Holographic Calculation of Rényi Entropies and Restrictions on Higher Derivative Terms

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Entanglement Rényi Entropy

- Consider a quantum system divided into two subsystems A and A^C
- Entanglement Entropy for subsystem A is defined as the von Neumann entropy for the reduced density matrix ρ_A

$$S_{\mathrm{EE}} := -\mathrm{Tr}\left(\rho_A \ln \rho_A\right)$$

$$\rho_A = \mathrm{Tr}_{A^C}\, \rho$$

Entanglement Rényi Entropy

- Entanglement Entropy measures the quantum entanglement of A and A^C provided that the total system lies in a pure state
- Example: consider a two-spinor system
  - Case 1: Tensor product state

$$|\psi\rangle = |\uparrow\rangle|\uparrow\rangle \rightarrow \rho_A = |\uparrow\rangle\langle\uparrow| \rightarrow S_{\text{EE}} = 0$$

  - Case 2: Maximally Entangled State

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\right) \rightarrow \rho_A = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \rightarrow S_{EE} = \ln 2$$

Entanglement Rényi Entropy

- Difficulties in representing the operator $\ln \rho_A$, do not appear if one calculate Rényi Entropy[1]

$$S_q := \frac{1}{1-q} \ln \mathrm{Tr}\, \rho_A^q$$

- Entanglement Entropy can be recovered as an appropriate limit of Rényi Entropy

$$S_{EE} = \lim_{q \to 1} S_q$$

- Bonus: The whole spectrum of ρ_A can be recovered, if all S_q with $q \geq 1$ are known

[1] A. Renyi, On the Foundations of Information Theory, Rev. Int. Stat. Inst. 33 1 (1965)

Entanglement Rényi Entropy

- Rényi Entropies obey some interesting inequalities by definition

$$S_q = \frac{1}{1-q} \ln \sum_i p_i^q$$

$$0 \le p_i \le 1$$

$$\sum_i p_i = 1$$

$\longrightarrow$

$$\frac{\partial S_q}{\partial q} \le 0$$

$$\frac{\partial}{\partial q}\left(\frac{q-1}{q} S_q\right) \ge 0$$

$$\frac{\partial}{\partial q}\left(\left(q-1\right) S_q\right) \ge 0$$

$$\frac{\partial^2}{\partial q^2}\left(\left(q-1\right) S_q\right) \le 0$$

Entanglement Rényi Entropy

- We consider theories with a conserved charge in the grand canonical ensemble
- Rényi Entropy definition is extended naturally[1]

$$S_q(\mu) := \frac{1}{1-q} \ln \mathrm{Tr}\, \rho_A^q(\mu)$$

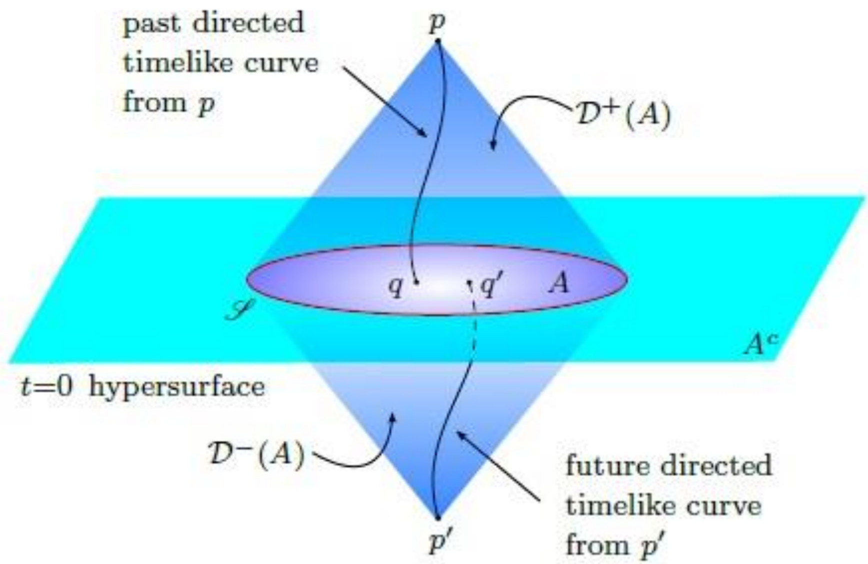
$$\rho_A(\mu) = \frac{\rho_A e^{\mu Q_A}}{Tr\left[\rho_A e^{\mu Q_A}\right]}$$

- The chemical potential μ can be considered imaginary. Such analysis can provide information about the confinement phase transition in QCD theories with fermions[2]

[1]A. Belin, et al, Holographic Charged Renyi Entropies, JHEP 1312, 059 (2013) [hep-th/1310.4180]
[2]A. Roberge et al, Gauge Theories With Imaginary Chemical Potential and the Phases of QCD, Nucl. Phys. B 275, 734 (1986)

CFT Calculation of Rényi Entropy

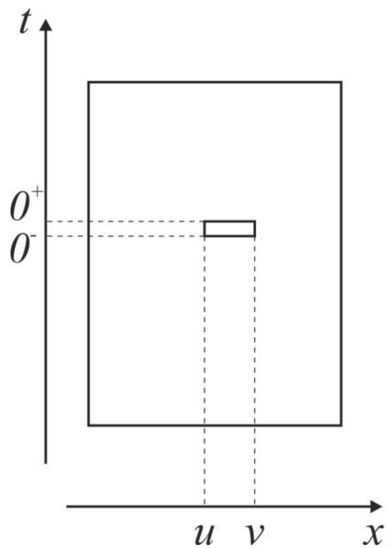
- Typically CFT calculations of Rényi Entropy involve the “Replica Trick”[1]
- The entangling surface is usually considered a sphere of radius R (just two points in 2d)



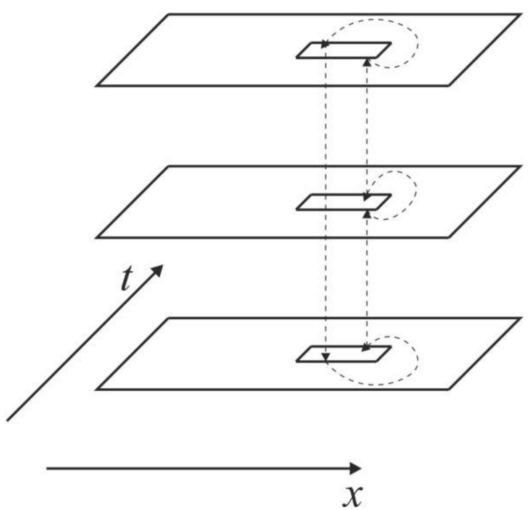
[1]P. Calabrese et al, Entanglement Entropy and Quantum Field Theory, J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152]

CFT Calculation of Rényi Entropy

In the path integral formalism $\mathrm{Tr}\rho_A{}^q = Z_q / Z_1{}^q$ and one computes the partition function Z_q by gluing together q copies of R_d along the boundary (∂A).



Path integral representation of the reduced density matrix



The q-sheeted surface

[1]P. Calabrese et al, Entanglement Entropy and Quantum Field Theory, J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152]

CFT Calculation of Rényi Entropy

Alternatively one can perform the coordinate transformation[1]

$$t = R\frac{\sinh\frac{\tau}{R}}{\cosh u + \cosh\frac{\tau}{R}}, \quad r = R\frac{\sinh u}{\cosh u + \cosh\frac{\tau}{R}}$$

and the Minkowski metric in $D(A)$ is written as

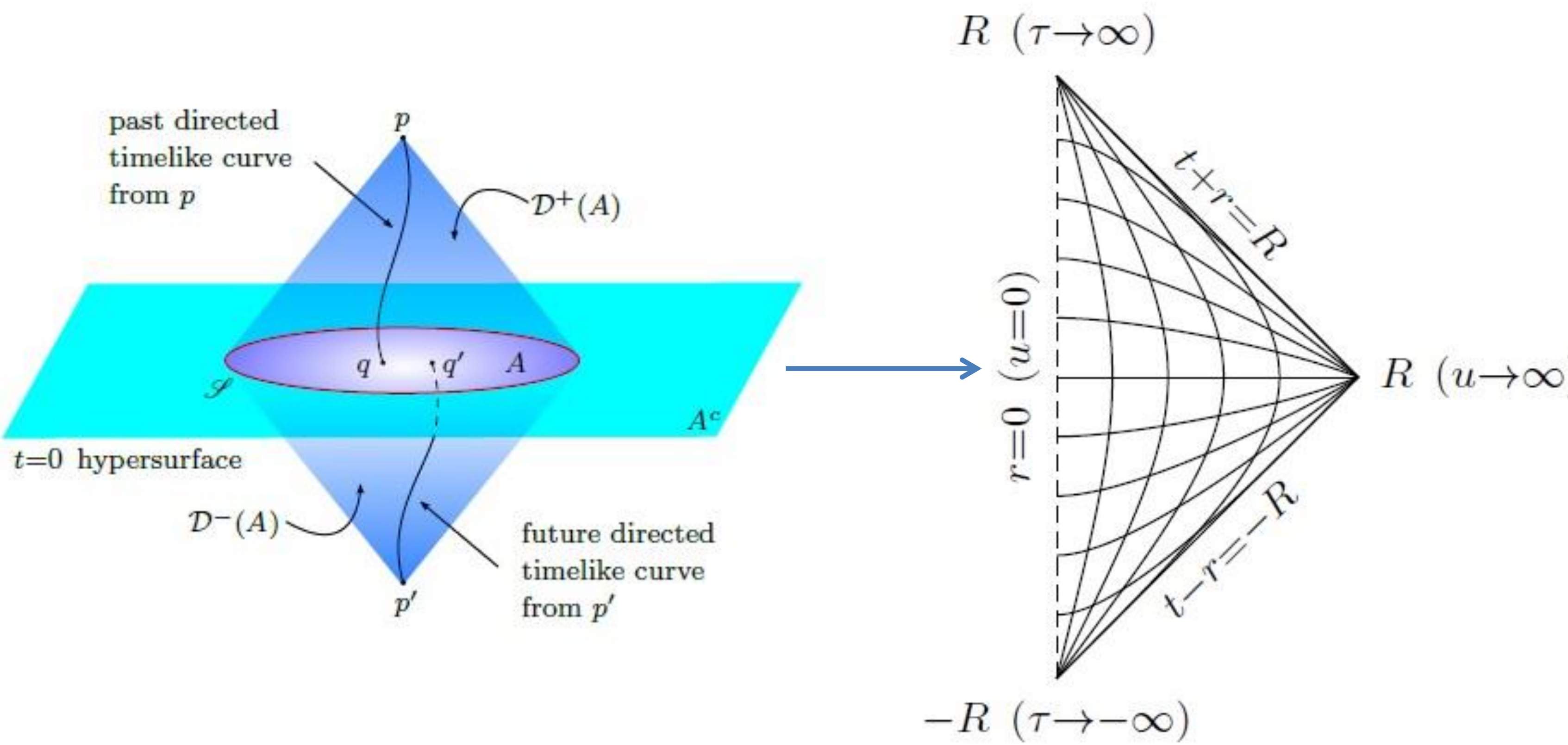
$$ds^2_{\mathbf{R}^{i,d-1}} = -dt^2 + dr^2 + r^2 dS^2_{d-2}$$

$$\downarrow$$

$$ds^2_{\mathbf{R}\times\mathbf{H}^{d-1}} = \Omega^2 ds^2_{\mathbf{R}^{1,d-1}} = -d\tau^2 + R^2\left(du^2 + \sinh^2 u dS^2_{d-2}\right), \quad \Omega = \frac{1}{\cosh u + \cosh\frac{\tau}{R}}$$

[1]H. Casini, et al, Towards a Derivation of Holographic Entanglement Entropy, JHEP 1105, 036 (2011) [hep-th/1102.0440]

CFT Calculation of Rényi Entropy



CFT in the Minkowski vacuum on $D(A)$ → Thermal state on $\mathbf{R}\times\mathbf{H}^{d-1}$, with temperature

$$T_0 = \frac{1}{2\pi R}$$

CFT Calculation of Rényi Entropy

The conformal transformation maps the reduced density matrix ρ_A to a thermal one (up to a unitary transformation)

$$\rho_A(\mu) \to U^{-1} \frac{e^{-\beta H + \mu Q_A}}{Z(\beta, \mu)} U$$

leading to an expression of **Rényi Entropy** for spherical entangling surface **in Minkowski** space **as function** of the **Thermal Entropy** on $\mathbf{R} \times \mathbf{H}^{d\text{-}1}$

$$S_q(\mu) = \frac{q}{q-1} \frac{1}{T_0} \int_{\frac{T_0}{q}}^{T_0} dT\, S(T, \mu)$$

Holographic Calculation of Rényi Entropy

AdS/CFT Correspondence

Boundary theory	Bulk theory
Thermal state on $\mathbf{R}\times\mathbf{H}^{d\text{-}1}$	Asymptotically AdS BH with hyperbolic horizon
Temperature T	Hawking temperature T
Thermal Entropy	Black Hole Entropy
Global $\mathrm{U}(1)$ symmetry	Gauge $\mathrm{U}(1)$ symmetry
Wide class of CFTs including $c\neq a$	Gauss-Bonnet corrections

Holographic Calculation of Rényi Entropy

Bulk action

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left(\frac{d(d-1)}{L^2} + R + \frac{\lambda L^2}{(d-2)(d-3)} \left(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\kappa\lambda}^2 \right) - \frac{\ell_*^2}{4} F^2 \right)$$

Bounds on GB coupling from causality[1]

$$-\frac{(3d+2)(d-2)}{4(d+2)^2} \le \lambda \le \frac{(d-2)(d-3)(d^2-d+6)}{4(d^2-3d+6)^2}$$

Central Charges as function of GB coupling[2,3]

$$\tilde{C}_T = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\tilde{L}}{l_p} \right)^{d-1} (1 - 2\lambda f_\infty), \quad a_d^* = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\tilde{L}}{l_p} \right)^{d-1} \left(1 - 2\frac{d-1}{d-3} \lambda f_\infty \right)$$

[1]X. O. Camanho et al, Causality Constraints in AdS/CFT from Conformal Collider Physics and GB Gravity, JHEP 1004, 007 (2010) [hep-th/0911.3160v2]

[2]A. Buchel et al, Holographic GB Gravity in Arbitrary Dimensions, JHEP 1003, 111 (2010) [hep-th/0911.4257]

[3] R. Myers et al, Holographic c-Theorems in Arbitrary Dimensions, JHEP 1101, 125 (2011) [hep-th/1011.5819]

Holographic Calculation of Rényi Entropy

BH Solution[1,2]

- Metric: $$ds^2 = -\left(\frac{r^2}{L^2} f(r) - 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} f(r) - 1} + r^2 dH_{d-1}^2$$

$$f(r) = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 4\lambda\left[1 + \frac{L^2}{d-1}\left(\frac{m}{r^{d-1}} - \frac{q^2}{2(d-2)r^{2d-2}}\right)\right]}\right)$$

Note that AdS scale is $\tilde{L} = \frac{L}{\sqrt{f_\infty}}, \quad f_\infty = \lim_{r\to\infty} f(r) = \frac{1}{2\lambda}\left(1 - \sqrt{1 - 4\lambda}\right)$

- EM field: $$A = \left(\frac{1}{d-2}\frac{\tilde{L}q}{R\ell_* r^{d-2}} - \frac{\mu}{2\pi R}\right) dt$$

[1]M. Cvetic, et al, Black Hole Thermodynamics and Negative Entropy in dS and AdS Einstein-GB Gravity, Nucl. Phys. B 628, 295 (2002) [hep-th/0112045]
[2]G. Pastras et al, Thermodynamics of the Maxwell-GB AdS BH with Higher Derivative Gauge Corrections, JHEP 0907, 030 (2009) [hep-th/0807.3478]

Holographic Calculation of Rényi Entropy

Parameters m and q are connected with energy and charge of the black hole

$$E = \frac{mV_{\mathbf{H}^{d-1}}}{2\ell_P^{\,d-1}}, \quad Q = \frac{q\ell_* V_{\mathbf{H}^{d-1}}}{\ell_P^{\,d-1}}$$

and can be expressed in terms of the horizon radius and the chemical potential

$$m = (d-1)\tilde{L}^{d-2}\left(\frac{x^d}{f_\infty} - \left(1 - \frac{d-2}{2(d-1)}\left(\frac{\mu\ell_*}{2\pi\tilde{L}}\right)^2\right)x^{d-2} + \lambda f_\infty x^{d-4}\right)$$

$$\mu = \frac{2\pi\tilde{L}^{d-1}q}{(d-2)\ell_* x^{d-2}}$$

$$x \equiv \frac{r_H}{\tilde{L}}$$

Holographic Calculation of Rényi Entropy

The above allow the expression of Hawking temperature as function of x, μ, λ

$$T(x,\mu,\lambda) = \frac{1}{2\pi\tilde{L}}\frac{1}{2f_\infty x}\frac{dx^4 - (d-2)\left(1+\frac{d-2}{2(d-1)}\left(\frac{\mu l_*}{2\pi L}\right)^2\right)f_\infty x^2 + (d-4)\lambda f_\infty{}^2}{x^2 - 2\lambda f_\infty}$$

GB terms alter the area law for BH entropy

$$S_1(x,\lambda) = V_{H^{d-1}}\left(\frac{\tilde{L}}{\ell_p}\right)^{d-1} 2\pi\left(x^{d-1} - 2\frac{d-1}{d-3}\lambda f_\infty x^{d-3}\right)$$

Holographic Calculation of Rényi Entropy

We result in the final expression[1] for S_q

$$S_q(\mu) = \frac{q}{q-1}\frac{1}{T_0}\int_{x_q}^{x_1} dx S(x,\mu)\frac{dT(x,\mu)}{dx}$$

↓

$$S_q(\mu,\lambda) = V_{\mathbf{H}^{d-1}} 2\pi \frac{q}{q-1}\left(\frac{\tilde{L}}{\ell_p}\right)^{d-1}\frac{d-1}{d-2}\times\left[\frac{x_1{}^d - x_q{}^d}{f_\infty} - \lambda f_\infty\left(x_1{}^{d-4} - x_q{}^{d-4}\right)\right.$$
$$\left. - \frac{1}{d-1}\left(x_1{}^{d-1} - \frac{x_q{}^{d-1}}{q}\right) - \frac{2\lambda f_\infty}{d-3}\left(x_1{}^{d-3} - \frac{x_q{}^{d-3}}{q}\right)\right]$$

where

$$d\frac{x_q^4}{f_\infty} - 2\frac{x_q^3}{q} - (d-2)\left(1 + \frac{d-2}{2(d-1)}\left(\frac{\mu\ell_*}{2\pi\tilde{L}}\right)^2\right)x_q^2 + 4\lambda f_\infty\frac{x_q}{q} + (d-4)\lambda f_\infty = 0$$

[1]G. Pastras et al, Charged Renyi Entropies in CFTs with Einstein-GB Holographic Duals, [hep-th/1404.1309]

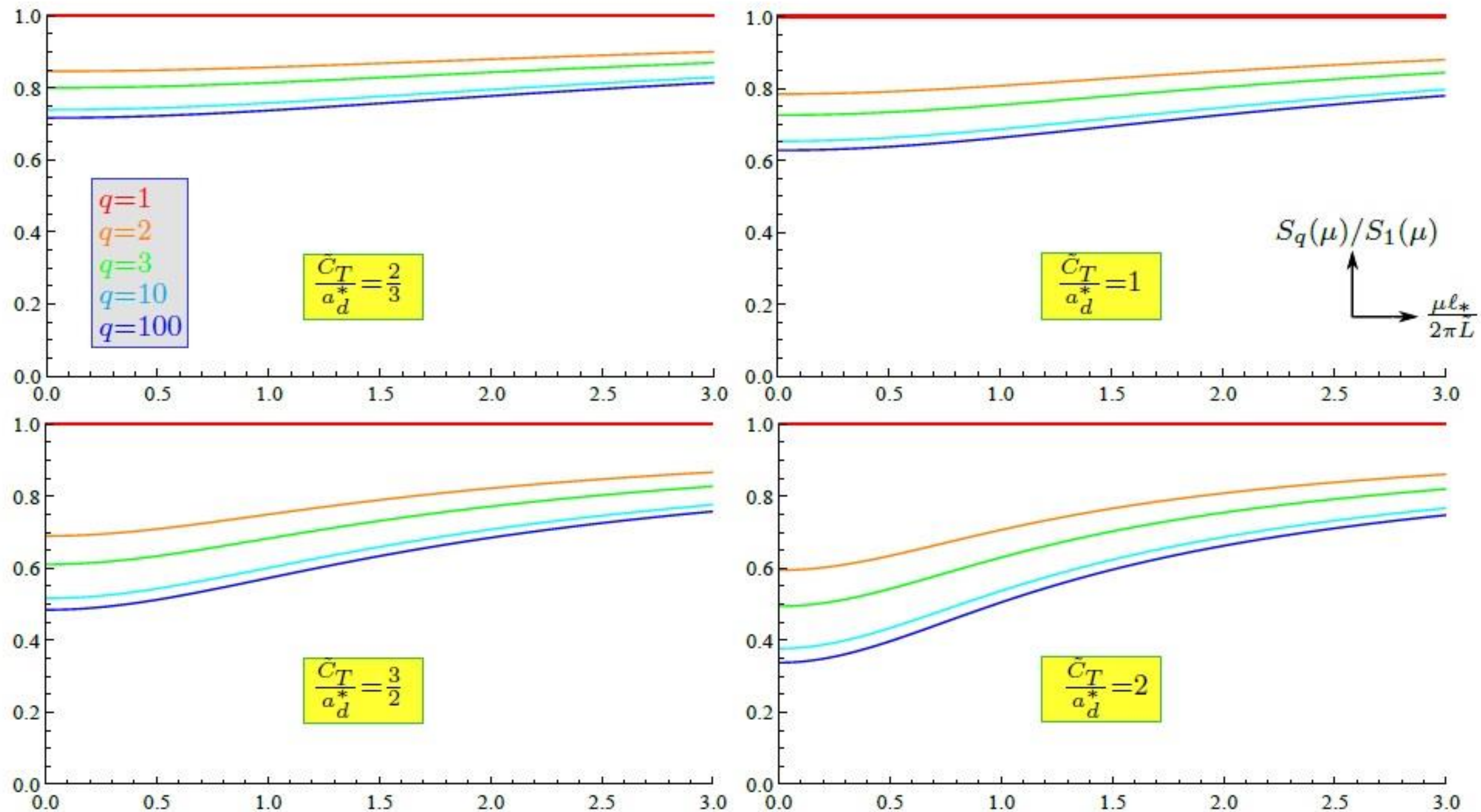
Holographic Calculation of Rényi Entropy-Dependence on μ

The limit of S_q for large chemical potentials depends only on the product $f_\infty \mu^2$ and it is independent of q. Specifically

$$\lim_{\mu\to\infty} S_q(\mu,\lambda) \sim V_{H^{d-1}}\, 2\pi \left(\frac{(d-2)\sqrt{f_\infty}}{\sqrt{2d(d-1)}} \frac{\mu \ell_*}{2\pi \ell_P} \right)^{d-1}$$

Thus $\lim_{\mu\to\infty} \frac{S_q(\mu,\lambda)}{S_{EE}(\mu,\lambda)} = 1$ as shown in next figure:

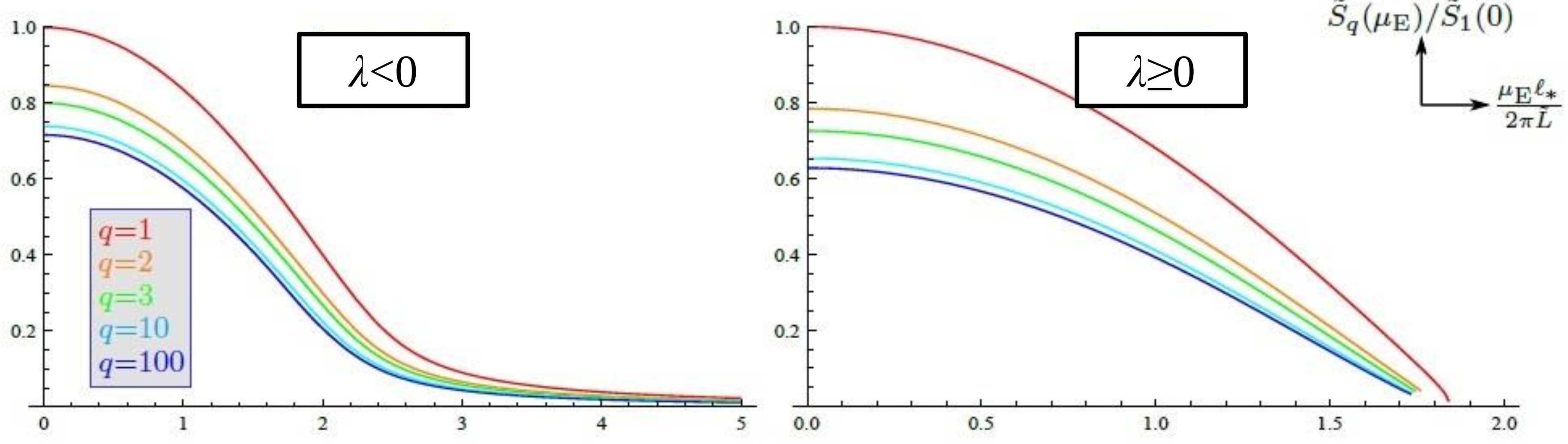
Holographic Calculation of Rényi Entropy-Dependence on μ



Holographic Calculation of Rényi Entropy-Dependence on μ_E

The dependence of S_q on an imaginary chemical potential μ_E presents an interesting discrimination between $\lambda<0$ and $\lambda\geq0$

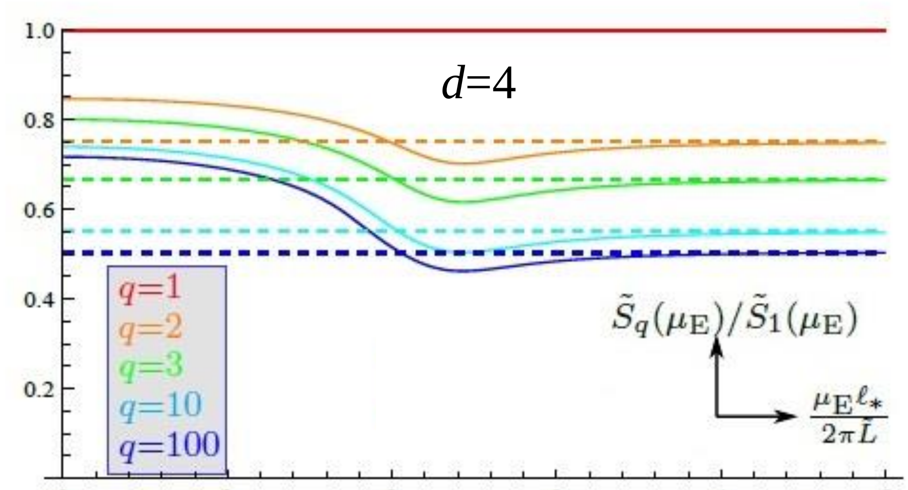
- for $\lambda < 0$, S_q is well defined for arbitrarily high μ_E
- for $\lambda \geq 0$, there is a branch cut related to a naked singularity

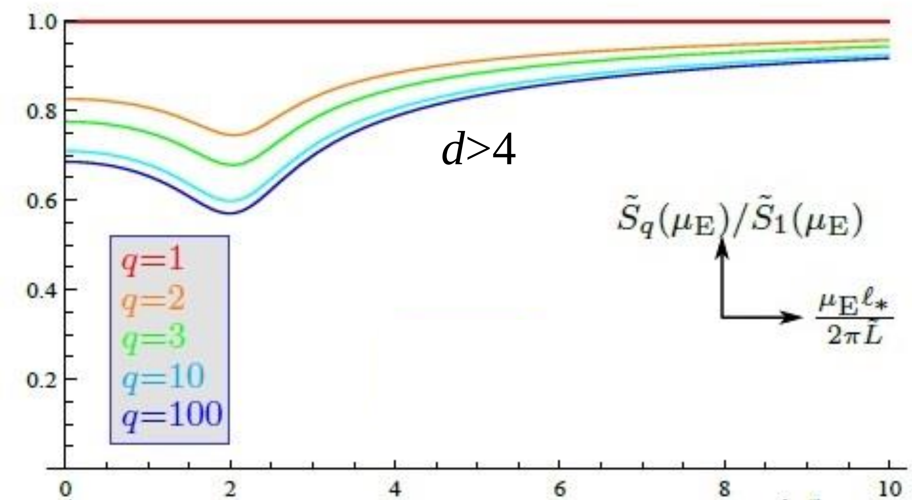


Holographic Calculation of Rényi Entropy-Dependence on μ_E

Moreover for $\lambda<0$, the asymptotic behavior of S_q for large μ_E presents a discrimination between d=4 and d>4

$$\lim_{\mu_E \to \infty} \frac{S_q(i\mu_E, \lambda)}{S_{EE}(i\mu_E, \lambda)} = \begin{cases} \dfrac{q+1}{2q}, & d = 4 \\ 1, & d > 4 \end{cases}$$



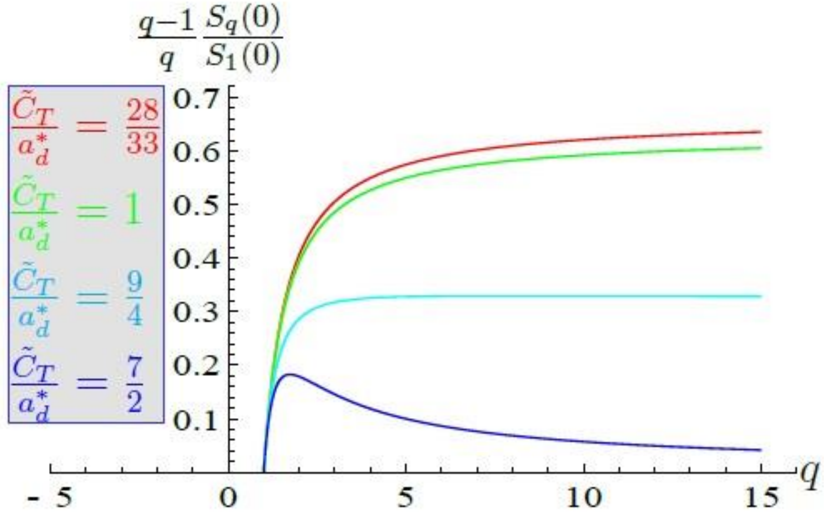


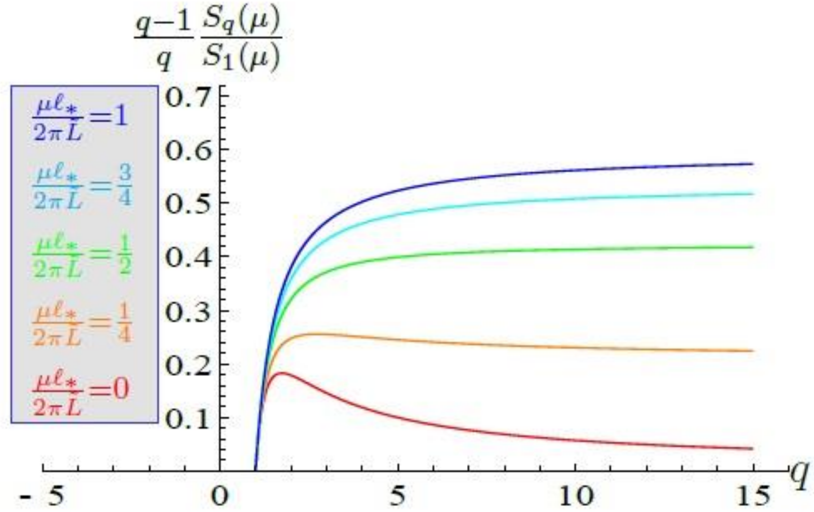
Holographic Calculation of Rényi Entropy-Rényi Inequality violation

Rényi entropy obeys by definition a set of inequalities including

$$\frac{\partial}{\partial q}\left(\frac{q-1}{q}S_q\right)\geq 0$$

However…





Holographic Calculation of Rényi Entropy-Rényi Inequality violation

The basic formula that relates Rényi entropy with black hole entropy

$$S_q(\mu) = \frac{q}{q-1}\frac{1}{T_0}\int_{\frac{T_0}{q}}^{T_0} dT\, S(T,\mu)$$

implies

$$\frac{\partial}{\partial q}\left(\frac{q-1}{q}S_q(\mu)\right) = \frac{1}{q^2}S\left(\frac{T_0}{q},\mu\right)$$

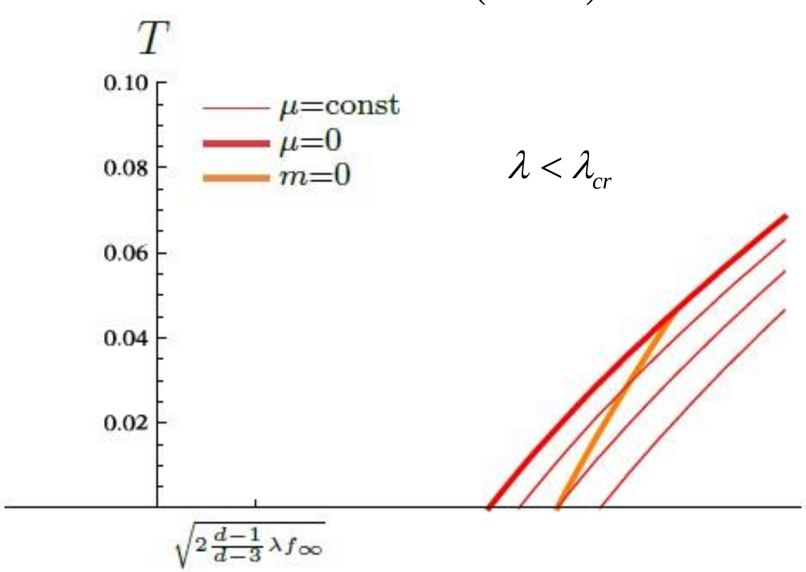
Thus, the observed inequality is occurring when negative entropy black holes are present[1]

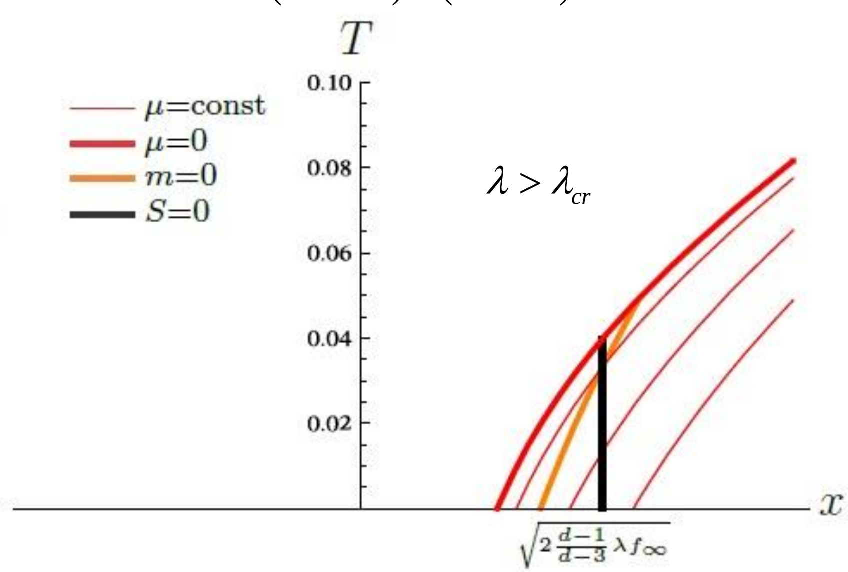
[1]L. -Y. Hung et al, Holographic Calculations of Renyi Entropy, JHEP 1112, 047 (2011) [hep-th/1110.1084]

Holographic Calculation of Rényi Entropy-Rényi Inequality violation

Higher derivative corrections to the area law, allow for negative entropy BHs when[1,2]

$$\lambda > \lambda_{cr} \equiv \frac{(d-3)(d^2+d-8)}{3d(d-1)^2} \quad \text{and} \quad \left(\frac{\mu \ell_*}{2\pi \tilde{L}}\right)^2 < \frac{d(d-1)^2}{(d-2)^2(d-3)}(\lambda - \lambda_{cr})$$

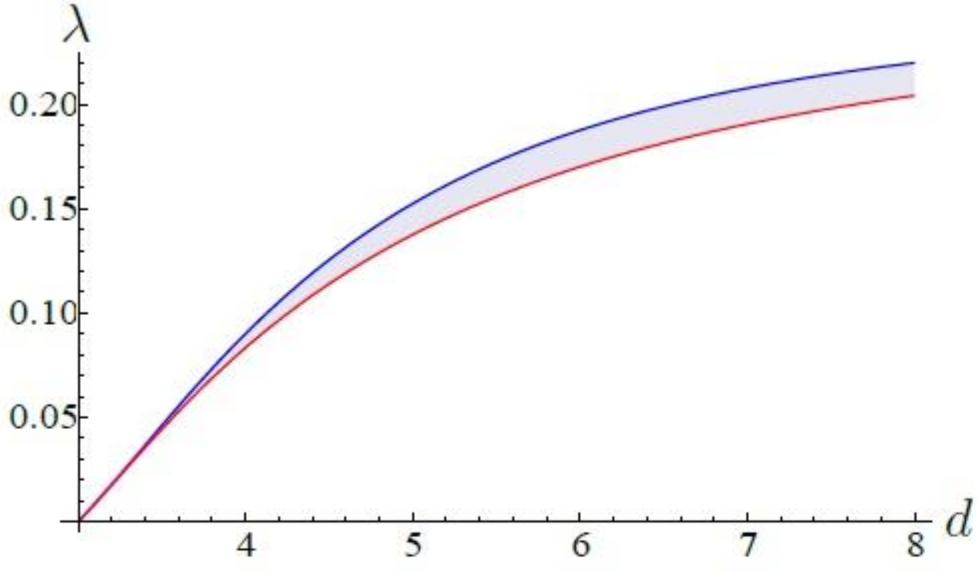




[1]M. Cvetic, et al, [hep-th/0112045]
[2]G. Pastras et al, [hep-th/0807.3478]

Holographic Calculation of Rényi Entropy-Rényi Inequality violation

The constraint imposed in GB coupling by Rényi inequality is stricter that the corresponding constraint by causality, but really close[1]



[1]G. Pastras et al, Charged Renyi Entropies in CFTs with Einstein-GB Holographic Duals, [hep-th/1404.1309]

Discussion

$$S_{AdS\,BH} < 0 \quad \text{and} \quad \frac{\partial}{\partial q}\left(\frac{q-1}{q} S_q\right) < 0$$

are an artifact of

$\mathcal{L}_{R^2}$ competing with $\mathcal{L}_R$

In full theory $\mathcal{L}_R + \mathcal{L}_{R^2} + \mathcal{L}_{R^3} + \ldots$ $\qquad \frac{\partial}{\partial q}\left(\frac{q-1}{q} S_q\right) \overset{?}{<} 0$

Discussion

$$\frac{\partial}{\partial q}\left(\frac{q-1}{q}S_q\right) \geq 0 \longrightarrow \lambda_{GB} < \lambda_{\text{critical entropy}}$$

causality $\longrightarrow \lambda_{GB} < \lambda_{\text{critical causality}}$

Why $\lambda_{\text{critical entropy}} \simeq \lambda_{\text{critical causality}}$?

In full theory $\mathcal{L}_R + \mathcal{L}_{R^2} + \mathcal{L}_{R^3} + \ldots \quad \lambda_{\text{critical entropy}} \overset{?}{=} \lambda_{\text{critical causality}}$

or...

Discussion

$$\lambda_{GB} > \lambda_{\text{critical entropy}} \quad : \quad S_{\text{AdS BH}} < 0 \quad \text{but} \quad S_{\text{dS BH}} > 0$$

$$\text{AdS BH} \xleftrightarrow{\text{phase transition } \mathbf{???}} \text{dS BH}^{1}$$

If yes, then $\lambda_{GB} > \lambda_{\text{critical entropy}}$ has information about **dS/CFT**?

[1]M. Cvetic, et al, Black Hole Thermodynamics and Negative Entropy in dS and AdS Einstein-GB Gravity, Nucl. Phys. B 628, 295 (2002) [hep-th/0112045]

Discussion

AdS BHs with **spherical** horizons $\xrightarrow{\textbf{?}}$ Entanglement Entropy

Richer phase structure (HP phase transition)

but

not a simple entangling surface (time dependent?)

Thank you