,\mathcal{J}^{2n-4}}_{\text{next-to-subleading}} + \ldots \Big] e^{-n(2+\mathcal{R})}, \quad \mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}}$$

- The first few leading terms are known:

$$\delta\epsilon_{\mathrm{cl}}\Big|_{\text{leading}} = -\frac{4}{\pi}\sin^3\frac{p}{2}\,e^{-L_{\mathrm{eff}}}\Big[1+2\,L_{\mathrm{eff}}^2\cos^2\frac{p}{2}\,e^{-L_{\mathrm{eff}}} + 8\,L_{\mathrm{eff}}^4\cos^4\frac{p}{2}\,e^{-2L_{\mathrm{eff}}} + \frac{128}{3}L_{\mathrm{eff}}^6\cos^6\frac{p}{2}\,e^{-3L_{\mathrm{eff}}} + \\ + \frac{800}{3}L_{\mathrm{eff}}^8\cos^8\frac{p}{2}\,e^{-4L_{\mathrm{eff}}} + \frac{9216}{5}L_{\mathrm{eff}}^{10}\cos^{10}\frac{p}{2}\,e^{-5L_{\mathrm{eff}}} + \ldots\Big].$$

Klose-McLoughlin, 2008

- We would like to go beyond these approximations...
- ... and have to tell something more about the structure of finite-size corrections.
- In the following we will obtain closed-form expressions for the leading, subleading and next-to-subleading coefficients of $\delta\epsilon_{\mathrm{cl}}$.

The Method

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$$\begin{aligned}
\mathcal{E} &= d(a,x)\ln x + h(a,x) \\
\mathcal{J} &= c(a,x)\ln x + b(a,x) \\
p &= f(a,x)\ln x + g(a,x)
\end{aligned}$$

where $v \equiv \cos a$, $x = x(\omega, v)$ and d, h, c, b, f, g are known power series of x and a.

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- Secondly, this expression is inverted for $a = a\,(x, p, \mathcal{J})$ which is plugged into the first 2 equations:

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- Thirdly, we eliminate x from these two equations leading to an expression for $\mathcal{E} = \mathcal{E}\,(p, \mathcal{J})$.

Classical Finite-Size Corrections via the W-Function

- For $\mathcal{E} \equiv \pi E/\sqrt{\lambda}$, $\mathcal{J} \equiv \pi J/\sqrt{\lambda}$, we find:

$$\mathcal{E} - \mathcal{J}\Big|_{\text{classical}} = \sin\frac{p}{2} + \frac{1}{4\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W + \frac{W^2}{2}\right] - \frac{1}{16\mathcal{J}^3}\tan^4\frac{p}{2}\sin^2\frac{p}{2}\Big[(3\cos p + 2)\,W^2 +$$

$$+\frac{1}{6}(5\cos p + 11)\,W^3\Big] - \frac{1}{512\mathcal{J}^4}\tan^6\frac{p}{2}\sin\frac{p}{2}\Big\{(7\cos p - 3)^2\frac{W^2}{1+W} -$$

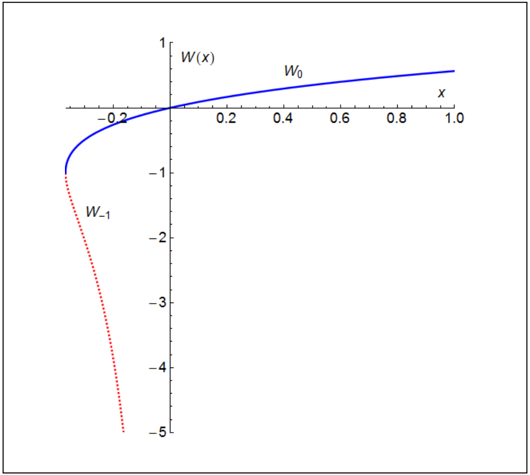
$$-\frac{1}{2}(25\cos 2p - 188\cos p - 13)\,W^2 - \frac{1}{2}(47\cos 2p + 196\cos p - 19)\,W^3 -$$

$$-\frac{1}{3}(13\cos 2p + 90\cos p + 137)\,W^4\Big\} + \ldots \qquad (\mathcal{J}, \lambda \to \infty)$$

Floratos-Georgiou-GL, 2013-2014

where the argument of the W-function is $W_0\left(\pm 16\mathcal{J}^2\cot^2(p/2)\, e^{-2\mathcal{J}\csc p/2 - 2}\right)$ and $\pm$ refers to the Elementary/Doubled Region.

Lambert W-Function



- The Lambert W-function is defined implicitly by the following relation:

$$W(z)\, e^{W(z)} = z \Leftrightarrow W\left(z\, e^{z}\right) = z$$

- Infinite exponential: ${}^{\infty}(e^z) = e^{z^{z^{\cdots}}} = \frac{W(-z)}{-z}$

- It has two real branches, $W_0(x)$ for $x \in \left[-e^{-1}, \infty\right)$ and $W_{-1}(x)$ for $x \in \left[-e^{-1}, 0\right]$ and a branch point at $W\left(-e^{-1}\right) = -1$.

- Its principal branch W_0 may be expanded according to the following Taylor series:

$$W_0(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-1}}{n!} \cdot x^n, \quad |x| \leq e^{-1}.$$

Closed-Form Expressions via the W-function

- Using the Taylor series of the Lambert W-function, we may compute the following terms:

$$\text{leading:} \quad \sum_{n=1}^{\infty} \mathfrak{a}_n(p)\, \mathcal{J}^{2n-2}\, e^{-n(2+\mathcal{R})} = \frac{1}{4\mathcal{J}^2} \tan^2\frac{p}{2} \sin^3\frac{p}{2} \left[W + \frac{W^2}{2}\right].$$

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$$\text{subleading:}\ \sum_{n=2}^{\infty} \mathfrak{b}_n(p)\, \mathcal{J}^{2n-3}\, e^{-n(2+\mathcal{R})} = -\frac{1}{16\mathcal{J}^3} \tan^4\frac{p}{2} \sin^2\frac{p}{2} \left[(3\cos p + 2)\, W^2 + \frac{1}{6}(5\cos p + 11)\, W^3\right].$$

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next-to-subleading: $$\sum_{n=2}^{\infty} \mathfrak{c}_n\,(p)\;\mathcal{J}^{2n-4}\,e^{-n(2+\mathcal{R})} = -\frac{1}{512\mathcal{J}^4}\tan^6\frac{p}{2}\sin\frac{p}{2}\Bigg\{(7\cos p-3)^2\,\frac{W^2}{1+W}- -\frac{1}{2}\,(25\cos 2p-188\cos p-13)\,W^2-\frac{1}{2}\,(47\cos 2p+196\cos p-19)\,W^3- -\frac{1}{3}\,(13\cos 2p+90\cos p+137)\,W^4\Bigg\}.$$

Floratos-GL, 2014

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- $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_2$, $\mathfrak{c}_2$ are the Arutyunov-Frolov-Zamaklar coefficients.
- $\mathfrak{a}_1$-$\mathfrak{a}_6$ are the Klose-Mcloughlin coefficients.

Section 4

Conclusions

Why Study Magnons?

- $\mathcal{N} = 4$ SYM possesses certain elementary excitations that are known as magnons.
- Giant magnons are the string theory duals of $\mathcal{N} = 4$ SYM magnon excitations, living in $\mathbb{R} \times \mathrm{S}^2 \subset \mathrm{AdS}_5 \times \mathrm{S}^5$.
- This identification allows to test the validity of AdS/CFT correspondence at the level of its elementary excitations.
- We also learn a lot about the way the two theories are equal – i.e. the AdS/CFT dictionary.
- Owing to the weak/strong nature of AdS/CFT we also learn a lot about each theory's structure in its non-perturbative sector.
- Going beyond AdS/CFT, we learn a lot about the structure of gauge theories and gravity in curved space-times.

Results Summary

- By inverting the elliptic integrals that furnish the conserved charges of GMs, we have succeeded in obtaining closed-form expressions for the leading, subleading and next-to-subleading series of the finite-size corrections to the dispersion relation of the HM giant magnon:

$$E - J = \epsilon_\infty + \sqrt{\lambda}\,\delta\epsilon_{\text{cl}} + \delta\epsilon_{\text{1-loop}} + \frac{1}{\sqrt{\lambda}}\delta\epsilon_{\text{2-loop}} + \ldots \qquad (J, \lambda \to \infty)\,, \quad \mathcal{J} \equiv \frac{\pi J}{\sqrt{\lambda}}$$

$$\delta\epsilon_{\text{cl}} = \frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \left[\mathfrak{a}_n(p)\,\mathcal{J}^{2n-2} + \mathfrak{b}_n(p)\,\mathcal{J}^{2n-3} + \mathfrak{c}_n(p)\,\mathcal{J}^{2n-4} + \ldots\right] e^{-n(2+\mathcal{R})}, \quad \mathcal{R} \equiv 2\mathcal{J}\csc\frac{p}{2}$$

- This results in compact expressions of the GM dispersion relations:

$$\sum_{n=1}^{\infty} \mathfrak{a}_n(p)\,\mathcal{J}^{2n-2}\,e^{-n(2+\mathcal{R})} = \frac{1}{4\mathcal{J}^2}\tan^2\frac{p}{2}\sin^3\frac{p}{2}\left[W + \frac{W^2}{2}\right], \quad \mathfrak{b}_n(p) = \ldots\,, \quad \mathfrak{c}_n(p) = \ldots$$

- We learn about the structure of the classical exponential corrections ...

- We may assume that the W-functions will keep appearing to all orders ...

Outlook

Many questions still unanswered.

- Can we find an iterative all-orders formula?
- How about quantum corrections?
- What about weak coupling?
- Can we make contact with other methods that account for wrapping effects (Lüscher corrections, TBA, Y-system, QSC)?
- Many interesting generalizations: ABJM, AdS, spiky strings, deformed backgrounds, correlation functions ...
- Also: do these new expressions teach us anything about the physics of GMs?
- Do 2+1-dimensional generalizations with M2-branes exist? (Axenides-Floratos-GL, 2013)

Ευχαριστώ!

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MIS-448332-ORASY (NSRF 2007–13 ACTION, KRIPIS)

Coordinate Bethe Ansatz Revisited

- Let $\text{Tr}\left[\mathcal{Z}^L\right] \sim |x>=|\uparrow \ldots \uparrow\uparrow\uparrow \ldots \uparrow>$ denote the ground state: $\Delta = J$.
- 1-magnon states $\left|\mathcal{Z}^J \mathcal{W}\right\rangle \sim |x>=|\uparrow \ldots \uparrow \underset{x}{\downarrow} \uparrow \ldots \uparrow>$ are diagonalized by:

$$|p>=\sum_{x=1}^{J+1} e^{ipx}|x> \longrightarrow \boxed{\Delta = J + 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + \mathcal{O}\left(\lambda^2\right)} \quad \& \quad p = \frac{2\pi k}{L}\,, \; k \in \mathbb{Z}.$$

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- 2-magnon states $\mathrm{Tr}\left[\mathcal{Z}^J\mathcal{W}^2\right] \sim |x_1,x_2>=|\uparrow\ldots\uparrow\underset{x_1}{\downarrow}\uparrow\ldots\uparrow\underset{x_2}{\downarrow}\uparrow\ldots\uparrow>$ are diagonalized by:

$$|p_1,p_2>=\sum_{x_2>x_1=1}^{J+2}\left(e^{ip_1x_1+ip_2x_2}+S_{21}e^{ip_1x_2+ip_2x_1}\right)|x_1,x_2>\longrightarrow \boxed{\Delta = J+2+\frac{\lambda}{2\pi^2}\sum_{j=1,2}\sin^2\frac{p_j}{2}+\mathcal{O}\left(\lambda^2\right)}$$

We also find, $S_{12} = \dfrac{u_1-u_2+i}{u_1-u_2-i} = e^{ip_1L}, \quad u_j = \dfrac{1}{2}\cot\dfrac{p_j}{2} \quad \& \quad p_1+p_2 = \dfrac{2\pi k}{L}, \; k\in\mathbb{Z}.$

Factorized Scattering

- For M-magnon states Tr $\left[\mathcal{Z}^J \mathcal{W}^M\right]$ the situation is similar, i.e.

$$|p_1, p_2, \ldots, p_M >= \sum_{x_M > \ldots > x_1 = 1}^{J+M} \left[\sum_{\sigma} S_{\sigma(1,2,\ldots,M)} e^{i p_j x_{\sigma_j}} \right] |x_1, x_2, \ldots, x_M >$$

- Scaling dimensions: $\Delta = J + M + \frac{\lambda}{2\pi^2} \sum \sin^2 \frac{p_j}{2} + \mathcal{O}\left(\lambda^2\right)$

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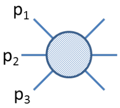
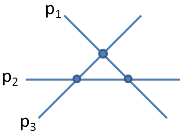
- S-matrix: $S_{12\ldots M}\left(p_1, p_2, \ldots, p_M\right) = S_{12} S_{23} \ldots S_{M1} \quad \rightarrow$ Factorized (Elastic) Scattering!





- Momenta are conserved: $\left\{p_1', p_2', \ldots, p_M'\right\} = \left\{p_1, p_2, \ldots, p_M\right\} \rightarrow$ L conservation laws $\rightarrow$ Integrability.
- Bethe ansatz equations (BAE): $e^{ip_j L} = S_{j1} S_{j2} \ldots S_{jM}\,, \quad j = 1, 2, \ldots, M.$

Going to Higher Loops

- Higher-loop contributions to the dilatation operator are known, however these become more and more complicated:

$$\mathcal{D} = L \cdot \mathbb{I} + \frac{\lambda}{8\pi^2}\hat{H} + \lambda^2 D_2 + \lambda^3 D_3 + \ldots$$

$$\hat{H} = \frac{1}{2}\sum_{j=1}^{L}(1 - 4\,\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1})\ , \quad \mathcal{D}_2 = \sum_{j=1}^{L}(-\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+2} + 4\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 3)\ , \quad \mathcal{D}_3 = \ldots$$

- They contain non-neighboring, as well as higher-order interactions (e.g. $\boldsymbol{\sigma}^4$ in $\mathcal{D}_3$).

- However Integrability is an all-loop property of $\mathcal{N} = 4$ SYM !

- We thus expect factorized scattering to persist in higher loops and all sectors of the gauge theory.

All-Loop Asymptotic Bethe Ansatz

- An all-loop, asymptotic BA for $\mathfrak{su}(2)$ has been proposed...

$$\Delta = J + M + \frac{\lambda}{8\pi^2}\sum_{j=1}^{M} E\left(p_j\right)\ ,\ E\left(p_j\right) = \frac{8\pi^2}{\lambda}\left[\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}}-1\right] = 4\sin^2\frac{p_j}{2} - \frac{\lambda}{\pi^2}\sin^4\frac{p_j}{2} + \mathcal{O}\left(\lambda^2\right)$$

$$e^{ip_jL} = \prod_{\substack{k=1\\k\neq j}}^{M} S_{jk}\ ,\quad S_{jk} = \frac{u_j - u_k + i}{u_j - u_k - i}\cdot S_{jk}^{D}\ ,\quad u\left(p_j\right) = \frac{1}{2}\cot\frac{p_j}{2}\sqrt{1+\lambda\sin^2\frac{p_j}{2}}\ ,\quad S_{jk}^{D} \equiv \text{"Dressing Factor"}$$

Beisert-Dippel-Staudacher (BDS), 2004

- Asymptotic means that there's a *"critical"* loop order equal to the length of the spin-chain L at which the ABA ceases to hold.

The Dressing Phase

- The $\mathfrak{su}(2)$ "Dressing Phase/Factor" of $\mathcal{N} = 4$ SYM,

$$S^D_{jk} = \sigma^2\left(p_j, p_k\right)$$

Arutyunov-Frolov-Staudacher (AFS), 2004

- ... was introduced in order to reconcile the weak and strong coupling limits of the BA.
- At weak coupling it is equal to unity up to 3-loops: $\sigma^2_{jk(\text{weak})} = 1 + \mathcal{O}\left(\lambda^3\right)$.
- At strong coupling it is given by the AFS phase: $\sigma^2_{jk(\text{strong})} = \sigma^2_{jk(\text{AFS})}$.

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- At strong coupling it is given by the AFS phase: $\sigma^{2}_{jk(\text{strong})} = \sigma^{2}_{jk(\text{AFS})}$.
- For $M = 2$ magnons at strong coupling $(\lambda \to \infty)$ one may calculate the AFS phase exactly:

$$\sigma^{2}_{(\text{AFS})}\left(p_1, p_2\right) = \exp\left\{i\,\frac{\sqrt{\lambda}}{\pi}\left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2}\right)\cdot\log\left[\frac{\sin^{2}\left(p_1 - p_2\right)/4}{\sin^{2}\left(p_1 + p_2\right)/4}\right]\right\}.$$

Pohlmeyer Reduction

- Classical strings in $\mathbb{R} \times S^2$ are dual to the sine-Gordon (sG) equation:

$$\partial_+ \mathbf{X} \cdot \partial_- \mathbf{X} = \dot{\mathbf{X}}^2 - \acute{\mathbf{X}}^2 = \cos 2\phi \Longrightarrow \boxed{\ddot{\phi} - \phi'' + \frac{1}{2}\sin 2\phi = 0}$$

Pohlmeyer, 1976

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Pohlmeyer, 1976

- Classical strings in $\mathbb{R} \times S^3$ are dual to the complex sine-Gordon (CsG) equation:

$$\begin{aligned} K_i &\equiv e_{ijkl} X_j \partial_+ X_k \partial_- X_l \\ \pm \mathbf{K} \cdot \partial_\pm^2 \mathbf{X} &= 4 \sin^2 \phi \, \partial_\pm \chi \end{aligned} \Longrightarrow \boxed{\ddot{\psi} - \psi'' + \frac{\psi^* \left(\dot{\psi}^2 - \psi'^2\right)}{1 - |\psi|^2} + \psi\left(1 - |\psi|^2\right) = 0}, \; \psi \equiv e^{i\chi} \sin \phi$$

Pohlmeyer & Lund-Regge, 1976

Pohlmeyer Reduction

- Classical strings in $\mathbb{R} \times S^2$ are dual to the sine-Gordon (sG) equation:

$$\partial_+ \mathbf{X} \cdot \partial_- \mathbf{X} = \dot{\mathbf{X}}^2 - \acute{\mathbf{X}}^2 = \cos 2\phi \Longrightarrow \boxed{\ddot{\phi} - \phi'' + \frac{1}{2}\sin 2\phi = 0}$$

Pohlmeyer, 1976

- Classical strings in $\mathbb{R} \times S^3$ are dual to the complex sine-Gordon (CsG) equation:

$$\begin{array}{l} K_i \equiv e_{ijkl} X_j \partial_+ X_k \partial_- X_l \\ \pm \mathbf{K} \cdot \partial_\pm^2 \mathbf{X} = 4 \sin^2 \phi \, \partial_\pm \chi \end{array} \Longrightarrow \boxed{\ddot{\psi} - \psi'' + \frac{\psi^* \left(\dot{\psi}^2 - \psi'^2\right)}{1 - |\psi|^2} + \psi \left(1 - |\psi|^2\right) = 0}, \; \psi \equiv e^{i\chi} \sin \phi$$

Pohlmeyer & Lund-Regge, 1976

- Also: classical strings in $AdS_{2,3,4}$ are equivalent to Liouville, sinh-Gordon and B_2 Toda model...
de Vega-Sanchez, 1993

- CsG equation can be written as a deformed gWZW model (Bakas, 1993).
IIB Superstring in $AdS_5 \times S^5$ has similarly been proven equivalent to a deformed gWZW model.
Grigoriev-Tseytlin & Mikhailov, Schäfer-Nameki 2007

Giant Magnon Scattering

- Plugging the GM ansatz into the string EoM & using the definition of the sG field ϕ, we find that GMs are the Pohlmeyer duals of the sG solitons:

$$\phi(\sigma,\tau) = 2\arctan e^{\pm\gamma(\sigma - v\tau)} = \text{arcsin}\,\text{sech}\left[\gamma(\sigma - v\tau)\right], \quad \gamma = \left(1 - v^2\right)^{-1/2}$$

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- We may scatter GMs in the sG picture and recover the gauge theory results. E.g. we may consider the sG kink-antikink solution (the result is the same for any 2-soliton solution),

$$\tan\frac{\phi}{2} = \frac{1}{v}\frac{\sinh\gamma v\tau}{\cosh\gamma\sigma}$$

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- The last term of δ_{12} depends on the choice of the world-sheet gauge and it may be omitted.

Magnons in AdS/CFT Correspondence

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$$S_{12} = \frac{u_1 - u_2 + i}{u_1 - u_2 - i} \cdot \sigma^2_{\text{AFS}}\left(p_1, p_2\right) \xrightarrow{\lambda\to\infty} \sigma^2_{\text{AFS}}\left(p_1, p_2\right)$$

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- By factorized scattering, the $\mathfrak{su}(2)$ S-matrices on both sides of the AdS/CFT match!

The BMN String

- The BMN string is a point-like string that rotates at the equator of $\mathbb{R} \times S^2 \subset AdS_5 \times S^5$:

$$\left\{ t = \tau, \rho = \overline{\theta} = \overline{\phi}_1 = \overline{\phi}_2 = 0 \right\} \times \left\{ \theta = \frac{\pi}{2}, \phi = \tau, \theta_1 = \phi_1 = \phi_2 = 0 \right\}$$

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- It is the AdS/CFT dual of the $\mathcal{N} = 4$ SYM BPS operator $\mathrm{Tr}\left[\mathcal{Z}^J\right]$.

Classical GKP String on S^2 at Finite-Size

- Two finite-sized GMs with $v = 0$, $p = \pi$ and angular momentum $J/2$ each, give the GKP string on S^2:

$$E - J\Big|_{\text{GKP}} = \frac{2\sqrt{\lambda}}{\pi}\left[1 - 4e^{-\pi J/\sqrt{\lambda}-2} + 4\left(\frac{2\pi J}{\sqrt{\lambda}} - 1\right) e^{-2\pi J/\sqrt{\lambda}-4} - \ldots\right], \quad J, \lambda \to \infty$$

Arutyunov-Frolov-Zamaklar, 2006

- In general, we may write ($\mathcal{E} \equiv \pi\, E/2\sqrt{\lambda}$ and $\mathcal{J} \equiv \pi\, J/2\sqrt{\lambda}$):

$$\mathcal{E} - \mathcal{J}\Big|_{\text{GKP}} = 1 + \sum_{n=1}^{\infty}\left[\widetilde{\mathfrak{a}}_n\, \mathcal{J}^{n-1} + \widetilde{\mathfrak{b}}_n\, \mathcal{J}^{n-2} + \widetilde{\mathfrak{c}}_n\, \mathcal{J}^{n-3} + \ldots\right]\left(e^{-2\mathcal{J}-2}\right)^n$$

- In terms of the W-function, we obtain:

$$\mathcal{E} - \mathcal{J}\Big|_{\text{GKP}} = 1 - \frac{1}{4\mathcal{J}}\left(2W + W^2\right) - \frac{1}{16\mathcal{J}^2}\left(W^2 + W^3\right) - \frac{1}{256\mathcal{J}^3}\,\frac{W^3\left(11\,W^2 + 26\,W + 16\right)}{1 + W} + \ldots$$

Floratos-Georgiou-GL, 2013

where the argument of the W-function is $W\left(\pm 8\mathcal{J}e^{-2\mathcal{J}-2}\right)$, with the plus sign corresponding to closed and folded strings ($\omega > 1$) and the minus sign to circular strings ($\omega < 1$).

Magnon & Giant Magnon Summary at Infinite Size

BMN String	Giant Magnon	GKP String
$E - J = 0$	$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2}$	$E - J = \frac{2\sqrt{\lambda}}{\pi}$
$M = 0$ Magnons	$M = 1$ Magnon	$M = 2$ Magnons
$\mathrm{Tr}\left[\mathcal{Z}^L\right]$	$\left\vert\mathcal{Z}^J\mathcal{W}\right\rangle$	$\mathrm{Tr}\left[\mathcal{Z}^J\mathcal{W}^2\right]$
$\vert\uparrow\ldots\uparrow\uparrow\uparrow\ldots\uparrow>$	$\vert\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow>$	$\vert\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow\downarrow\uparrow\ldots\uparrow>$
$\Delta - J = 0$	$\Delta - J = 1 + \frac{\lambda}{2\pi^2} \sin^2 \frac{p}{2}$	$\Delta - J = 2 + \frac{\lambda}{\pi^2} \sin^2 \frac{n\pi}{J+1}$

Symbolic Computations

- Using Mathematica, we obtain the following results for GMs in the elementary region:

$$\begin{aligned}\mathcal{E}-\mathcal{J}\Big|_{\text{classical}} =& \sin\frac{p}{2}-4\sin^3\frac{p}{2}\,e^{-(2+\mathcal{R})}-\left[8\mathcal{J}^2\csc\frac{p}{2}\sin^2 p-\mathcal{J}\left(12\cos 2p-8\cos p-4\right)+4\left(6\cos p+7\right)\sin^3\frac{p}{2}\right]e^{-2(2+\mathcal{R})}-\\ &-\left[32\mathcal{J}^4\csc^5\frac{p}{2}\sin^4 p+\frac{32}{3}\mathcal{J}^3\left(31\cos 2p+88\cos p+57\right)+32\mathcal{J}^2\left(9\sin\frac{5p}{2}+11\sin\frac{3p}{2}+6\sin\frac{p}{2}\right)-\right.\\ &\left.-\mathcal{J}\left(96\cos 3p+44\cos 2p-112\cos p-28\right)+\frac{8}{3}\left(37\cos 2p+97\cos p+72\right)\sin^3\frac{p}{2}\right]e^{-3(2+\mathcal{R})}-\\ &-\left[\frac{512}{3}\mathcal{J}^6\csc^9\frac{p}{2}\sin^6 p+2048\mathcal{J}^5\left(19\cos p+5\right)\cos^2\frac{p}{2}\cot^2\frac{p}{2}+\frac{64}{3}\mathcal{J}^4\left(1273\cos 2p+1824\cos p+\right.\right.\\ &\left.+1319\right)\cos\frac{p}{2}\cot\frac{p}{2}+\frac{64}{3}\mathcal{J}^3\left(441\cos 3p+1242\cos 2p+1983\cos p+1118\right)+8\mathcal{J}^2\left(431\sin\frac{7p}{2}+\right.\\ &\left.+734\sin\frac{5p}{2}+544\sin\frac{3p}{2}+273\sin\frac{p}{2}\right)-\frac{4}{3}\mathcal{J}\left(511\cos 4p+360\cos 3p-88\cos 2p-588\cos p-195\right)+\\ &\left.+4\left(118\cos 3p+322\cos 2p+532\cos p+349\right)\sin^3\frac{p}{2}\right]e^{-4(2+\mathcal{R})}-\ldots, \qquad \mathcal{R}\equiv 2\mathcal{J}\csc\frac{p}{2} \qquad (\mathcal{J},\lambda\to\infty)\end{aligned}$$

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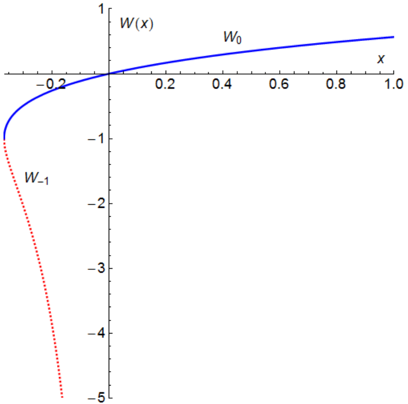
More about the Lambert W-Function

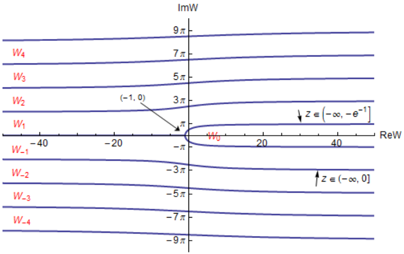
- The Lambert W-function is defined implicitly by the following relation:

$$W(z)\, e^{W(z)} = z \Leftrightarrow W\left(z\, e^{z}\right) = z$$



Branch Structure

- The general branch structure is reminiscent of that of the logarithm:



- The curves separating the branches are not straight lines, but the "Quadratrix of Hippias":

$$\left\{ -\eta \cot \eta + i\eta, \quad -\pi < \eta < \pi, \quad 2k\pi < \pm\eta < (2k+1)\pi \right\}, \quad k = 1, 2, 3, \ldots$$

- Two real branches, $W_0\left(-e^{-1} \leq x < \infty\right) \in [-1, \infty)$ and $W_{-1}\left(-e^{-1} \leq x \leq 0\right) \in [-1, -\infty)$.
- Triple branch point at $W_{0,\pm1}\left(-e^{-1}\right) = -1$.
- Branch cuts: $\left(-\infty, -e^{-1}\right]$ for $W_{0,\pm1}$ and $(-\infty, 0]$ for $W_{k\neq0}$.

More about the Lambert W-Function

- Its real branches W_0, W_{-1} may be expanded according to the following Taylor series:

$$W_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^n}{(n+1)!} \cdot x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-1}}{n!} \cdot x^n, \quad |x| \leq e^{-1}$$

$$W_{-1}(x) = \ln|x| - \ln\ln|x| + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^n}{m!} \begin{bmatrix} n+m \\ n+1 \end{bmatrix} (\ln|x|)^{-n-m} (\ln\ln|x|)^m,$$

where the unsigned Stirling numbers of the first kind, $\begin{bmatrix} n+m \\ n+1 \end{bmatrix}$ are defined recursively as:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \& \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = 0, \ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad n, k \geq 1.$$