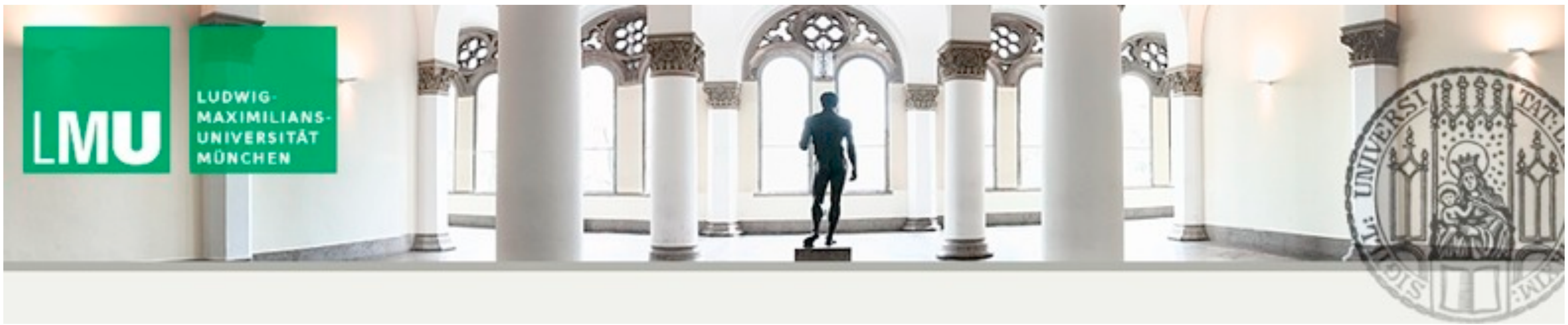
Non-Associative Geometry and Double Field Theory

DIETER LÜST (LMU, MPI)



Corfu, September 15, 2014

Outline:

I) Introduction

II) Non-geometric Backgrounds &
Non-Commutativity/Non-Associativity
(world sheet point of view)

III) Double field theory (target space point of view)

IV) Dimensional Reduction of DFT

I) Introduction

Jacobi identity for three operators in QM:

$$\begin{aligned}\mathrm{Jac}(F,G,H) &= [F,G,H] \\ &= [F,[G,H]] + [H,[F,G]] + [G,[H,F]] \\ &= \big(F(GH) - (FG)H\big) - \big(F(HG) - (FH)G\big) + \ldots\end{aligned}$$

Algebraically zero for associative operators.

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Algebraically zero for associative operators.

Poisson bracket: $\{f,g\} = \omega^{IJ}\partial_I f \partial_J g$

ω^{IJ} .... Poisson structure

The Jacobi identity of the Poisson bracket is zero if

$$d\omega = 0$$

Why is $d\omega = 0$ in standard Hamiltonian mechanics?

Choose local coordinates q^i, p_i

Hamiltonian dynamics is defined on the cotangent bundle $T^\star M$, which defines a 2D-dim manifold:

$$Q_I = (\underbrace{q^1, \ldots, q^D}_{\in M}; \underbrace{p_1, \ldots, p_D}_{\in T^\star M})$$

Poisson structure = Symplectic structure:

$$\omega = \omega^{IJ} dQ_I \wedge dQ_J = dv \,, \quad v = q^i dp_i - p_i dq^i$$

$$\omega^{ij} = \begin{pmatrix} 0 & 1_D \\ -1_D & 0 \end{pmatrix} .$$

The symplectic structure of $T^\star M$ is exact.

$$\Longrightarrow \quad d\omega = 0$$

Deformation quantization - Star product:

$$f \star g = e^{\frac{i}{2}\omega^{IJ}\partial_I f \partial_J g} = fg + \frac{i}{2}\omega^{IJ}\partial_I f \partial_J g + \ldots$$

e.g. $\omega^{IJ} = \begin{pmatrix} 0 & 1_D \\ -1_D & 0 \end{pmatrix}$, $f = q^i \, , \; g = p_j$

Then the commutator is obtained: $[q^i, p_j]_\star = \delta^i_j .$

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Quasi Poisson tensor: $d\omega \neq 0$

$$(f \star g) \star h - f \star (g \star h) \sim \omega^{[IL}\partial_L \omega^{JK]}\partial_I f \partial_J g \partial_K h \neq 0$$

Non-associativity is possible if $d\omega \neq 0$!

Non-associativity in physics:

- Jordan & Malcev algebras, octonions

  M. Günaydin, F. Gürsey (1973); M. Günaydin, D. Minic, arXiv:1304.0410

- Nambu dynamics

  Y. Nambu (1973); D. Minic, H.Tze (2002); M.Axenides, E. Floratos (2008)

- Magnetic monopoles R. Jackiw (1985); M. Günaydin, B. Zumino (1985)

- Closed string field theory A. Strominger (1987), B. Zwiebach (1993)

- T-duality and principle torus bundles

  P. Bouwknegt, K. Hannabuss, Mathai (2003)

- D-branes in curved backgrounds

  L. Cornalba, R. Schiappa (2001)

- Multiple M2-branes and 3-algebras J. Bagger, N. Lambert (2007)

- Non-geometric backgrounds:

- Non-geometric backgrounds:

- They are only consistent in string theory.

- Make use of string symmetries, T-duality $\Rightarrow$ T-folds,

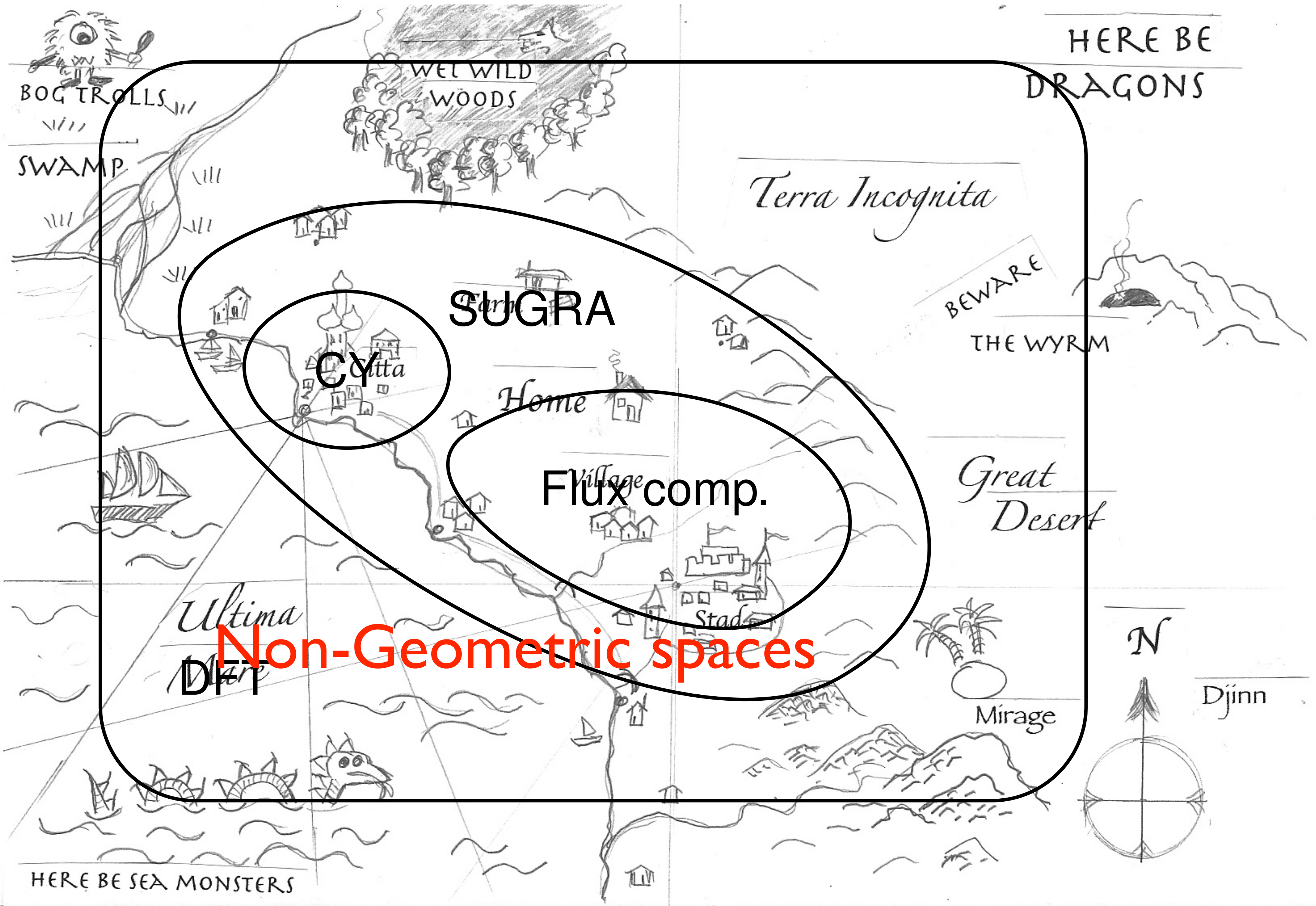
- Left-right asymmetric spaces $\Rightarrow$ Asymmetric orbifolds

(Kawai, Lewellen, Tye, 1986; Lerche, D.L. Schellekens, 1986, Antoniadis, Bachas, Kounnas, 1987; Narain, Sarmadi, Vafa, 1987; Ibanez, Nilles, Quevedo, 1987; ......, Faraggi, Rizos, Sonmez, 2014)

- Are related to non-commutative/non-associative geometry

(Blumenhagen, Plauschinn; Lüst, 2010; Blumenhagen, Deser, Lüst, Rennecke, Pluaschin, 2011; Condeescu, Florakis, Lüst; 2012, Andriot, Larfors, Lüst, Patalong, 2012))

Non-geometric spaces ⇔ Double Field Theory



II) Non-geometric backgrounds & non-commutativity/non-associativity (word-sheet)

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Consider D-dimensional toroidal string backgrounds:

Doubling of closed string coordinates and momenta:

- Coordinates: O(D,D) vector $X^M = (\tilde{X}_i, X^i)$

$$(X^i = X_L^i(\tau+\sigma) + X_R(\tau-\sigma) \quad \tilde{X}_i = X_L^i(\tau+\sigma) - X_R(\tau-\sigma))$$

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winding ← T-duality → momentum

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- O(D,D) transformations: Mix in general X^i with $\tilde{X}_i$.

☐ Act asymmetrically on X_L and X_R .

$$\begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix} \to \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X^i \\ \tilde{X}_i \end{pmatrix}, \quad \Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(D,D)$$

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Generalized metric: $\mathcal{H}_{MN} = \begin{pmatrix} G^{ij} & -G^{ik}B_{kj} \\ B_{ik}G^{kj} & G_{ij} - B_{ik}G^{kl}B_{lj} \end{pmatrix}$

$$\mathcal{H}_{MN} \to \Lambda^P_M \, \mathcal{H}_{PQ} \, \Lambda^Q_N$$

Non-geometric backgrounds & non-geometric fluxes:

- Non-geometric Q-fluxes: spaces that are locally still Riemannian manifolds but not anymore globally.

(Hellerman, McGreevy, Williams (2002); C. Hull (2004); Shelton, Taylor, Wecht (2005); Dabholkar, Hull, 2005)

Transition functions between two coordinate patches are given in terms of O(D,D) T-duality transformations:

C. Hull (2004)

$$\mathrm{Diff}(M_D) \quad \rightarrow \quad O(D,D)$$

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Q-space will become non-commutative:

- Non-geometric R-fluxes: spaces that are even locally not anymore manifolds.

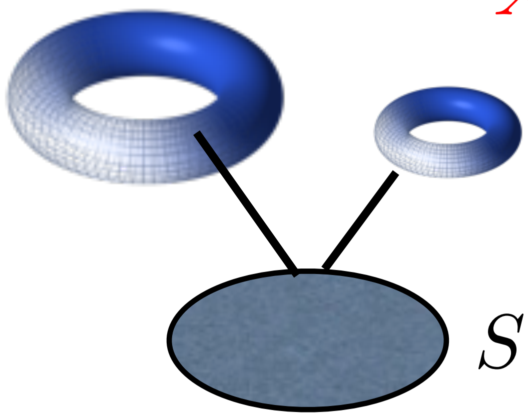
R-space will become non-associative:

$$\begin{aligned} [X^i, X^j, X^k] \quad &:= \quad [[X^i, X^j], X^k] + \text{cycl. perm.} = \\ &= \quad (X^i \cdot X^j) \cdot X^k - X^i \cdot (X^j \cdot X^k) + \cdots \neq 0 \end{aligned}$$

Example: Three-dimensional flux backgrounds:

Fibrations: 2-dim. torus that varies over a circle:

$$T^2_{X^1,X^2} \hookrightarrow M^3 \hookrightarrow S^1_{X^3}$$



Metric, B-field of T^2 : depends on X^3

$$\Rightarrow \quad \mathcal{H}_{MN}(X^3) \quad (M, N = 1, \ldots, 4)$$

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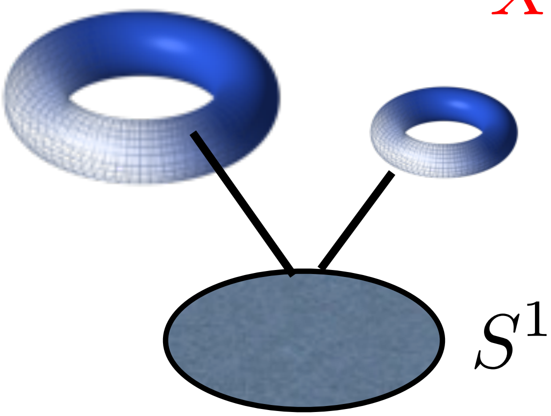

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The fibration is specified by its monodromy properties.

O(2,2) monodromy:

$$\mathcal{H}_{MN}(X^3+2\pi) \;=\; \Lambda_{O(2,2)}\,\mathcal{H}_{PQ}(X^3)\,\Lambda^{-1}_{O(2,2)}$$

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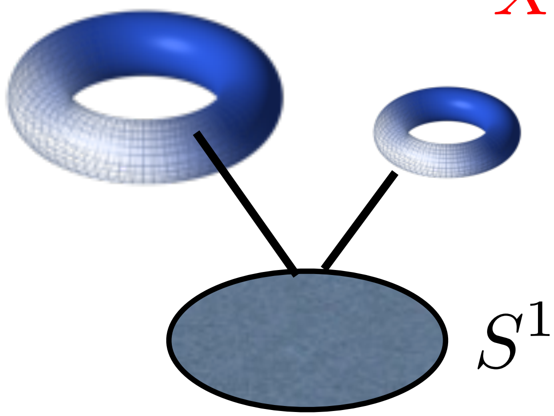

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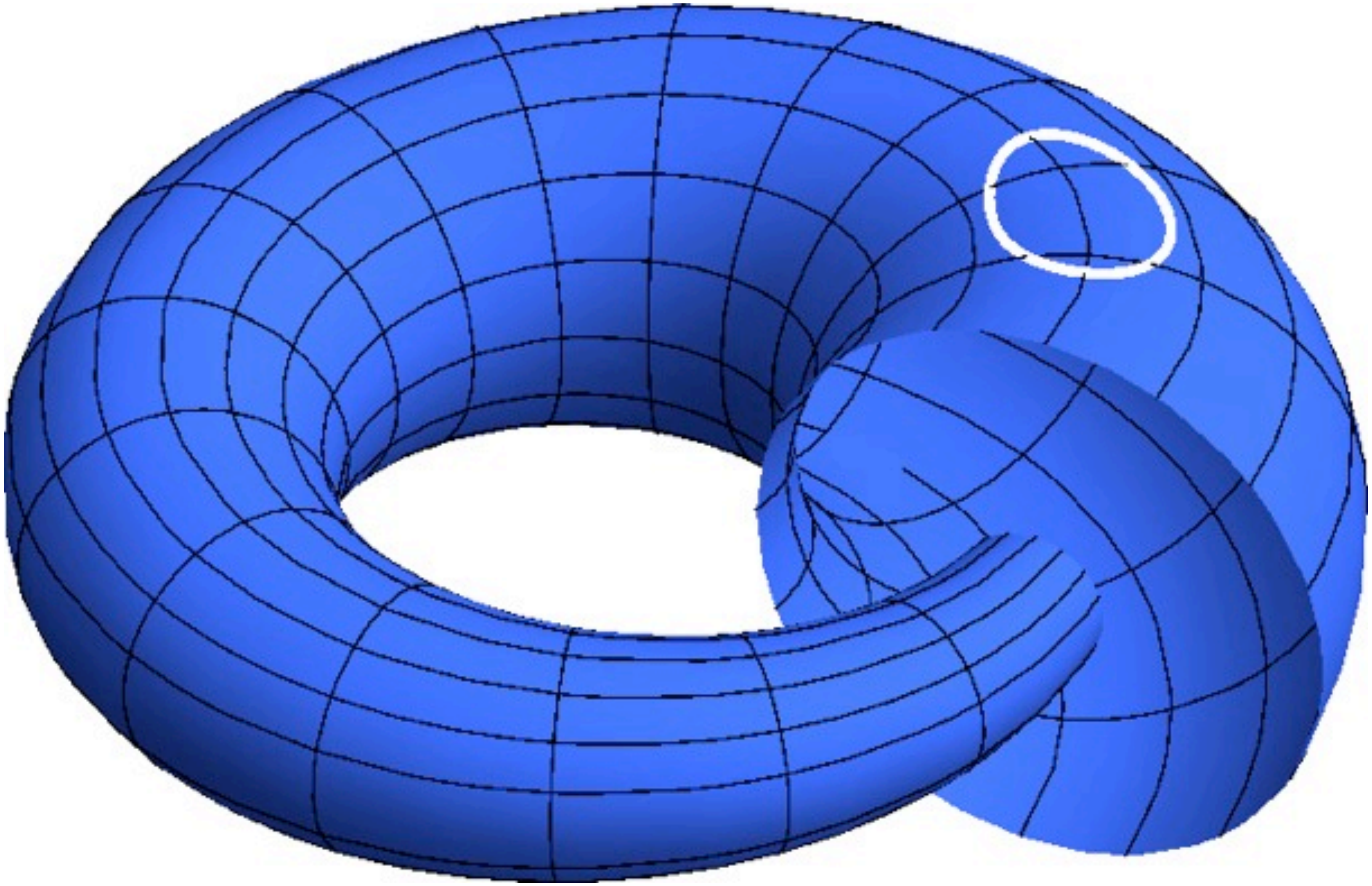
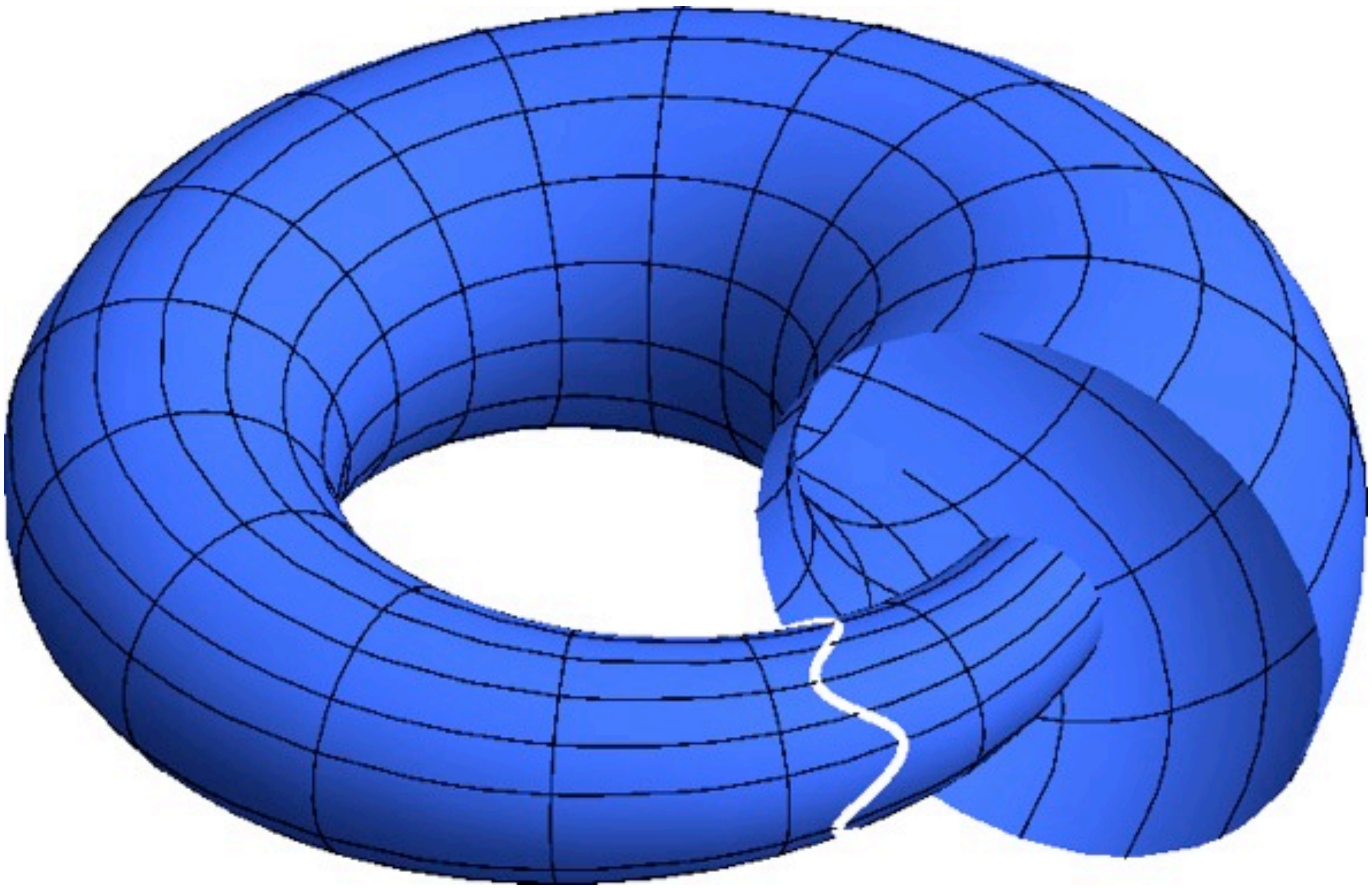
Non-geometric spaces:

- Monodromy mixes $X^i \leftrightarrow \tilde{X}_i$
- Acts asymmetrically on $X^i_L, X^i_R \quad (i=1,2)$



Non geometric torus, metric is patched together by a T-duality transformation: $G_{ij} \;\rightarrow\; G^{ij}$



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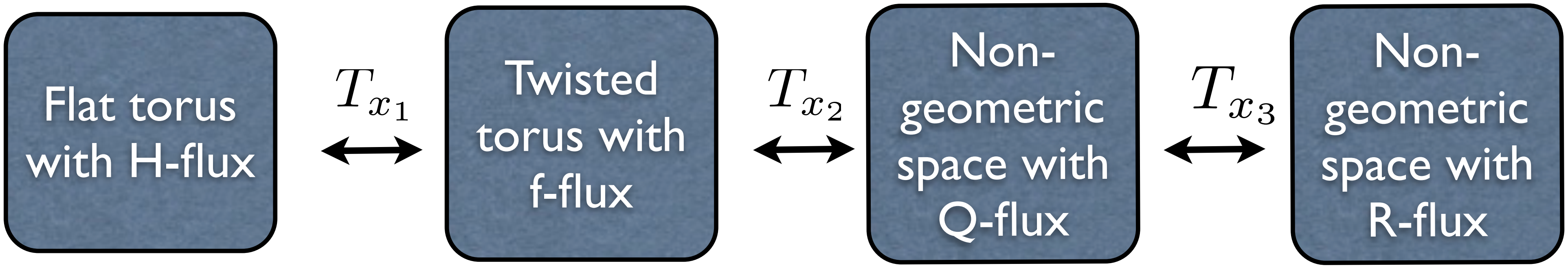
(Non-)geometric backgrounds with parabolic monodromy and single 3-form fluxes:



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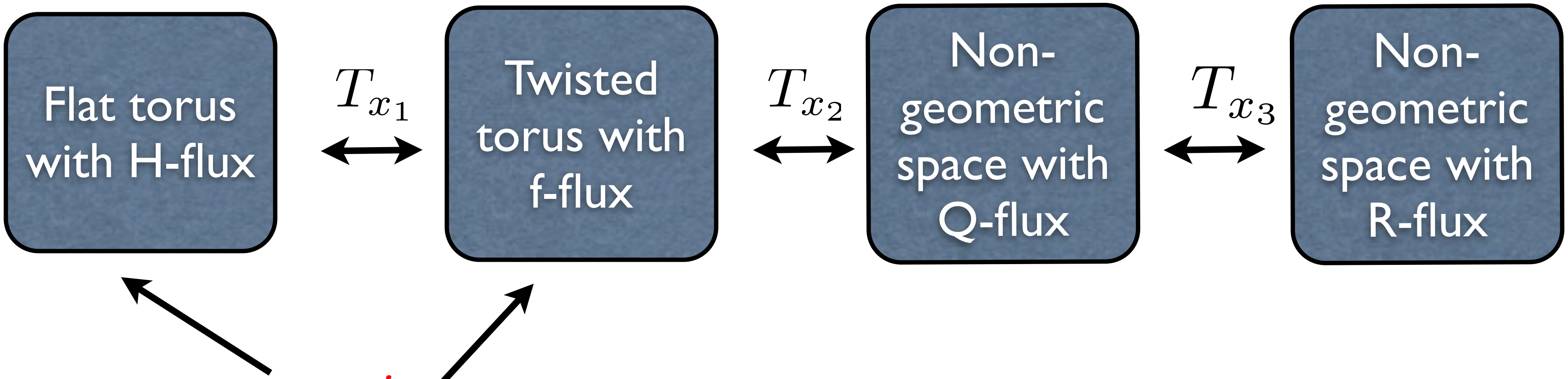

$$[X^i_{H,f}, X^j_{H,f}] = 0$$

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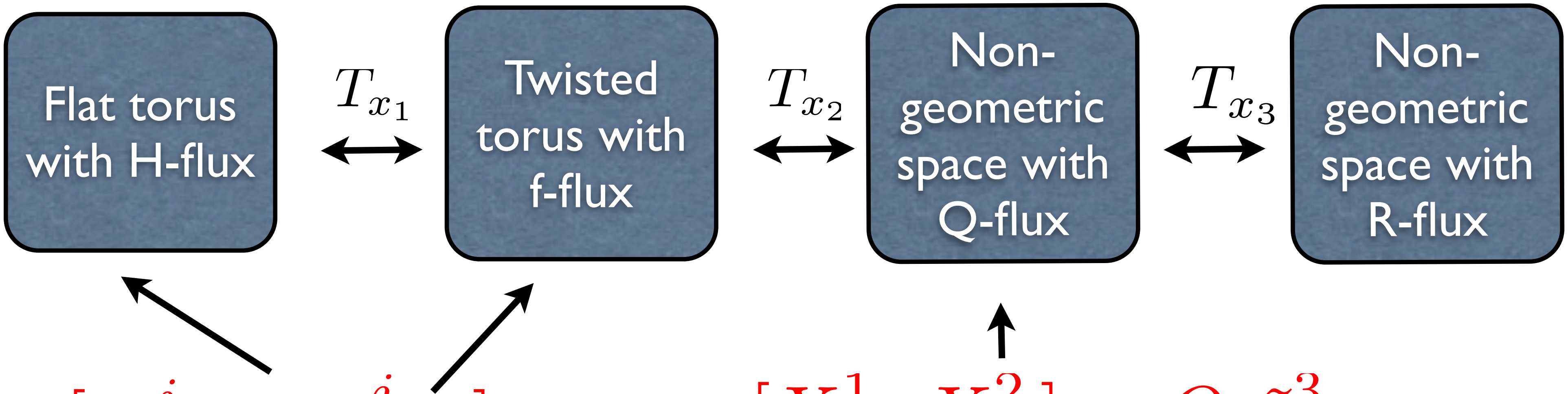

$$[X^i_{H,f}, X^j_{H,f}] = 0 \qquad [X^1_Q, X^2_Q] \simeq Q\ \tilde{p}^3$$

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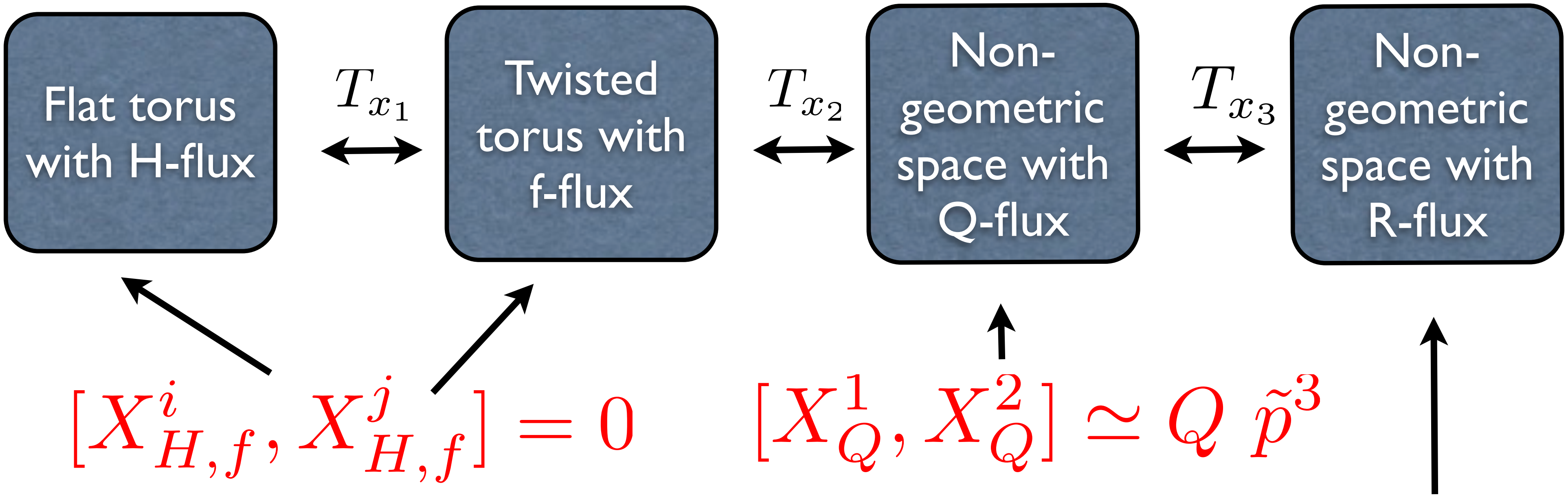

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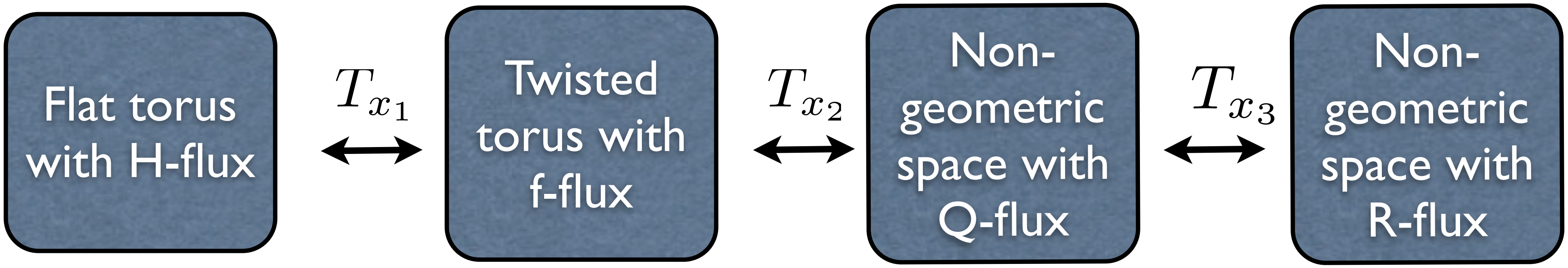

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They can be computed by

- standard world-sheet quantization of the closed string

- CFT & canonical T-duality

D. Andriot, M. Larfors, D. L., P. Patalong, arXiv:1211.6437

C. Blair, arXiv:1405.2283

I. Bakas, D.L. to appear soon

Q-flux: O(2,2) monodromy ⇒

mixed closed string boundary (DN) conditions:

$$X_Q^3(\tau,\sigma+2\pi) = X_Q^3(\tau,\sigma) + 2\pi\ \tilde{p}^3 \quad \Longrightarrow$$

$$O(2,2) \begin{cases} X_Q^1(\tau,\sigma+2\pi) = X_Q^1(\tau,\sigma) - 2\pi\ \tilde{p}^3\ Q\ \tilde{X}_{Q2}(\tau,\sigma)\,, \\ X_Q^2(\tau,\sigma+2\pi) = X_Q^2(\tau,\sigma) + 2\pi\ \tilde{p}^3\ Q\ \tilde{X}_{Q1}(\tau,\sigma)\,, \\ \tilde{X}_{Q1}(\tau,\sigma+2\pi) = \tilde{X}_{Q1}(\tau,\sigma)\,, \\ \tilde{X}_{Q2}(\tau,\sigma+2\pi) = \tilde{X}_{Q2}(\tau,\sigma)\,. \end{cases}$$

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winding number along base direction

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$$[X_Q^1(\tau,\sigma), X_Q^2(\tau,\sigma')] =$$

$$-\frac{i}{2}Q\ \tilde{p}^3\left(\sum_{n\neq 0}\frac{1}{n^2}e^{-in(\sigma'-\sigma)} - (\sigma'-\sigma)\sum_{n\neq 0}\frac{1}{n}e^{-in(\sigma'-\sigma)} + \frac{i}{2}(\sigma'-\sigma)^2\right)$$

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The non-commutativity of the torus (fibre) coordinates is determined by the winding in the circle (base) direction.

Corresponding uncertainty relation:

$$(\Delta X_Q^1)^2 (\Delta X_Q^2)^2 \geq L_s^6 \, Q^2 \ \langle \tilde{p}^3 \rangle^2$$

The spatial uncertainty in the X_1, X_2 - directions grows with the dual momentum in the third direction: non-local strings with winding in third direction.

R-flux background: T-duality in x^3-direction $\Rightarrow$ R-flux

$$\tilde{p}^3 \longleftrightarrow p_3\,, \qquad \tilde{X}_{Q,3} \equiv X_R^3$$

$\Rightarrow$ For the case of non-geometric R-fluxes one gets:

$$[X_R^1, X_R^2] = -i\frac{\pi^2}{6}\,R\,p_3 \qquad R \equiv Q$$

Use $[X_R^3, p_3] = i$ $\Longrightarrow$ Non-associative algebra:

$$[[X_R^1(\tau,\sigma), X_R^2(\tau,\sigma)], X_R^3(\tau,\sigma)] + \text{perm.} = \frac{\pi^2}{6}\,R$$

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Corresponding classical „uncertainty relations“:

$$(\Delta X_R^1)^2 (\Delta X_R^2)^2 \geq L_s^6\, R^2\, \langle p^3 \rangle^2$$

Volume: $(\Delta X_R^1)^2 (\Delta X_R^2)^2 (\Delta X_R^3)^2 \geq L_s^6\, R^2$

(see also: D. Mylonas, P. Schupp, R.Szabo, arXiv:1312.1621)

The algebra of commutation relation looks different in each of the four duality frames.

Non-vanishing commutators and 3-brackets:

T-dual frames	Commutators	Three-brackets
H-flux	$[\tilde{x}^1, \tilde{x}^2] \sim H\tilde{p}^3$	$[\tilde{x}^1, \tilde{x}^2, \tilde{x}^3] \sim H$
f-flux	$[x^1, \tilde{x}^2] \sim f\tilde{p}^3$	$[x^1, \tilde{x}^2, \tilde{x}^3] \sim f$
Q-flux	$[x^1, x^2] \sim Q\tilde{p}^3$	$[x^1, x^2, \tilde{x}^3] \sim Q$
R-flux	$[x^1, x^2] \sim Rp^3$	$[x^1, x^2, x^3] \sim R$

However: R-flux & winding coordinates:

$$[\tilde{x}^i, \tilde{x}^j, \tilde{x}^k] = 0$$

⇒ Star products, 3-products and CFT

$\Rightarrow$ Star products, 3-products and CFT

Open string non-commutativity:

Constant Poisson structure: $[x_i, x_j] = \theta_{ij}$

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Vertex operators at boundary of disc:

Worldsheet has $SL(2,\mathbb{R})/\mathbb{Z}_2$ symmetry.

Non-commutativity is possible:

$$12 = 21\,, \quad 123 = 231, \quad 123 \neq 132$$

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$$12 = 21\,, \quad 123 = 231, \quad 123 \neq 132$$

Moyal-Weyl star-product:

$$(f_1 \star f_2)(\vec{x}) = e^{i\theta^{ij}\,\partial_i^{x1}\,\partial_j^{x2}}\, f_1(\vec{x}_1)\, f_2(\vec{x}_2)|_{\vec{x}}$$

2-cyclicity: $\int d^n x\,(f \star g) = \int d^n x\,(g \star f)$

Non-commutative gauge theories: $S \simeq \int d^n x\, \mathrm{Tr} \hat{F}_{ab} \star \hat{F}^{ab}$

(N. Seiberg, E. Witten (1999); J. Madore, S. Schraml, P. Schupp, J. Wess (2000); ....)

Closed strings:

Vertex operators on sphere:

Worldsheet has $SL(2,\mathbb{C})/\mathbb{Z}_2$ symmetry.

Non-commutativity appears impossible:

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Non-associative algebra:

$$[x^i,\ x^j] = \epsilon^{ijk} p_k\ , [x^i,\ p^j] = i\hbar\delta^{ij}\ , \qquad [p^i,\ p^j] = 0$$

$$[x^i,\ x^j,\ x^k] = [[x^i,\ x^j],\ x^k] + \text{cycl. perm.} = R^{ijk}$$

Non-commutativity $\Rightarrow$ $\star_p$ 2-product:

D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926, arXiv:1312.162, arXiv:1402.7306.

I. Bakas, D.Lüst, arXiv:1309.3172

$$(f_1 \star_p f_2)(\vec{x}, \vec{p}) = e^{\frac{i}{2}\theta^{I\,J}(p)\,\partial_I \otimes \partial_J} \, (f_1 \otimes f_2)\big|_{\vec{x};\ \vec{p}}$$

6-dimensional Poisson tensor:

$$\theta^{I\,J}(p) = \begin{pmatrix} R^{ijk} p_k & \delta^i_j \\ & \\ -\delta^j_i & 0 \end{pmatrix} ; \; R^{ijk} = \frac{\pi^2 R}{6}\, \epsilon^{ijk}$$

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Mathematical framework for non-associative algebras:

Group cohomology,
3-Cocycles, 2-cochains and 3-products

(talk of 2013)

3-product:

R. Blumenhagen, A. Deser, D.Lüst, E. Plauschinn, F. Rennecke, arXiv:1106.0316

D. Mylonas, P. Schupp, R.Szabo, arXiv:1207.0926, arXiv:1312.162, arXiv:1402.7306.

I. Bakas, D.Lüst, arXiv:1309.3172

$$(f_1 \triangle_3 f_2 \triangle_3 f_3)(\vec{x}) = ((f_1 \star_p f_2) \star_p f_3)\,(\vec{x})$$

$$(f_1 \triangle_3 f_2 \triangle_3 f_3)(\vec{x}) = e^{iR^{ijk}\,\partial_i^{x_1}\,\partial_j^{x_2}\,\partial_k^{x_3}}\, f_1(\vec{x}_1)\, f_2(\vec{x}_2)\, f_3(\vec{x}_3)|_{\vec{x}}$$

This 3-product is non-associative.

It is consistent with the 3-bracket among the coordinates:

$$f_1 = X^i\,,\ f_2 = X^j\,,\ f_3 = X^k\ :$$

$$f_1 \triangle_3 f_2 \triangle_3 f_3 = [X^i, X^j, X^k] = R^{ijk}$$

It obeys the 3-cyclicity property:

$$\int d^n x\,(f_1 \triangle_3 f_2) \triangle_3 f_3 = \int d^n x\, f_1 \triangle_3 (f_2 \triangle_3 f_3)$$

Three point function in CFT:

R. Blumenhagen, A. Deser, D.Lüst, E. Plauschinn, F. Rennecke, arXiv:1106.0316

$$\left\langle X^i(z_1,\overline{z}_1)\, X^j(z_2,\overline{z}_2)\, X^c(z_3,\overline{z}_3)\right\rangle = R^{ijk}\left[\mathcal{L}\left(\tfrac{z_{12}}{z_{13}}\right) + \mathcal{L}\left(\tfrac{\overline{z}_{12}}{\overline{z}_{13}}\right)\right]$$

$$\Rightarrow \quad [X^i, X^j, X^k] := \lim_{z_i \to z}\Big[X^i(z_1,\overline{z}_1), [X^b(z_2,\overline{z}_2), X^c(z_3,\overline{z}_3)]\Big] + \text{cycl.} = R^{ijk}$$

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$\triangle_3$: Scattering of 3 momentum states in R-background:

(corresponds to 3 winding states in H-background)

$$V_i(z,\bar z) =: \exp\big(ip_i X^i(z,\bar z)\big) :$$

$$\langle\, V_{\sigma(1)}\; V_{\sigma(1)}\; V_{\sigma(1)}\,\rangle_R = \langle\, V_1\; V_2\; V_3\,\rangle_R \times \exp\Big(-i\eta_\sigma\, R^{ijk}\, p_{1,i}p_{2,j}p_{3,k}\Big).$$

$$(\eta_\sigma = 0, 1)$$

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$$(\eta_\sigma = 0, 1)$$

However this non-associative phase is vanishing, when going on-shell in CFT and using momentum conservation:

$$p_1 = -(p_2 + p_3)$$

On-shell CFT amplitudes are associative!

III) Double field theory (target space point of view)

W. Siegel (1993); C. Hull, B. Zwiebach (2009); C. Hull, O. Hohm, B. Zwiebach (2010,...)

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- O(D,D) invariant effective string action containing momentum and winding coordinates at the same time:

$$S_{\mathrm{DFT}} = \int \mathrm{d}^{2D} X \, e^{-2\phi'} \mathcal{R} \qquad X^M = (\tilde{x}_m, x^m)$$

$$\mathcal{R} = 4\mathcal{H}^{MN}\partial_M\phi'\partial_N\phi' - \partial_M\partial_N\mathcal{H}^{MN} \quad -4\mathcal{H}^{MN}\partial_M\phi'\partial_N\phi' + 4\partial_M\mathcal{H}^{MN}\partial_N\phi'$$
$$+\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} \quad -\tfrac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK}$$

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- Covariant fluxes of DFT:

(Geissbuhler, Marques, Nunez, Penas; Aldazabal, Marques, Nunez)

$$\mathcal{F}_{ABC} = \mathcal{D}_{[A} E_B{}^M \, E_{C]M} \, , \quad \mathcal{D}^A = E^A{}_M \, \partial^M \, .$$

Comprise all fluxes (Q,f,Q,R) into one covariant expression:

$$\mathcal{F}_{abc} = H_{abc} \, , \quad \mathcal{F}^a{}_{bc} = F^a{}_{bc} \, , \quad \mathcal{F}_c{}^{ab} = Q_c{}^{ab} \, , \quad \mathcal{F}^{abc} = R^{abc}$$

DFT action in flux formulation:

$$S_{\rm DFT} = \int dX\, e^{-2d}\Big[\mathcal{F}_A\mathcal{F}_{A'}S^{AA'} + \mathcal{F}_{ABC}\mathcal{F}_{A'B'C'}\Big(\frac{1}{4}S^{AA'}\eta^{BB'}\eta^{CC'} - \frac{1}{12}S^{AA'}S^{BB'}S^{CC'}\Big) - \frac{1}{6}\mathcal{F}_{ABC}\,\mathcal{F}^{ABC} - \mathcal{F}_A\mathcal{F}^A\Big]$$

(Looks similar to scalar potential in gauged SUGRA.)

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(Looks similar to scalar potential in gauged SUGRA.)

- Strong constraint (string level matching condition):

(CFT origin of the strong constraint: A. Betz, R. Blumenhagen, D. Lüst, F. Rennecke, arXiv:1402.1686)

$$\partial_M\partial^M \cdot = 0\,, \quad \partial_M f\,\partial^M g = \mathcal{D}_A f\,\mathcal{D}^A g = 0$$

Functions depend only on one kind of coordinates.

The strong constraint defines a D-dim. hypersurface (brane) in 2D-dim. double geometry.

Non-associative deformations in double field theory:

(R. Blumenhagen, M. Fuchs, F. Hassler, D.Lüst, R. Sun, arXiv:1312.0719)

DFT generalization of the 3-product:

$$\big(f \,\triangle_3\, g \,\triangle_3\, h\big)(X) = f\, g\, h + \frac{\ell_s^4}{6} \mathcal{F}_{ABC}\, \mathcal{D}^A f\, \mathcal{D}^B g\, \mathcal{D}^C h + O(\ell_s^8)$$

(For general functions $f(X,P)$ the phase space is 4D-dimensional.)

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Non-vanishing R-flux:

I. Bakas, D. L., arXiv:1309.3172

$$f = x^i\,,\ g = x^j\,,\ h = x^k\ :$$

$$f \,\triangle_3\, g \,\triangle_3\, h = [x^i, x^j, x^k] = \ell_s^4\, R^{ijk}$$

$$f = \tilde{x}_i\,,\ g = \tilde{x}_j\,,\ h = \tilde{x}_k\ : \quad f \,\triangle_3\, g \,\triangle_3\, h = [\tilde{x}_i, \tilde{x}_j, \tilde{x}_k] = 0$$

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Non-vanishing H-flux:

$$f = \tilde{x}_i\,,\; g = \tilde{x}_j\,,\; h = \tilde{x}_k\;:$$

$$f \,\triangle_3\, g \,\triangle_3\, h = [\tilde{x}_i, \tilde{x}_j, \tilde{x}_k] = \ell_s^4\, H_{ijk}$$

$$f = x^i\,,\; g = x^j\,,\; h = x^k\;: \quad f \,\triangle_3\, g \,\triangle_3\, h = [x^i, x^j, x^k] = 0$$

General functions f, g and h (conformal fields in CFT):

Consider the additional term in the DFT tri-product:

$$\mathcal{F}_{ABC}\,\mathcal{D}^A f\,\mathcal{D}^B g\,\mathcal{D}^C h$$

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Consider the additional term in the DFT tri-product:

$$\mathcal{F}_{ABC}\,\mathcal{D}^A f\,\mathcal{D}^B g\,\mathcal{D}^C h$$

Imposing the strong constraint on f, g and h the additional term vanishes and the tri-product becomes the normal product.

IV) Dimensional Reduction of DFT

O. Hohm, D. Lüst, B. Zwiebach, arXiv:1309.2977;
F. Hassler, D. Lüst, arXiv:1401.5068.

see also: A. Dabholkar, C. Hull, 2002, 2005;
C. Hull, R. Reid-Edwards, 2005, 2006, 2007

- Consistent DFT solutions: $R_{MN} = 0$

- 2(D-d) linear independent Killing vectors:

$$\mathcal{L}_{K_I{}^J} \mathcal{H}^{MN} = 0$$

- DFT and generalized Scherk-Schwarz ansatz (O(D,D) twists) gives rise to effective theory in D-d dimensions:

$$S_{\text{eff}} = \int \mathrm{d}x^{(D-d)} \sqrt{-g} e^{-2\phi} \Big(\mathcal{R} + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4} \mathcal{H}_{MN} F^{M\mu\nu} F^N_{\mu\nu} + \frac{1}{8} D_\mu \mathcal{H}_{MN} D^\mu \mathcal{H}^{MN} - V \Big)$$

The corresponding backgrounds are in general non-geometric and go beyond dimensional reduction of SUGRA.

(i) DFT on spaces satisfying the strong constraint (SC)

Rewriting of SUGRA, geometric spaces

(ii) Mild violation of SC:

- Killing vectors violate the SC.
- Patching of coordinate charts correspond to generalized coordinate transformations that violate the SC.

(iii) Strong violation of SC:

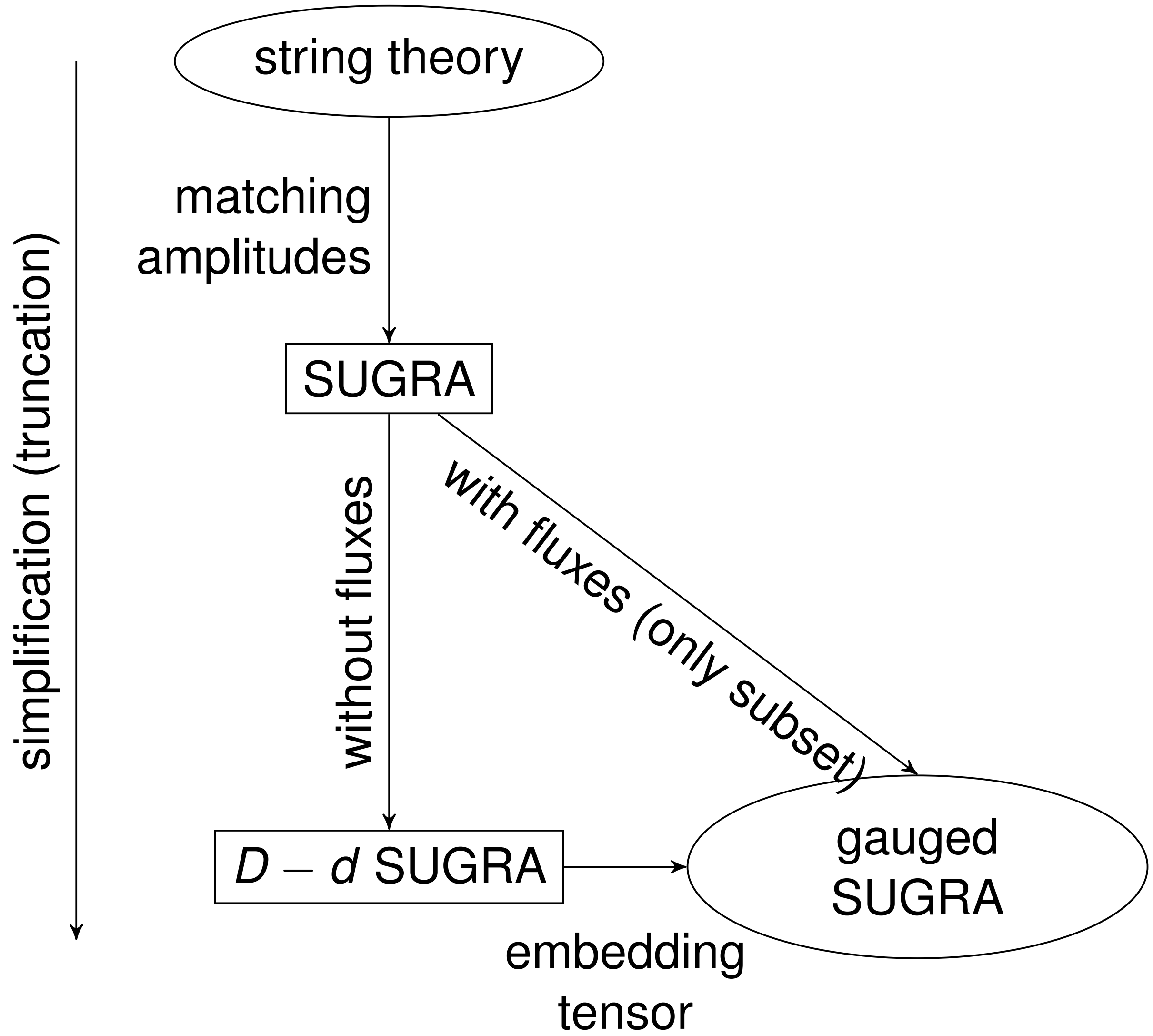
- Background fields violate the SC.

However the fluxes have to obey the closure constraint - consistent gauge algebra in the effective theory.

M. Grana, D. Marques, arXiv:1201.2924

Dimensional reduction of double field theory:

Generalized Scherk-Schwarz compactifications



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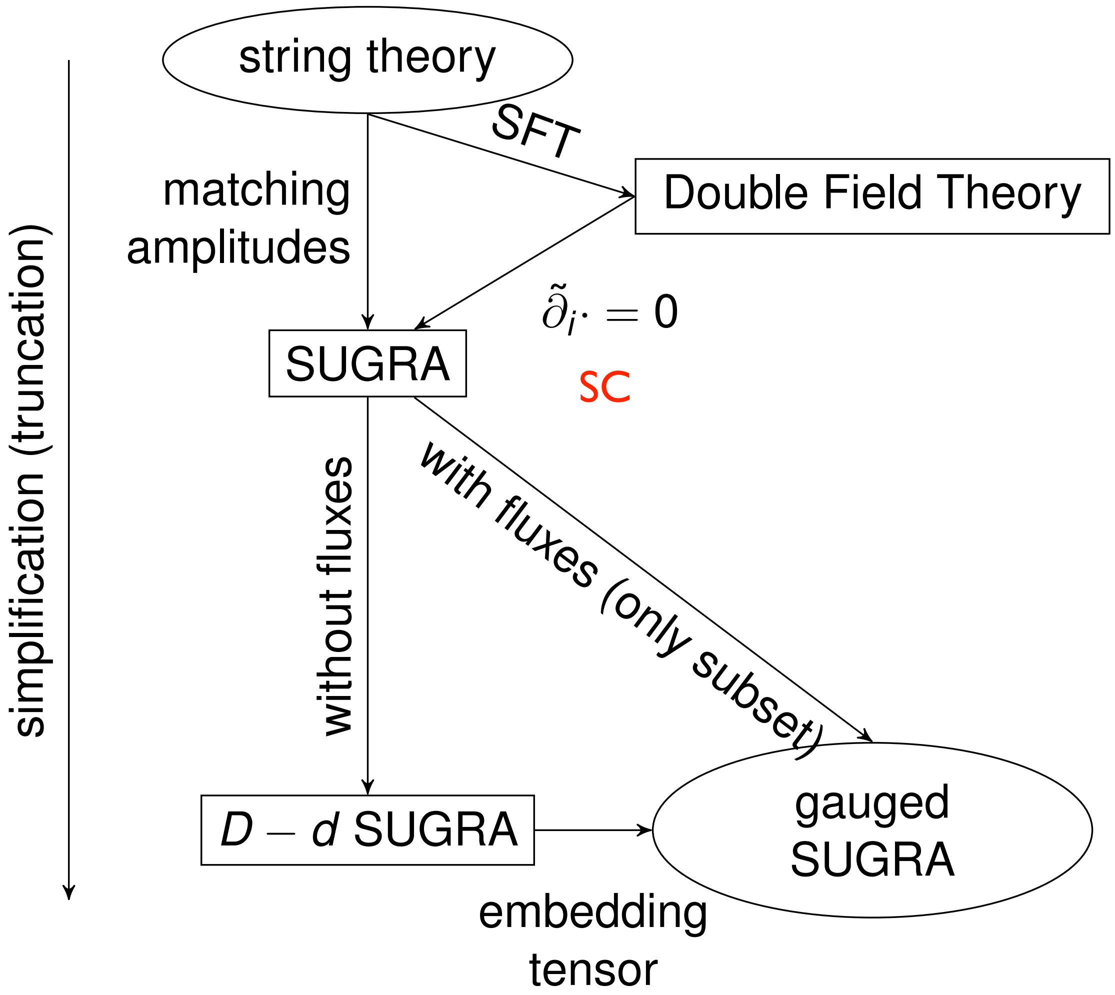

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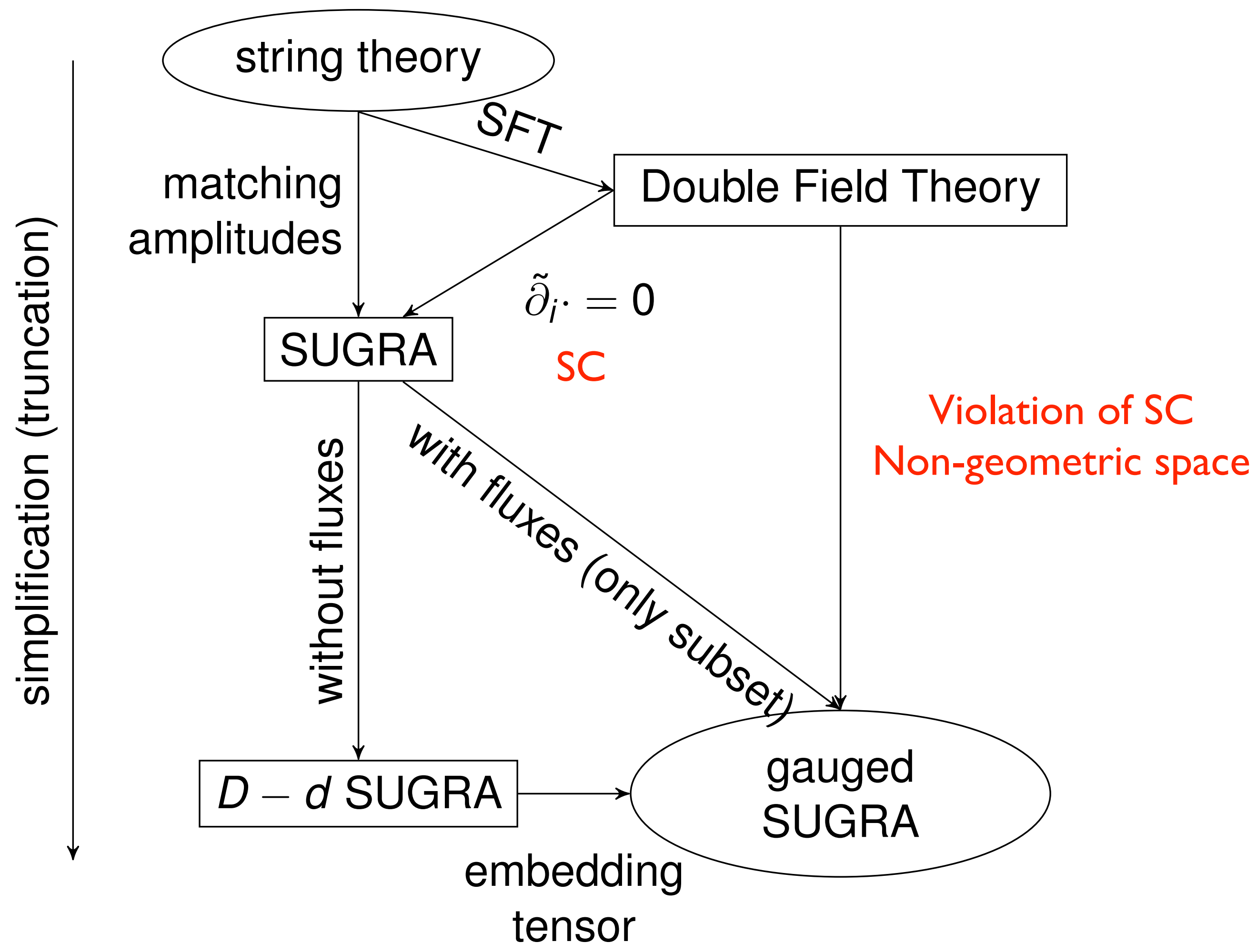

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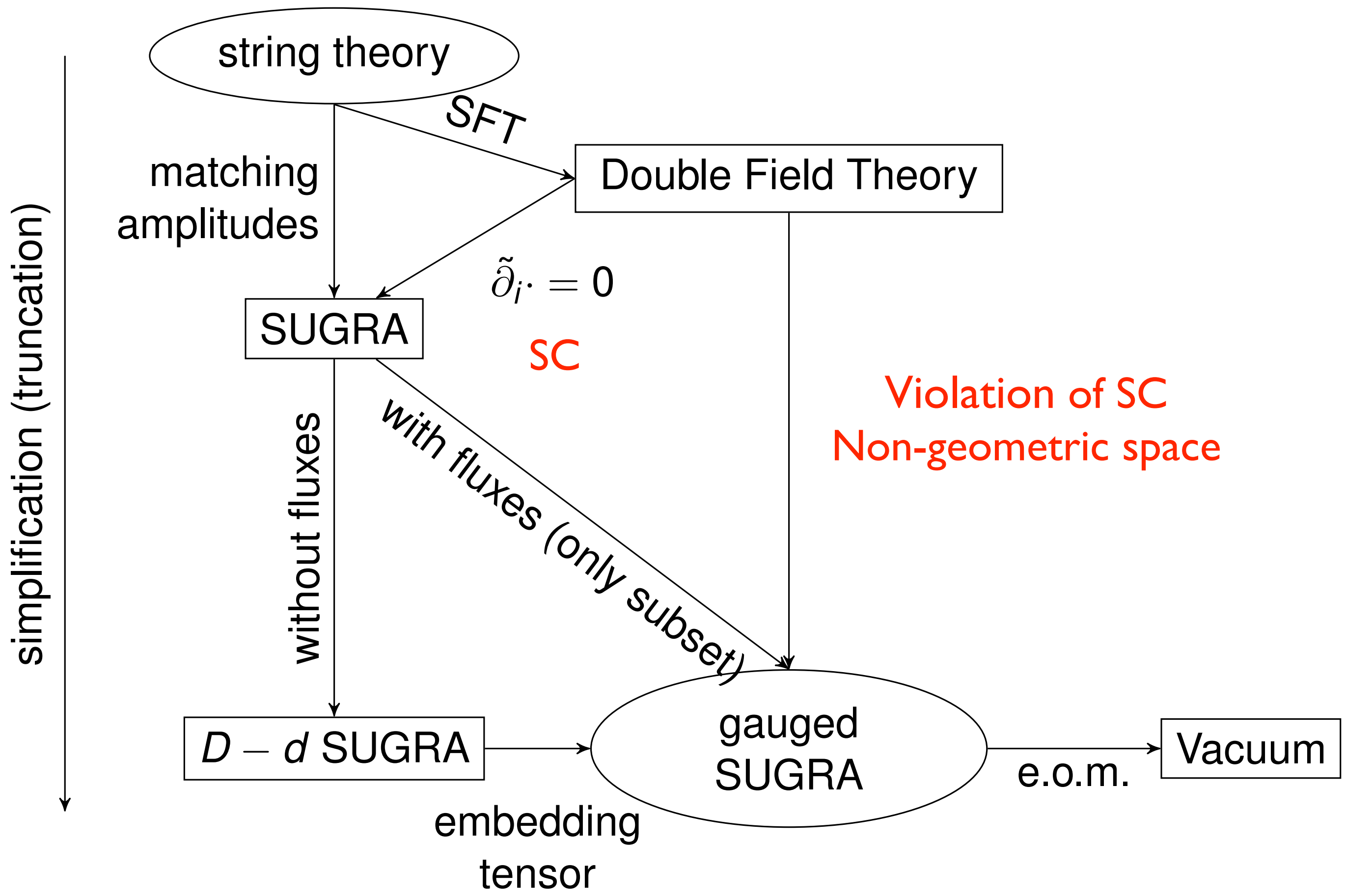

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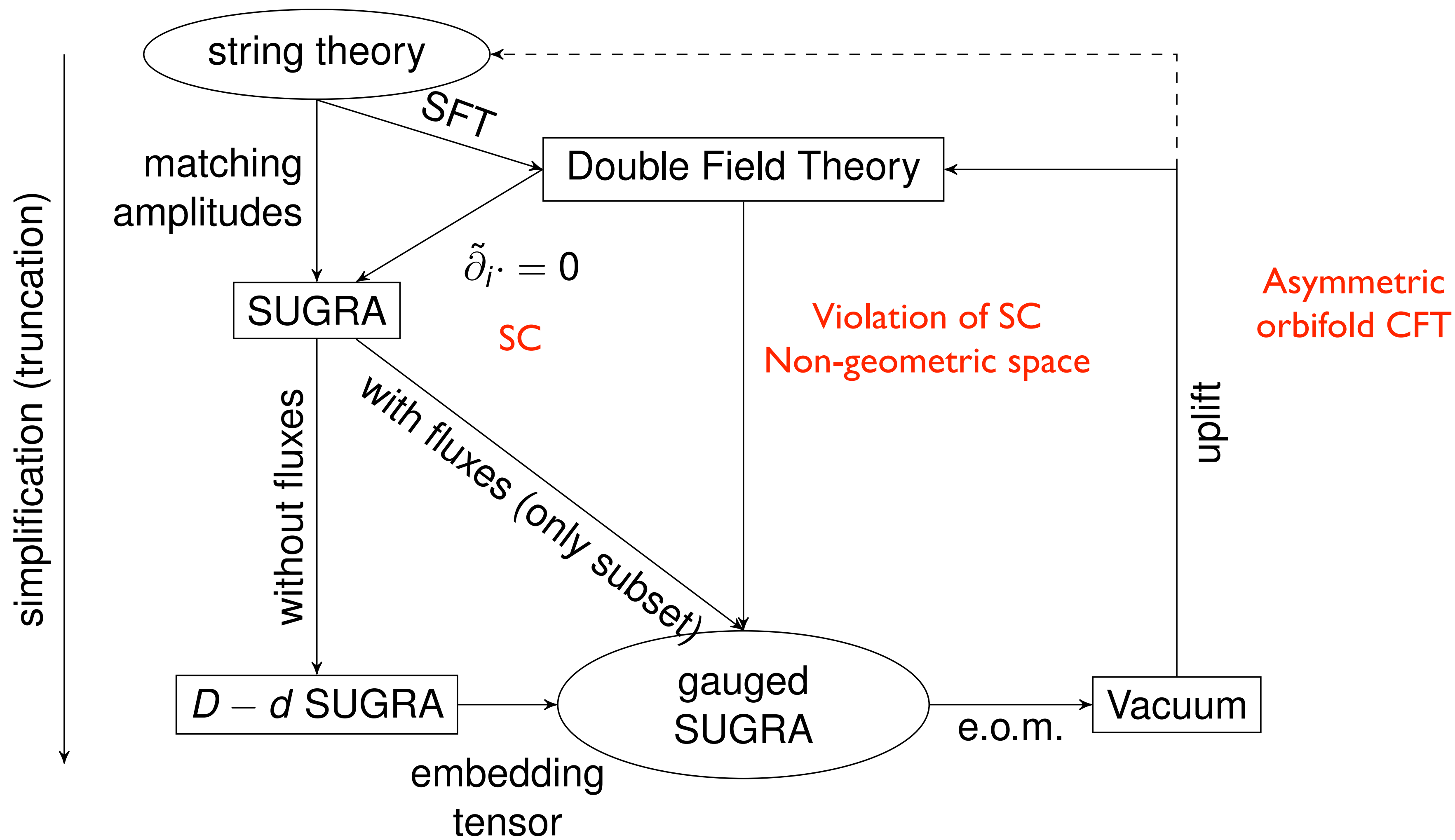


- Effective scalar potential:

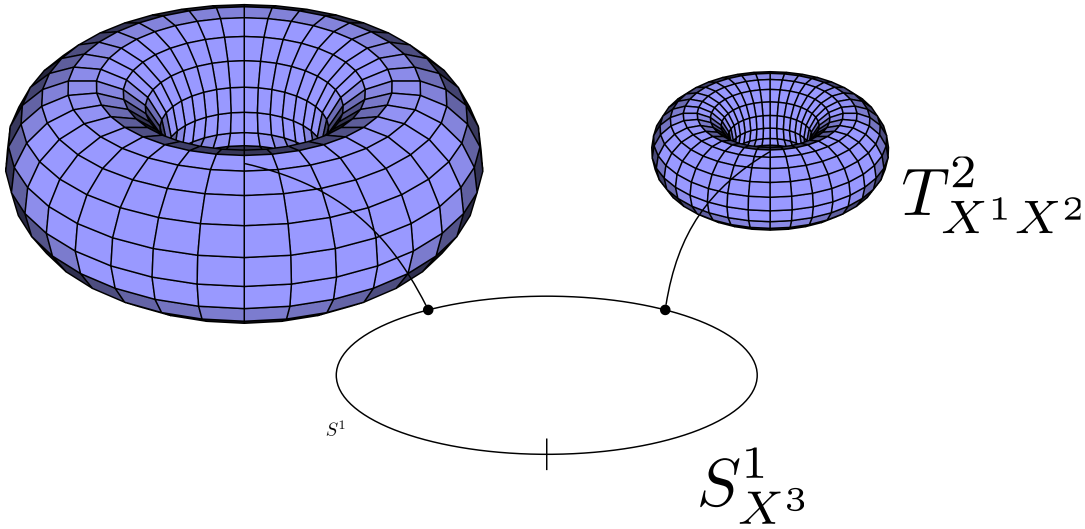
$$V = -\frac{1}{4}\mathcal{F}_I^{KL}\mathcal{F}_{JKL}\mathcal{H}^{IJ} + \frac{1}{12}\mathcal{F}_{IKM}\mathcal{F}_{JLN}\mathcal{H}^{IJ}\mathcal{H}^{KL}\mathcal{H}^{MN}$$

- $R_{MN} = 0 \quad \Rightarrow$ Minkowski vacua:

$$V = 0 \quad \text{and} \quad \mathcal{K}^{MN} = \frac{\delta V}{\delta \mathcal{H}_{MN}} = 0$$

This leads to additional conditions on the fluxes $\mathcal{F}_{IKM}$.

Simplest non-trivial solutions: d=3 dim. backgrounds:



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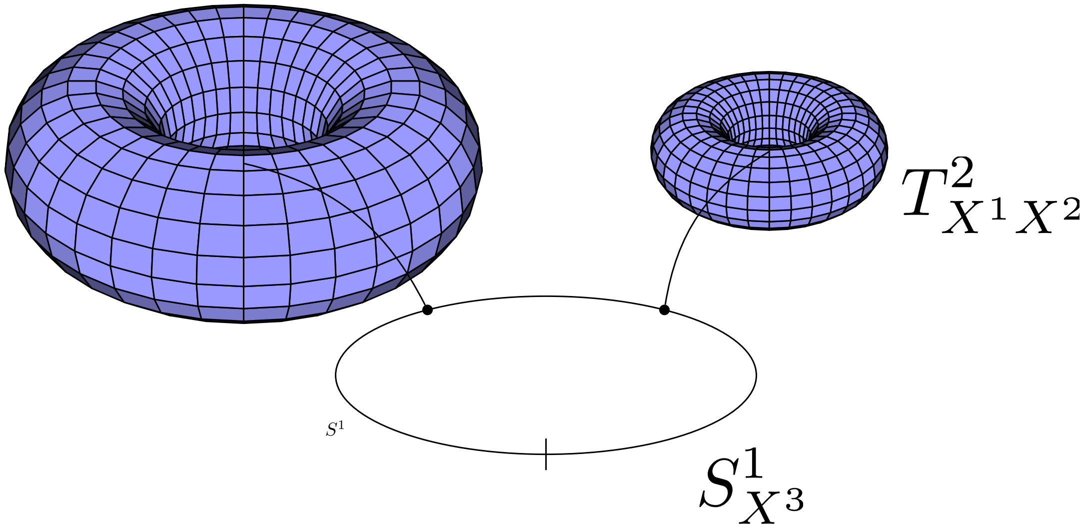

Parabolic background spaces: Single fluxes:

$$H_{123} \quad \text{or} \quad f^1_{23} \quad \text{or} \quad Q^{12}_3 \quad \text{or} \quad R^{123}$$

These backgrounds do not satisfy $R^{MN} = 0\,.$

- CFT: beta-functions are non-vanishing at quadratic order in fluxes.

- Effective scalar potential: no Minkowski minima (⇒ AdS)

Elliptic background spaces: Multiple fluxes:

These backgrounds do satisfy $R^{MN} = 0\,.$

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- Single elliptic geometric space (Solvmanifold):

$f^2_{13} = f^1_{23} = f \quad \Rightarrow$ Symmetric $\mathbb{Z}^L_4 \times \mathbb{Z}^R_4$ orbifold.

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- Single elliptic T-dual, non-geometric space:

$$H_{123} = Q_3^{12} = H$$

$$\Rightarrow \quad \text{Asymmetric} \quad \mathbb{Z}_4^L \times \mathbb{Z}_4^R \ \text{orbifold.}$$

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- Single elliptic T-dual, non-geometric space:

$$H_{123} = Q^{12}_3 = H$$

$\Rightarrow$ Asymmetric $\mathbb{Z}^L_4 \times \mathbb{Z}^R_4$ orbifold.

- Double elliptic, genuinely non-geometric space:

$$H_{123} = Q^{12}_3 = H\,, \quad f^2_{13} = f^1_{23} = f$$

$\Rightarrow$ Asymmetric $\mathbb{Z}^L_4$ orbifold.

Monodromy of double elliptic background:

c.s.: $$\tau(x_3) = \frac{\tau_0 \cos(f x_3) + \sin(f x_3)}{\cos(f x_3) - \tau_0 \sin(f x_3)}\,, \qquad f \in \frac{1}{4} + \mathbb{Z}\,,$$

Kahler: $$\rho(x_3) = \frac{\rho_0 \cos(H x_3) + \sin(H x_3)}{\cos(H x_3) - \rho_0 \sin(H x_3)}\,, \qquad H \in \frac{1}{4} + \mathbb{Z}\,.$$

$$\Longrightarrow \quad \tau(2\pi) = -\frac{1}{\tau(0)}\,, \qquad \rho(2\pi) = -\frac{1}{\rho(0)}$$

Monodromy of double elliptic background

Background satisfies strong constraint

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Corresponding Killing vectors of background:

$$K_{\hat{I}}^{\ \hat{J}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}(Hx^3 + f\tilde{x}^3) & \frac{1}{2}(Hx^2 + f\tilde{x}^2) & -\frac{1}{2}(fx^3 + H\tilde{x}^3) & \frac{1}{2}(fx^2 + H\tilde{x}^2) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Killing vectors do not satisfy strong constraint
However their algebra closes!

- There situations, where the strong constraint even for the background can be violated. - This seems to be the case for certain very asymmetric orbifolds. C. Condeescu, I. Florakis, C. Kounnas, D.Lüst, arXiv:1307.0999

$$\tau(x_3,\tilde{x}_3) = \frac{\tau_0 \cos(f_4 x_3 + f_2 \tilde{x}_3) + \sin(f_4 x_3 + f_2 \tilde{x}_3)}{\cos(f_4 x_3 + f_2 \tilde{x}_3) - \tau_0 \sin(f_4 x_3 + f_2 \tilde{x}_3)} \,, \qquad f_4, g_4 \in \frac{1}{8} + \mathbb{Z}$$

$$\rho(x_3,\tilde{x}_3) = \frac{\rho_0 \cos(g_4 x_3 + g_2 \tilde{x}_3) + \sin(g_4 x_3 + g_2 \tilde{x}_3)}{\cos(g_4 x_3 + g_2 \tilde{x}_3) - \rho_0 \sin(g_4 x_3 + g_2 \tilde{x}_3)} \,, \qquad f_2, g_2 \in \frac{1}{4} + \mathbb{Z}$$

Fluxes:

Parameter	Fluxes
f_4	$f, \tilde{f}$
f_2	$Q, \tilde{Q}$
g_4	H, Q
g_2	$\tilde{f}, R$

Asymmetric $\mathbb{Z}_4^L \times \mathbb{Z}_2^R$ orbifold with H, f ,Q,R-fluxes.

This (partially?) solves a so far existing puzzle between effective SUGRA and uplift/string compactification.

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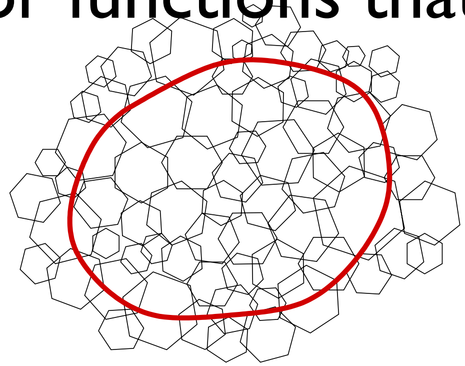

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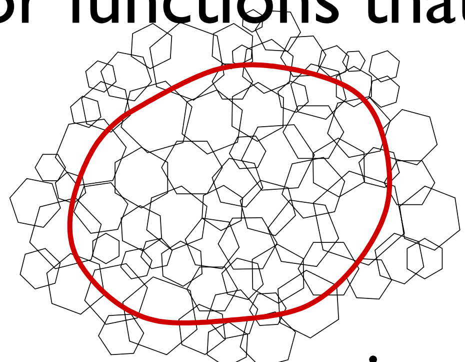

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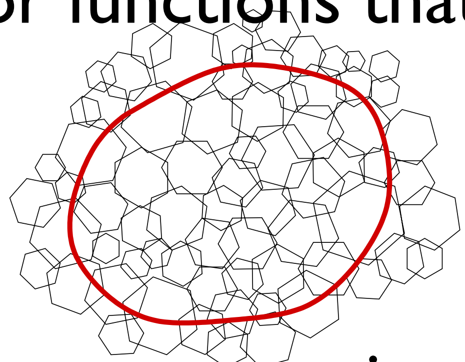

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- DFT allows for consistent reduction on non-geometric backgrounds that go beyond Sugra and also beyond generalized geometry $\Rightarrow$ interesting applications for cosmology.