Minimal Model Holography

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Quantum Symmetries and Strings
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Based mainly on work with Rajesh Gopakumar,

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AdS / CFT duality

Much recent progress in string theory has been related to AdS/CFT duality

[Maldacena '97, ...]

superstrings on $\mathrm{AdS}_5 \times S^5$ = SU(N) super Yang-Mills theory in 4 dimensions

4d non-abelian gauge theory similar to that appearing in the standard model of particle physics.

Relation of parameters

The relation between the parameters of the two theories is

$$\left(\frac{R}{l_{\mathrm{Pl}}}\right)^4 = N \qquad g_{\mathrm{string}} = g_{\mathrm{YM}}^2 \qquad \left(\frac{R}{l_{\mathrm{s}}}\right)^4 = g_{\mathrm{YM}}^2 N = \lambda$$

AdS radius in Planck units

AdS radius in string units

‘t Hooft parameter

Strong weak duality

For example, in the large N limit of gauge theory at large ‘t Hooft coupling

$$\left(\frac{R}{l_{\mathrm{Pl}}}\right)^4 = N \qquad g_{\mathrm{string}} = g^2_{\mathrm{YM}} \qquad \left(\frac{R}{l_{\mathrm{s}}}\right)^4 = g^2_{\mathrm{YM}} N = \lambda$$

large small large

Supergravity (point particle) approximation is good for AdS description.

AdS/CFT duality

This is interesting since it gives insights into strongly coupled gauge theories using supergravity methods.

Many applications and insights:

- anomalous dimensions in N=4 SYM
  [Minahan,Zarembo,Beisert,Staudacher,...]
- structural insights into amplitudes
  [Witten,Cazacho,Arkani-Hamed,Alday,Maldacena, Korchemsky,Drummond,Sokatchev,...]
- quark gluon plasma
  [Liu,Rajagopal,Wiedemann,Gubser,...]
- quantum critical systems
  [Hartnoll,Herzog,Horowitz,Kachru,Sachdev,Son,...]
- ...

Conceptual understanding

However, at present, we are far away from a conceptual understanding of why the duality works, and what ingredients are crucial for it, e.g. whether it requires

supersymmetry
integrability
...

This is obviously an important question since in many applications these features are absent.

Weakly coupled gauge theory

In order to make progress in this direction analyse another corner of AdS/CFT: consider case where gauge theory is weakly coupled

$$\left(\frac{R}{l_{\rm Pl}}\right)^4 = N \qquad g_{\rm string} = g^2_{\rm YM} \qquad \left(\frac{R}{l_{\rm s}}\right)^4 = g^2_{\rm YM} N = \lambda$$

large

small

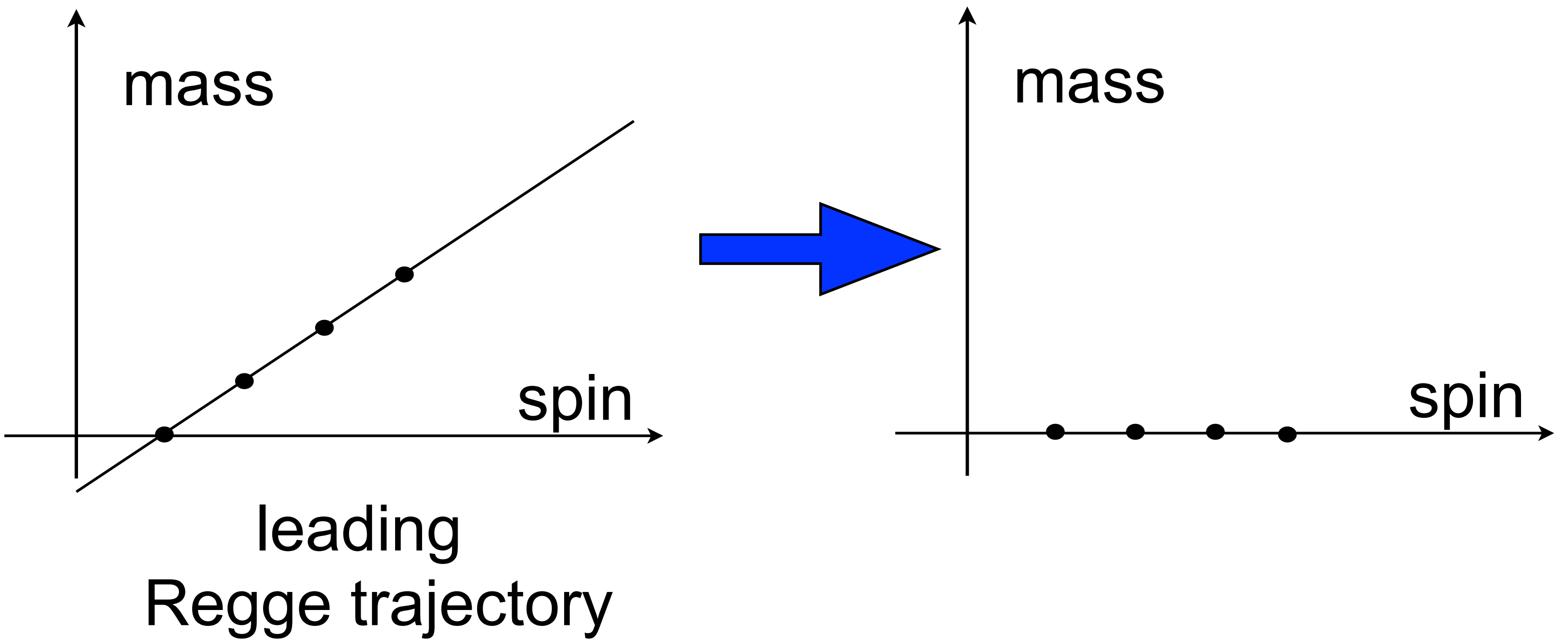
$$l_{\rm s} \to \infty$$

`tensionless strings'

[Sundborg '01] [Witten '01] [Sezgin,Sundell '01]

Tensionless limit

In tensionless limit all string excitations become massless:



Higher spin theory

Resulting theory has an infinite number of massless higher spin fields, which generate a very large gauge symmetry.

$\rightarrow$ effective description in terms of Vasiliev Higher Spin Theory.

maximally unbroken phase of string theory

Leading Regge trajectory

On the dual CFT side, the traces of bilinears of elementary Yang-Mills fields form closed subsector in free theory.

This subsector is believed to correspond to the leading Regge trajectory from the string point of view:

vector-like HS -- CFT duality

State of the Art

In the past this idea was taken as a general motivation to consider dualities relating

Vasiliev HS theory on AdS	$\longleftrightarrow$	vector-like weakly coupled CFT

However, recently interesting progress about how these dualities fit actually into stringy AdS/CFT correspondence has been made....

[Chang, Minwalla, Sharma, Yin '12]
[MRG, Gopakumar '14]

Higher spin theories

Higher spin (HS) theories have a long history.

- Fronsdal (1978): free HS theory in flat space with gauge symmetry

$$\delta\varphi_{\mu_1\cdots\mu_s} = \partial_{\mu_1}\,\epsilon_{\mu_2\cdots\mu_s} + \text{symmetrisations}$$

doubly traceless

traceless

Generalisation to AdS straightforward: $\partial \to \nabla$

Higher spin theories

- Fradkin & Vasiliev (1987): interacting HS theory on AdS (or dS) background.

    - involves infinitely many higher spin fields

    - cosmological constant allows for higher derivative interactions

(Evades various no-go theorems a la Coleman-Mandula.)

Higher spin CFT duality

First concrete proposal for HS - CFT duality

[Klebanov-Polyakov '02]
[Sezgin-Sundell '02]

higher spin theory on AdS4	$\longleftrightarrow$	3d O(N) vector model in large N limit

Different versions: vector model fields bosons or fermions;
free or interacting fixed point.

More recently: generalisation to family of parity-violating theories.

[Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin '11]
[Aharony, Gur-Ari, Yacoby '11]

Checks of the proposal

During the last few years impressive checks of the proposal have been performed, in particular

3-point functions of HS fields on AdS4

have been matched to

3-point functions of HS currents in O(N) model to leading order in 1/N.

[Giombi, Yin ’09-’10]

Furthermore, the symmetries have been identified.

[Giombi, Prakash, Yin ’11], [Giombi, Yin ’11]
[Maldacena, Zhiboedov ’11-’12]

3d proposal

More recently also lower dimensional version was found:

[MRG,Gopakumar ’10]

AdS3:
higher spin theory
with a complex
scalar of mass M

$\longleftrightarrow$

2d CFT:
$\mathcal{W}_{N,k}$ minimal models
in large N ‘t Hooft limit
with coupling λ

where $\lambda = \dfrac{N}{N+k}$ and $M^2 = -(1-\lambda^2)$

Outline

This version of the duality is bosonic, but can nevertheless be tested in quite some detail. In particular, we can match

- quantum symmetries
- spectrum

In the second lecture I will also explain relation of these dualities to stringy AdS3 -- CFT2 duality.

The HS theory on AdS3

The AdS3 HS theory can be described very simply.

Recall that pure gravity in AdS3: Chern-Simons theory based on

$$\mathfrak{sl}(2,\mathbb{R})$$

[Achucarro, Townsend '86]
[Witten '88]

Higher spin description: replace

[Prokushkin, Vasiliev '98]

[one spin field for each spin $s = 2, 3, \ldots$]

$$\mathfrak{sl}(2,\mathbb{R}) \rightarrow \mathrm{hs}[\lambda] \cong \mathfrak{sl}(\lambda)$$

where $\mathrm{hs}[\lambda] \oplus \mathbb{C} \cong \dfrac{U(\mathfrak{sl}(2))}{\langle C_2 - \frac{1}{4}(\lambda^2 - 1)\mathbf{1}\rangle}$

[Bordemann et.al. '98]
[Bergshoeff et.al. '90]
[Pope, Romans, Shen '90]

Higher spin algebra

Generators of $\mathrm{hs}[\lambda]$:

$$V_n^s \quad \text{with } |n| < s \ , \quad s = 2, 3, \ldots$$

\`wedge algebra'

For these higher spin theories asymptotic symmetry algebra can be determined following Brown & Henneaux, leading to classical

$$\mathcal{W}_\infty[\lambda] \quad \text{algebra}$$

[Henneaux & Rey '10]
[Campoleoni et al '10]
[MRG, Hartman '11]

Asymptotic symmetry algebra

Asymptotic symmetry algebra extends hs algebra
`beyond the wedge':

pure gravity: $L_0, L_{\pm 1} \;\longrightarrow\; L_n \;,\; n \in \mathbb{Z}$

$\mathfrak{sl}(2, \mathbb{R})$ (Virasoro)

higher spin: $\mathrm{hs}[\lambda] \;\longrightarrow\; \mathcal{W}_\infty[\lambda]$

[Figueroa-O'Farrill et.al. '92]

generated by V_n^s

$s = 2, \ldots, \infty, \;\; n \in \mathbb{Z}$

Dual CFT

By the usual arguments, dual CFT should therefore have

$$\mathcal{W}_\infty[\lambda] \text{ symmetry.}$$

Basic idea:

$$\mathcal{W}_\infty[\lambda] = \lim_{N\to\infty} \mathcal{W}_{N,k} \qquad \text{with} \qquad \lambda = \frac{N}{N+k} \ .$$

't Hooft limit of 2d CFT!

The minimal models

The minimal model CFTs are the cosets

$$\mathcal{W}_{N,k}: \quad \frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}}$$

[Bais et.al. '88]
[Bouwknegt, Schoutens '92]

e.g. Ising model (N=2, k=1)
tricritical Ising (N=2, k=2)
3-state Potts (N=3,k=1),..

with central charge

$$c_N(k) = (N-1)\Big[1 - \frac{N(N+1)}{(N+k)(N+k+1)}\Big] \ .$$

General N: higher spin analogue of Virasoro minimal models. [Spin fields of spin s=2,3,..,N.]

Relation of symmetries

Note that asymptotic symmetry algebra

$$\mathcal{W}_\infty[\lambda]$$

is a classical (commutative) Poisson algebra.

In order to understand relation to minimal model W-algebras, need to understand how to quantise it.

Quantisation

Quantisation is quite subtle since the Poisson algebra is non-linear --- cannot just replace Poisson brackets by commutators without violating Jacobi identities...

However, there is essentially a unique way of defining a consistent quantum W-algebra (whose classical limit reduces to Poisson algebra).

[MRG, Gopakumar '12]
[Blumenhagen, et.al. '94]
[Hornfeck '92-'93]

Quantum symmetry

There are two steps to this argument. To illustrate them consider an example. Naive quantisation of classical algebra leads to

[MRG, Gopakumar '12]

$$
\begin{aligned}
[W^3_m, W^3_n] &= 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \\
&+ \frac{8N_3}{c}(m-n)\,(LL)_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}
\end{aligned}
$$

spin-3 field

non-linear term

Jacobi identity

$$[W_m^3, W_n^3] = 2(m-n)W_{m+n}^4 + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n}$$
$$+ \frac{8N_3}{c}(m-n)\,(LL)_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}$$

spin-3 field

non-linear term

Jacobi identity

$$[W^3_m, W^3_n] = 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{8N_3}{c}(m-n)(LL)_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}$$

Jacobi identity determines quantum correction

$$[W^3_m, W^3_n] = 2(m-n)W^4_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{8N_3}{c+\frac{22}{5}}(m-n)\Lambda^{(4)}_{m+n} + \frac{N_3 c}{144}m(m^2-1)(m^2-4)\delta_{m,-n}$$

where

$$\Lambda^{(4)}_n = \sum_p :L_{n-p}L_p: + \tfrac{1}{5}x_n L_n$$

Similar considerations apply for the other commutators.

Structure constants

The second step concerns structure constants. The fields can be rescaled so that

$$W^{(3)} \cdot W^{(3)} \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + \frac{32}{(5c+22)} \cdot \Lambda^{(4)} + 4 \cdot W^{(4)}$$

but then coupling constant

$$W^{(3)} \cdot W^{(4)} \sim C^{4}_{33} \cdot W^{(3)} + \cdots$$

characterises algebra. Classical analysis determines

[MRG, Hartman '11]

$$\left(C^{4}_{33}\right)^2 = \frac{64}{5} \frac{\lambda^2 - 9}{\lambda^2 - 4} + \mathcal{O}(\tfrac{1}{c}) \ .$$

Structure constants

Classical analysis determines

$$\left(C_{33}^{4}\right)^{2} = \frac{64}{5}\,\frac{\lambda^2 - 9}{\lambda^2 - 4} + \mathcal{O}\left(\tfrac{1}{c}\right)\,.$$

Structure constants

Classical analysis determines

$$\left(C_{33}^{4}\right)^{2} = \frac{64}{5}\,\frac{\lambda^2-9}{\lambda^2-4} + \mathcal{O}\left(\tfrac{1}{c}\right) \; .$$

Requirement that representation theory agrees for $\lambda = N$ with $\mathcal{W}_N$:

$$\left(C_{33}^{4}\right)^{2} = \frac{64\,(c+2)\,(\lambda-3)\,\big(c(\lambda+3)+2(4\lambda+3)(\lambda-1)\big)}{(5c+22)\,(\lambda-2)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)} \; .$$

[Note: $\mathrm{hs}[\lambda]\Big|_{\lambda=N} \cong sl(N,\mathbb{R})$ implies $\mathcal{W}_\infty[\lambda]|_{\lambda=N} = \mathcal{W}_N$.]

Explicit check

This formula has also been checked explicitly for the two special cases:

[Bergshoeff et.al '90]
[Bakas, Kiritsis '90]

$\lambda = 0:$ N complex fermions with u(1) coset giving c = N-1

$\lambda = 1:$ N complex bosons giving c=2N

[MRG, Jin, Li '13]

Higher Structure Constants

Similarly, higher structure constants can be determined

[Blumenhagen, et.al.'94]
[Hornfeck '92-93]

$$C_{33}^4 C_{44}^4 = \frac{48\big(c^2(\lambda^2-19)+3c(6\lambda^3-25\lambda^2+15)+2(\lambda-1)(6\lambda^2-41\lambda-41)\big)}{(\lambda-2)(5c+22)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)}$$

$$(C_{34}^5)^2 = \frac{25(5c+22)(\lambda-4)\big(c(\lambda+4)+3(5\lambda+4)(\lambda-1)\big)}{(7c+114)(\lambda-2)\,\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)}$$

$$C_{45}^5 = \frac{15}{8(\lambda-3)(c+2)(114+7c)\big(c(\mu+3)+2(4\lambda+3)(\lambda-1)\big)} C_{33}^4 \times \Big[c^3(3\lambda^2-97)+c^2(94\lambda^3-467\lambda^2-483)+c(856\lambda^3-5192\lambda^2+4120) + 216\lambda^3-6972\lambda^2+6756\Big]\,.$$

Higher Structure Constants

Actually, can rewrite all of them more simply as

[MRG, Gopakumar '12]

$$C^4_{44} = \frac{9(c+3)}{4(c+2)}\,\gamma - \frac{96(c+10)}{(5c+22)}\,\gamma^{-1}$$

$$(C^5_{34})^2 = \frac{75(c+7)(5c+22)}{16(c+2)(7c+114)}\,\gamma^2 - 25$$

$$C^5_{45} = \frac{15\,(17c+126)(c+7)}{8\,(7c+114)(c+2)}\,\gamma - 240\,\frac{(c+10)}{(5c+22)}\,\gamma^{-1}$$

where $$\gamma^2 \equiv \left(C^4_{33}\right)^2$$

These structure constants (and probably all) are actually determined in terms of γ^2 by Jacobi identity.

[Candu, MRG, Kelm, Vollenweider, unpublished]

Quantum algebra

Thus full quantum algebra is characterised by two free parameters: central charge c and

[MRG, Gopakumar '12]

$$\gamma^2 \text{ (rather than } \lambda)$$

But

$$(C_{33}^4)^2 \equiv \gamma^2 = \frac{64(c+2)(\lambda-3)\big(c(\lambda+3)+2(4\lambda+3)(\lambda-1)\big)}{(5c+22)(\lambda-2)\big(c(\lambda+2)+(3\lambda+2)(\lambda-1)\big)} \, .$$

Thus there are three roots that lead to the same algebra:

$$\mathcal{W}_\infty[\lambda_1] \cong \mathcal{W}_\infty[\lambda_2] \cong \mathcal{W}_\infty[\lambda_3] \qquad \text{at fixed } c$$

`Triality'

Triality

In particular,

$$\mathcal{W}_\infty[N] \cong \mathcal{W}_\infty[\tfrac{N}{N+k}] \cong \mathcal{W}_\infty[-\tfrac{N}{N+k+1}] \quad \text{at } c = c_N(k)$$

minimal model

asymptotic symmetry
algebra of hs theory

This is even true at finite N and k, not just in the
‘t Hooft limit!

This triality generalises level-rank duality of coset models
of [Kuniba, Nakanishi, Suzuki ’91] and [Altschuler, Bauer, Saleur ’90].

Symmetries

So the symmetries give strong evidence for the duality

HS on AdS3	$\longleftrightarrow$	2d CFT with
=CS with		$\mathcal{W}_\infty[\lambda]$
$\mathrm{hs}[\lambda]$		symmetry
		=
		minimal models

Semiclassical limit: take c large, e.g. via 't Hooft limit.

Spectrum

Higher spin fields themselves correspond only to the vacuum representation of the W-algebra!

To see this, calculate partition function of massless spin s field on thermal AdS3

[MRG, Gopakumar, Saha '10]

$$Z^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \ . \qquad q = \exp(-\frac{1}{k_{\rm B}T})$$

Generalisation of [Giombi, Maloney, Yin '08] to higher spin, using techniques developed in [David, MRG, Gopakumar '09].

1-loop partition function

The complete higher spin theory therefore contributes

$$Z_{\text{hs}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \, .$$

MacMahon function!

This reproduces precisely contribution of CFT vacuum representation in ‘t Hooft limit.

Representations

This is only a small part of full CFT: it also has the representations labelled by

$$(\rho, \mu; \nu)$$

rep of $\mathfrak{su}(N)_k$ $\mathfrak{su}(N)_1$ $\mathfrak{su}(N)_{k+1}$

Compatibility constraint: $\rho + \mu - \nu \in \Lambda_R(\mathfrak{su}(N))$

fixes μ uniquely: label representations by $(\rho; \nu)$.

Simple representations

Simplest reps that generate all W-algebra reps upon fusion: (0;f) and (f;0) (& conjugates).

't Hooft limit: $\boxed{h(\mathrm{f};0)} = \frac{1}{2}(1+\lambda)$ $\quad$ $\boxed{h(0;\mathrm{f})} = \frac{1}{2}(1-\lambda)$

semiclassical: (for fixed N) $\boxed{h(\mathrm{f};0)} = \frac{1}{2}(1-N)$ $\quad$ $\boxed{h(0;\mathrm{f})} = -\frac{c}{2N^2}$

dual to perturbative scalar

non-perturbative

Proposal

Contribution from all representations of the form (*;0) is accounted for by adding to the hs theory a complex scalar field of the mass

[MRG, Gopakumar '10]

$$-1 \leq M^2 \leq 0 \quad \text{with} \quad M^2 = -(1-\lambda^2) \ .$$

[Compatible with hs symmetry since hs theory has massive scalar multiplet with this mass.]

[Prokushkin, Vasiliev, '98]

Corresponding conformal dimension then

$$M^2 = \Delta(\Delta - 2) \ \Rightarrow \ \Delta = 1 + \lambda \ .$$

(standard quantisation)

Checks of proposal

Main evidence from 1-loop calculation:

Contribution of single real scalar to thermal partition function is

[Giombi, Maloney, Yin '08]

$$Z^{(1)}_{\text{scalar}} = \prod_{l=0,l'=0}^{\infty} \frac{1}{(1 - q^{h+l}\bar{q}^{h+l'})} ,$$

where

$$h = \frac{1}{2}\Delta = \frac{1}{2}(1+\lambda) .$$

Total 1-loop partition function

The total perturbative 1-loop partition function of our AdS theory is then:

$$Z^{(1)}_{\mathrm{pert}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{|1-q^n|^2} \times \prod_{l,l'=0}^{\infty} \frac{1}{(1-q^{h+l}\bar{q}^{h+l'})^2}$$

higher spin fields

scalar fields

Total 1-loop partition function

The total perturbative 1-loop partition function of our AdS theory is then:

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$\mathcal{W}_N$ modes

$$(\mathrm{f}; 0)^{\otimes r_1} \otimes (\bar{\mathrm{f}}; 0)^{\otimes r_2}$$

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$\mathcal{W}_N$ modes

$$(\text{f};0)^{\otimes r_1}\otimes(\bar{\text{f}};0)^{\otimes r_2}$$

We have shown analytically that this agrees exactly with CFT partition function of (*;0) representations in 't Hooft limit!

[MRG,Gopakumar, '10]
[MRG,Gopakumar,Hartman,Raju, '11]

Lowest orders

For example, for single scalar first non-trivial terms (including higher spin mode contributions) are

$$
\begin{aligned}
Z^{(1)} = \;\; & q^h \bar{q}^h \left(1 + q + 2q^2 + 4q^3 + 8q^4 + \cdots\right)\left(1 + \bar{q} + 2\bar{q}^2 + 4\bar{q}^3 + 8\bar{q}^4 + \cdots\right) \\
& + q^{2h}\bar{q}^{2h}\left(1 + q + 3q^2 + 5q^3 + \cdots\right)\left(1 + \bar{q} + 3\bar{q}^2 + 5\bar{q}^3 + \cdots\right) \\
& + q^{2h+1}\bar{q}^{2h+1}\left(1 + q + 3q^2 + \cdots\right)\left(1 + \bar{q} + 3\bar{q}^2 + \cdots\right) + \cdots \; .
\end{aligned}
$$

This is of the form

$$
Z^{(1)} = \chi_{h_1}(q)\chi_{h_1}(\bar{q}) + \chi_{h_2}(q)\chi_{h_2}(\bar{q}) + \chi_{h_3}(q)\chi_{h_3}(\bar{q}) + \cdots
$$

characters of $\mathcal{W}_N$ reps

Lowest orders

$$Z^{(1)} = \quad q^h \bar{q}^h \Big(1 + q + 2q^2 + 4q^3 + 8q^4 + \cdots\Big)\Big(1 + \bar{q} + 2\bar{q}^2 + 4\bar{q}^3 + 8\bar{q}^4 + \cdots\Big)$$
$$+q^{2h}\bar{q}^{2h}\Big(1 + q + 3q^2 + 5q^3 + \cdots\Big)\Big(1 + \bar{q} + 3\bar{q}^2 + 5\bar{q}^3 + \cdots\Big)$$
$$+q^{2h+1}\bar{q}^{2h+1}\Big(1 + q + 3q^2 + \cdots\Big)\Big(1 + \bar{q} + 3\bar{q}^2 + \cdots\Big) + \cdots \ .$$

with

$$\chi_{h_1}(q) = q^h\Big(1 + q + 2q^2 + 4q^3 + 8q^4 + \cdots\Big) = \chi(0;\mathrm{f})$$
$$\chi_{h_2}(q) = q^{2h}\Big(1 + q + 3q^2 + 5q^3 + \cdots\Big) = \chi(0;[0,1,0^{N-3}])$$
$$\chi_{h_3}(q) = q^{2h+1}\Big(1 + q + 3q^2 + \cdots\Big) = \chi(0;[2,0^{N-2}]) \ ,$$

calculated from first principles in CFT!

Non-perturbative states

The remaining states, i.e. those of the form

$$(*;\nu) \qquad \text{with } \nu \neq 0$$

seem to correspond to conical defect solutions (possibly dressed with perturbative excitations).

[Castro, Gopakumar, Gutperle, Raeymaekers ’11]
[MRG, Gopakumar ’12]
[Perlmutter, Prochazka, Raeymaekers ’12]

Summary of First Lecture

Explained proposal of bosonic hs -- CFT duality and evidence in favour of it:

- matching of symmetries
- agreement of partition functions

Next step: identify as a consistent subsector of stringy AdS -- CFT duality:

$\longrightarrow$ making duality susy.... (tomorrow)

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1011.2986, 1205.2472, **1305.4181**, **1406.6103**

Summary from last time

Have duality of symmetries:

HS on AdS3

=CS with

$\mathrm{hs}[\lambda]$

$\longleftrightarrow$

2d CFT with

$\mathcal{W}_\infty[\lambda]$

symmetry

=

minimal models

Furthermore, massive complex scalar field is necessary in order to account for (half of the) non-trivial CFT representations.

$\mathcal{N} = 4$ Supersymmetry

In order to relate these hs dualities to stringy dualities, consider situation with $\mathcal{N} = 4$ supersymmetry.

Best studied example:

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times M_4 \qquad \text{with } M_4 = \mathbb{T}^4 \text{ or } M_4 = \mathrm{K3}$$

Then dual CFT is believed to have small $\mathcal{N} = 4$ superconformal symmetry.

The small $\mathcal{N} = 4$ algebra

The small superconformal algebra is generated by an affine su(2) algebra, as well as 4 supercharges:

$$
\begin{aligned}
[A^i_m, A^j_n] &= \tfrac{k}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A^l_{m+n} \\
[A^i_m, G^a_r] &= \mathrm{i}\, \alpha^i_{ab}\, G^b_{m+r} \\
\{G^a_r, G^b_s\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\quad + 4\, (r-s)\, \mathrm{i}\, \alpha^i_{ab}\, A^i_{r+s} \ .
\end{aligned}
$$

The central charge of the Virasoro algebra is then c=6k.

Dual CFT

At one point in moduli space, the dual CFT of string theory is described by the symmetric orbifold theory

$$\mathrm{Sym}_N(\mathbb{T}^4) \equiv \left(\mathbb{T}^{4\otimes(N)}\right)/S_N$$

This CFT is essentially free --- could this be the

CFT dual in the tensionless limit?

Small $\mathcal{N} = 4$ Supersymmetry

Unfortunately, no coset CFTs with small $\mathcal{N} = 4$ (that would naturally appear in a hs--CFT duality) are known....

$$\boxed{\text{`tensionless'}\ \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4}$$

hs theory

$\updownarrow$

coset dual ??

string theory

$\updownarrow$

symmetric orbifold

$\mathrm{Sym}_N(\mathbb{T}^4)$

Large $\mathcal{N}=4$ Supersymmetry

We shall therefore first explore the situation with large $\mathcal{N}=4$ superconformal symmetry --- this is the symmetry algebra of the CFT dual to string theory on

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$$

Towards String Theory

Indeed, dual CFT is expected to have

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$$

Large $\mathcal{N} = 4$

$$\mathrm{Vir} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$$

with 4 supercharges

[Boonstra, Peeters, Skenderis '98; Elitzur, Feinerman, Giveon, Tsabar '99; de Boer, Pasquinucci, Skenderis '99; Gukov, Martinec, Moore, Strominger '04; ...]

The large $\mathcal{N} = 4$ algebra

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\; \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \\
[U_m, G_r^a] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, G_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, G_s^b\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{G_r^a, G_s^b\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\quad + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\Big) \ .
\end{aligned}
$$

The large $\mathcal{N}=4$ algebra

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \\
[U_m, G_r^a] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, G_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, G_s^b\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{G_r^a, G_s^b\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\qquad + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\Big)\ .
\end{aligned}
$$

U(1) current

The large $\mathcal{N} = 4$ algebra

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \\
[U_m, G_r^a] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, G_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, G_s^b\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{G_r^a, G_s^b\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\qquad + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\Big)\ .
\end{aligned}
$$

4 free fermions

The large $\mathcal{N} = 4$ algebra

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \qquad \longleftarrow \qquad 2\ \mathfrak{su}(2)'\mathrm{s} \\
[U_m, G_r^a] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, G_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, G_s^b\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{G_r^a, G_s^b\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\quad + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\Big)\ .
\end{aligned}
$$

The large $\mathcal{N}=4$ algebra

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \\
[U_m, \boxed{G_r^a}] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, \boxed{G_r^a}] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, \boxed{G_{m+r}^b} \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, \boxed{G_s^b}\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{\boxed{G_r^a}, \boxed{G_s^b}\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\quad + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} \Big)\ .
\end{aligned}
$$

4 supercharges

parameter

Large $\mathcal{N} = 4$

Since there are two current algebras, the algebra is actually characterised by two parameters: in addition to the central charge

$$c = \frac{6k^+k^-}{k^+ + k^-}$$

[Sevrin, Troost, Van Proeyen, Schoutens, Spindel, Theodoridis '88-'90; Goddard, Schwimmer '88]

have parameter

$$\gamma = \frac{k^-}{k^+ + k^-} \ .$$

($k^\pm$: size of the two S3s.)

Dual CFT

In this case the situation is reversed: the dual CFT of this string background is not known.

[Gukov, Martinec, Moore, Strominger '04]
see however [Tong '14]

However, one family of $\mathcal{N} = 4$ coset CFTs is known: it is based on the Wolf symmetric spaces

$$\frac{\mathfrak{su}(N+2)^{(1)}_{k+N+2}}{\mathfrak{su}(N)^{(1)}_{k+N+2} \oplus \mathfrak{u}(1)^{(1)}_{\kappa}} \oplus \mathfrak{u}(1)^{(1)}_{\kappa} \cong \frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N+4)_1}{\mathfrak{su}(N)_{k+2} \oplus \mathfrak{u}(1)_{\kappa}} \oplus \mathfrak{u}(1)_{\kappa} \ .$$

[Sevrin, Troost, Van Proeyen, Schoutens, Spindel, Theodoridis '88-'90]

Dual CFT

They contain large $\mathcal{N}=4$ algebra with

$$\begin{aligned} k^+ &= k+1 \\ k^- &= N+1 \end{aligned} \quad \Rightarrow \quad \boxed{\gamma = \frac{N+1}{N+k+2}}$$

as well as certain higher spin currents.

Thus it is natural to look for a hs dual of these cosets in 't Hooft limit.

[MRG,Gopakumar '13]

Large $\mathcal{N} = 4$

$$\boxed{\text{`tensionless'}\ \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1}$$

hs theory

$\updownarrow$ $\longleftarrow$ try to find

Wolf space coset

string theory

$\updownarrow$

???

From the large $\mathcal{N} = 4$ .....

To begin with consider wedge algebra of the large $\mathcal{N} = 4$ algebra.

$$
\begin{aligned}
[U_m, U_n] &= \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n} \\
[A_m^{\pm,i}, Q_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, Q_s^b\} &= \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s} \\
[A_m^{\pm,i}, A_n^{\pm,j}] &= \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l} \\
[U_m, G_r^a] &= m\, Q_{m+r}^a \\
[A_m^{\pm,i}, G_r^a] &= \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b \\
\{Q_r^a, G_s^b\} &= 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s} \\
\{G_r^a, G_s^b\} &= \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} \\
&\quad + 4\,(r-s)\, \left(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\right) \ .
\end{aligned}
$$

free fermions (h=1/2) do not contribute

u(1) current: only zero mode --- central

...to the wedge algebra

Surviving wedge generators: $L_0 \ , L_{\pm 1} \ , \quad G^a_{\pm\frac{1}{2}} \ , \quad A_0^{\pm,i}$

$$
\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} \\
[L_m, G^a_r] &= (\tfrac{m}{2} - r) G^a_{m+r} \\
[A_0^{\pm,i}, G^a_r] &= \mathrm{i}\, \alpha^{\pm\, i}_{ab} G^b_r \\
[A_0^{\pm,i}, A_0^{\pm,j}] &= \mathrm{i}\, \epsilon^{ijl} A_0^{\pm,l} \\
\{G^a_r, G^b_s\} &= 2\delta^{ab}\, L_{r+s} \\
&\quad + 4\,(r-s) \left(\gamma\, \mathrm{i}\, \alpha^{+\, i}_{ab}\, A^{+,i}_{r+s} + (1-\gamma)\, \mathrm{i}\, \alpha^{-\, i}_{ab}\, A^{-,i}_{r+s} \right) \ .
\end{aligned}
$$

free parameter

related to

$$\gamma = \frac{\alpha}{1+\alpha}$$

Isomorphic to exceptional super Lie algebra

$$D(2,1|\alpha)$$

$\mathcal{N} = 2$ version

Thus we need to find hs algebra that contains this exceptional superalgebra...

Can be constructed naturally, starting from $\mathcal{N} = 2$ version of hs theory: [Prokushkin,Vasiliev '98]

$$sB[\mu] = \frac{U(\mathfrak{osp}(1|2))}{\langle C^{\mathfrak{osp}} - \frac{1}{4}\mu(\mu - 1)\mathbf{1}\rangle}$$

$$= \mathrm{shs}[\mu] \oplus \mathbb{C}$$

$\mathcal{N} = 2$ hs algebra

$\mathcal{N} = 4$ version

Introduce Chan-Paton factors:

$$sB_M[\mu] = sB[\mu] \otimes \mathrm{M}_M(\mathbb{C})$$
$$= \mathrm{shs}_M[\mu] \oplus \mathbb{C}$$

For M=2 the resulting Lie algebra contains

[MRG,Gopakumar '13]

$$\mathrm{shs}_2[\mu] \supset D(2,1|\alpha) \quad \text{where} \quad \gamma = \frac{\alpha}{1+\alpha} = \mu$$

(Different strategy from [Henneaux, Lucena Gomez, Park, Rey '12].)

Vasiliev theory

Since just added Chan-Paton factors, corresponding hs theory can be constructed by usual methods.

[Prokushkin,Vasiliev '98]
cf. [Chang, Minwalla, Sharma, Yin '12]

The spin content of the asymptotic symmetry algebra is then determined by the full spin content of

$$\mathrm{shs}_2[\mu]$$

$\mathcal{N} = 4$ hs algebra

This hs algebra contains 8 spin fields for each spin $s \geq 1$;
in terms of representations of superalgebra:

$$\mathrm{shs}_2[\mu] = D(2,1|\alpha) \oplus \bigoplus_{s=1}^{\infty} R^{(s)}$$

where

$$R^{(s)}: \quad \begin{array}{ll} s: & (\mathbf{1},\mathbf{1}) \\ s+\frac{1}{2}: & (\mathbf{2},\mathbf{2}) \\ s+1: & (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \\ s+\frac{3}{2}: & (\mathbf{2},\mathbf{2}) \\ s+2: & (\mathbf{1},\mathbf{1}) \; . \end{array}$$

reps w.r.t
$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

‘t Hooft limit

The corresponding asymptotic symmetry algebra then has the same generators as the $\mathcal{N}=4$ Wolf space cosets in the large (N,k) limit, where we take

$$\lambda \equiv \frac{N+1}{N+k+2} = \gamma = \mu \ .$$

In fact this identification is fixed by requiring the large $\mathcal{N}=4$ algebra to match, i.e. by identifying γ .

$\mathcal{N} = 4$ hs duality

This therefore suggests that we have the duality:

higher spin theory based on $\mathrm{shs}_2[\lambda]$ $\longleftrightarrow$ Wolf symmetric space cosets

with

$$\lambda = \frac{N+1}{N+k+2} = \gamma$$

[MRG,Gopakumar '13]

Further checks

More evidence has been recently found for this proposal: asymptotic symmetry algebra of hs theory matches precisely with the W-algebra of the Wolf space cosets in the ‘t Hooft limit. [MRG, Peng ’14]

In fact, the most general quantum W-algebra with this spin spectrum is uniquely determined by the levels of the two su(2)’s. [Beccaria, Candu, MRG ’14]

[This holds both for the `linear’ as well as the `non-linear’ version of the W-algebra.]

Spectrum

Also, the 1-loop thermal partition function of the hs theory with one complex scalar multiplet was successfully matched with the `perturbative half' of the CFT partition function of the Wolf space cosets in the 't Hooft limit.

$$Z_{\mathrm{pert}} = \sum_{\Lambda} \left|\chi_{(0;\Lambda)}\right|^2 \ .$$

[Creutzig, Hikida, Roenne '13]
[Candu, Vollenweider '13]

Basic situation

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$

hs theory based on

$\mathrm{shs}_2[\lambda]$

$\updownarrow$

Wolf space cosets

$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$

string theory

$\updownarrow$

symmetric orbifold

Contraction

While we cannot compare these two dualities directly, the large superconformal symmetry contracts to the small superconformal symmetry in the limit in which one of the two levels, say k^+, goes to infinity,

$$k^+ \to \infty \qquad\qquad \left(\lambda = \gamma = \frac{k^-}{k^+ + k^-} \to 0\right)$$

Indeed, this just describes the case where the radius of the corresponding 3-sphere goes to infinity, and hence the sphere approximates flat space.

Basic idea

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 \quad \xrightarrow{\lambda \to 0} \quad \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

hs theory based on

$$\mathrm{shs}_2[\lambda]$$

string theory

$$\updownarrow \qquad\qquad \updownarrow$$

Wolf space cosets $\quad \overset{?}{\subset} \quad$ symmetric orbifold

The free limit

Thus we should analyse the

$$k \to \infty \qquad\qquad (k^+ = k + 1)$$

limit of the Wolf space cosets --- and then compare to the symmetric orbifold.

[MRG, Gopakumar '14]

First let us understand the limit of the large superconformal algebra in detail:

The large $\mathcal{N} = 4$ algebra

$$[U_m, U_n] = \tfrac{k^+ + k^-}{2}\, m\, \delta_{m,-n}$$

$$[A_m^{\pm,i}, Q_r^a] = \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b$$

$$\{Q_r^a, Q_s^b\} = \tfrac{k^+ + k^-}{2}\, \delta^{ab}\, \delta_{r,-s}$$

$$[A_m^{\pm,i}, A_n^{\pm,j}] = \tfrac{k^\pm}{2}\, m\, \delta^{ij}\, \delta_{m,-n} + \mathrm{i}\, \epsilon^{ijl}\, A_{m+n}^{\pm,l}$$

$$[U_m, G_r^a] = m\, Q_{m+r}^a$$

$$[A_m^{\pm,i}, G_r^a] = \mathrm{i}\, \alpha_{ab}^{\pm\, i}\, G_{m+r}^b \mp \tfrac{2k^\pm}{k^+ + k^-}\, m\, \alpha_{ab}^{\pm\, i}\, Q_{m+r}^b$$

$$\{Q_r^a, G_s^b\} = 2\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} - 2\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i} + \delta^{ab}\, U_{r+s}$$

$$\{G_r^a, G_s^b\} = \tfrac{c}{3}\, \delta^{ab}\, (r^2 - \tfrac{1}{4})\delta_{r,-s} + 2\, \delta^{ab}\, L_{r+s} + 4\,(r-s)\, \Big(\gamma\, \mathrm{i}\, \alpha_{ab}^{+\, i}\, A_{r+s}^{+,i} + (1-\gamma)\, \mathrm{i}\, \alpha_{ab}^{-\, i}\, A_{r+s}^{-,i}\Big)\ .$$

need to rescale generators

$$U_m\ ,\ Q_r^a$$

$$A_n^{+,i} \quad (n \neq 0)$$

by $(k^+)^{-1/2}$

Contraction

Then the 4 free fermions and the u(1) field decouple completely, and the $\mathfrak{su}(2)_{k^+}$ affine algebra gives rise to 3 free bosons, as well as a custodial global su(2) symmetry.

Altogether we therefore get the small $\mathcal{N}=4$ algebra as well as 4 free bosons and fermions; they transform as

$$\begin{array}{ll} \text{bosons:} & 4\cdot\mathbf{1} \\ \text{fermions:} & 2\cdot\mathbf{2} \end{array}$$

w.r.t R-symmetry su(2) algebra

Wolf space cosets

What happens to the Wolf space cosets in this limit? --- similar to bosonic case

[MRG, Suchanek '11]

$$\frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N+4)_1}{\mathfrak{su}(N)_{k+2} \oplus \mathfrak{u}(1)_\kappa} \oplus \mathfrak{u}(1)_\kappa \ .$$

Including the above 4 free bosons and fermions, the `perturbative' part of the spectrum can be identified with the U(N)-singlet sector

$$\mathcal{H}_{\rm pert} = \bigoplus_{\Lambda} (0;\Lambda) \otimes (0;\Lambda) = \begin{pmatrix} 4(N+1) \ \text{free bosons} \\ 4(N+1) \ \text{free fermions} \end{pmatrix} / \mathrm{U}(N)$$

Continuous orbifold

Here the additional 4 N free bosons and fermions transform as

$$\begin{array}{ll} \text{bosons:} & 2\cdot(\mathbf{N},\mathbf{1})\oplus 2\cdot(\bar{\mathbf{N}},\mathbf{1}) \\ \text{fermions:} & (\mathbf{N},\mathbf{2})\oplus(\bar{\mathbf{N}},\mathbf{2}) \end{array}$$

$$\mathrm{U(N)} \qquad \mathfrak{su}(2)$$

The other coset representations can be interpreted as twisted sectors (and descendants) of this continuous orbifold --- actually, can give very concrete identification....

[MRG, Gopakumar '14]
see also [MRG, Kelm '14] for similar analysis for $\mathcal{N}=2$

Untwisted sector

Thus altogether the `perturbative' part of CFT consists of free bosons and fermions that transform as

$$
\begin{aligned}
\text{bosons:} \qquad & 2\cdot(\mathbf{N},\mathbf{1}) \oplus 2\cdot(\bar{\mathbf{N}},\mathbf{1}) \oplus 4\cdot(\mathbf{1},\mathbf{1}) \\
\text{fermions:} \qquad & (\mathbf{N},\mathbf{2}) \oplus (\bar{\mathbf{N}},\mathbf{2}) \oplus 2\cdot(\mathbf{1},\mathbf{2})
\end{aligned}
$$

subject to a U(N) singlet condition.

Comparison

This now looks very similar to the untwisted sector of the symmetric orbifold

$$\mathrm{Sym}_{N+1}(\mathbb{T}^4) \equiv \left(\mathbb{T}^{4\otimes(N+1)}\right)/S_{N+1}$$

Indeed, this sector is generated by free bosons and fermions in

$$\begin{aligned} \text{bosons:} \qquad & 4\cdot(\mathbf{N}+\mathbf{1},\mathbf{1}) = 4\cdot(\mathbf{N},\mathbf{1})\oplus 4\cdot(\mathbf{1},\mathbf{1}) \\ \text{fermions:} \qquad & 2\cdot(\mathbf{N}+\mathbf{1},\mathbf{2}) = 2\cdot(\mathbf{N},\mathbf{2})\oplus 2\cdot(\mathbf{1},\mathbf{2}) \end{aligned}$$

S_{N+1} $\quad \mathfrak{su}(2)$

Branching rule

In fact

$$S_{N+1} \subset \mathrm{U}(N)$$

and under this embedding, we have the branching rules

$$\mathbf{N}_{\mathrm{U}(N)} \rightarrow \mathbf{N}_{S_{N+1}} \qquad \bar{\mathbf{N}}_{\mathrm{U}(N)} \rightarrow \mathbf{N}_{S_{N+1}}$$

Comparison

Wolf coset:

$$\text{bosons:} \qquad 2 \cdot (\mathbf{N}, \mathbf{1}) \oplus 2 \cdot (\bar{\mathbf{N}}, \mathbf{1}) \oplus 4 \cdot (\mathbf{1}, \mathbf{1})$$

$$\text{fermions:} \qquad (\mathbf{N}, \mathbf{2}) \oplus (\bar{\mathbf{N}}, \mathbf{2}) \oplus 2 \cdot (\mathbf{1}, \mathbf{2})$$

Symmetric orbifold:

$$\text{bosons:} \qquad 4 \cdot (\mathbf{N}, \mathbf{1}) \oplus 4 \cdot (\mathbf{1}, \mathbf{1})$$

$$\text{fermions:} \qquad 2 \cdot (\mathbf{N}, \mathbf{2}) \oplus 2 \cdot (\mathbf{1}, \mathbf{2})$$

Thus the action of the permutation group on the free bosons and fermions is induced from the U(N) action!

Subtheory

It therefore follows that

[MRG, Gopakumar ’14]

untwisted sector
of cts orbifold

$\subset$

untwisted sector
of symmetric orbifold

$\uparrow$

perturbative part
of CFT dual of
hs theory for
$\lambda \to 0$

$\uparrow$

CFT dual of
string theory
in this limit

Thus **hs theory is naturally subsector of string theory**!

Stringy modular invariant

From the hs point of view, the symmetric orbifold (i.e. the stringy CFT dual) is a non-diagonal modular invariant.

For example, it contains the extended vacuum sector for which we have

$$Z_{\text{vac}}(q,y) = \sum_{\Lambda} D(\Lambda)\, \chi_{(0;\Lambda)}(q,y)$$

multiplicity of singlet representation of S_{N+1}

Stringy chiral algebra

Explicitly, these multiplicities are

$$\begin{aligned} Z_{\mathrm{vac}}(q,y) \quad = \quad & \chi_{(0;0)}(q,y) + \chi_{(0;[2,0,\ldots,0])}(q,y) + \chi_{(0;[0,0,\ldots,0,2])}(q,y) \\ & + \chi_{(0;[3,0,\ldots,0,0])}(q,y) + \chi_{(0;[0,0,0,\ldots,0,3])}(q,y) \\ & + \chi_{(0;[2,0,\ldots,0,1])}(q,y) + \chi_{(0;[1,0,0,\ldots,0,2])}(q,y) \\ & + 2\cdot\chi_{(0;[4,0,\ldots,0,0])}(q,y) + 2\cdot\chi_{(0;[0,0,0,\ldots,0,4])}(q,y) \\ & + \chi_{(0;[0,2,0,\ldots 0,0])}(q,y) + \chi_{(0;[0,0,\ldots 0,2,0])}(q,y) \\ & + \chi_{(0;[3,0,\ldots,0,1])}(q,y) + \chi_{(0;[1,0,0,\ldots,0,3])}(q,y) \\ & + 2\cdot\chi_{(0;[2,0,0,\ldots,0,2])}(q,y) + \mathcal{O}(q^{5/2}) \end{aligned}$$

Reproduces precisely vacuum character of symmetric orbifold (calculated from DMVV)!

Light States

However, it does not contain any of the `light' states any longer since they do not give rise to allowed (untwisted) representations of the extended chiral algebra.

Spacetime intepretation:

classical hs solutions do not lift to string theory.

Quantising Vasiliev

Seems to imply that direct quantisation of Vasiliev higher spin theory is problematic:

either:
- light states from non-perturbative hs solutions

or:
- embedding into string theory: add low lying degrees of freedom by hand.

Stringy completion

At $\lambda \to 0$ stringy description characterised by extended chiral algebra (which could be directly obtained from symmetric orbifold).

It is natural to believe that the same idea should also work away from this special point --- this suggests a new avenue for how to find the CFT dual of string theory on

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$$

Conclusion

- Explained evidence for bosonic minimal model holography and its large $\mathcal{N}=4$ generalisation.

- In the supersymmetric case found natural embedding of CFT dual of hs theory into CFT dual of string theory.

- Suggests a procedure for how to construct CFT dual to string theory on

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$$

Open problems & future directions

Open problems:

- Find branching rules $S_{N+1} \subset \mathrm{U}(N)$
- Interpretation from D1-D5 viewpoint
- Prove character identities
- Stringy modular invariants
- Characterise stringy symmetry
- Generalisation to K3 [Baggio, MRG, Peng, in progress]

HS viewpoint: new perspective on stringy CFT

- single vs multi-particle states from hs & string
- study perturbation theory
- higher dimensional analogue?

cf. [Chang, Minwalla, Sharma, Yin '12]
cf. [Beisert,Bianchi,Morales,Samtleben '04]