Integration of NCQFT through Ward identities

Ricardo Kullock*

Workshop on Noncommutative Quantum Field Theory and Gravity, Corfu, Greece
*Faculty of Physics, University of Vienna
ricardo.kullock@univie.ac.at

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Noncommutative action

- The action is typicaly defined through the star product

$$S = \int dx \left(\frac{1}{2}\partial_i\phi \star \partial^j\phi + \frac{1}{2}\mu^2\phi \star \phi + V \right)$$

where

$$f \star g = m\left(e^{i\theta^{ij}\partial_i\partial_j}(f,g) \right),$$

- This is such that

$$[x_i, x_j]_\star = i\theta_{ij}$$

- Another way of writting the noncommutativity is

$$(f \star g)(x) = \int \frac{dk}{(2\pi)^4} \int dy\ f(x + \frac{1}{2}\theta \cdot k)g(x+y)e^{ik\cdot y}$$

- The action previously defined is non-renormalizable. Grosse and Wulkenhaar solved this by adding

$$\frac{\Omega^2}{2}(\tilde{x}_i\phi)\star(\tilde{x}^i\phi)$$

to the lagrangian, where $\tilde{x} = 2\theta^{-1}\cdot x$.

- The treatment simplifies if we use the matrix basis

$$(f_{mn}\star f_{kl})(x) = \delta_{nk} f_{ml}(x),$$

with

$$\phi(x) = \sum \phi_{mn} f_{mn}(x)$$

- In the matrix basis we have

$$\partial_i f(x) = [\tilde{x}_i, f(x)]$$

- The kinetic term becomes

$$x_i x_i \phi^2 - x_i \phi x_i \phi$$

- With the extra potential and the mass term we are left with

$$(x_i x_i + \mu^2)\phi^2$$

- We can choose an apropriate basis so that $(x_i x_i + \mu^2)$ leads to a harmonic oscillator spectrum

Ward Identity

- The action for $\lambda\phi^4$ is written as

$$S = \sum \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi),$$

with a cut-off in the summation, and $H_{nm} = (m + n + \mu^2)$. The $\lambda\phi^3$ has also a linear term $M\phi$, $M_{ab} = m\delta_{ab}$

- The partition function is

$$\mathcal{Z}[J] = N\int \mathcal{D}\phi\; e^{-S+Tr(\phi J)} = Ne^{(-V[\partial/\partial J])}e^{\frac{1}{2}<J,H^{-1}J>},$$

in the $\lambda\phi^4$, with $< J, H^{-1}J >= \sum_{p,q} J_{pq}H^{-1}_{pq}J_{qp}$, and

$$\mathcal{Z}[J] = Ne^{(-V[\partial/\partial J])}e^{\frac{1}{2}<(J+M),H^{-1}(J+M)>}$$

in the $\lambda\phi^3$.

- We consider the variation of $W = \ln \mathcal{Z}$ with respect to infenitesimal unitary transformations, which leads to

$$\left(\sum_n (H_{nb} - H_{an}) \frac{\partial^2}{\partial J_{nb} \partial J_{an}} + \left(J_{na} \frac{\partial}{\partial J_{nb}} - J_{bn} \frac{\partial}{\partial J_{an}}\right)\right) e^{(-V[\partial/\partial J])} e^{\frac{1}{2} < J, H^{-1} J >} = 0,$$

- The Identity can be further developed to

$$\sum_n \frac{\partial^2}{\partial J_{nb} \partial J_{an}} \mathcal{Z} = \sum_n \left(\delta_{ab} G_{an} + \frac{1}{b-a} \left(J_{bn} \frac{\partial}{\partial J_{an}} - J_{na} \frac{\partial}{\partial J_{nb}}\right)\right) \mathcal{Z}$$

Two-point funtion for $\lambda\phi^4$

- The two-point function is given by

$$G_{ab} = \frac{N}{\mathcal{Z}[0]} \left[e^{-V} \frac{\partial^2}{\partial J_{ab} \partial J_{ba}} e^{\frac{1}{2}<J, H^{-1} J>} \right]_{J=0},$$

- This leads to (zero genus)

$$G_{ab} = H^{-1}_{ab} + \lambda H^{-1}_{ab} G_{ab} \sum G_{an} + \lambda H^{-1}_{ab} \sum \frac{G_{bn} - G_{ab}}{a - n}$$

- Notice it is not explicitly symmetric.

The $\lambda\phi^3$

- The two-point function can be calculated by

$$G_{ab} = \frac{N}{\mathcal{Z}[0]}\left[e^{-V}\frac{\partial}{\partial J_{ab}}H^{-1}_{ab}J_{ab}e^{\frac{1}{2}<J,H^{-1}J>}\right]_{J=0} =$$

$$G_{ab} = H^{-1}_{ab} + \frac{NH^{-1}_{ab}}{\mathcal{Z}[0]}\left[e^{-V}J_{ab}\frac{\partial}{\partial J_{ab}}e^{\frac{1}{2}<J,H^{-1}J>}\right]_{J=0} =$$

$$G_{ab} = H^{-1}_{ab} + \frac{H^{-1}_{ab}}{\mathcal{Z}[0]}\left[\frac{\partial(-V)}{\partial\phi_{ab}}\left[\frac{\partial}{\partial J}\right]\frac{\partial}{\partial J_{ab}}\mathcal{Z}[J]\right]_{J=0}$$

- The derivative of V will give

$$\lambda \sum \frac{\partial}{\partial J_{bn}} \frac{\partial}{\partial J_{na}} \mathcal{Z}[J].$$

- Here enters the Ward identity, so that we have, for $a \neq b$

$$G_{ab} = H_{ab}^{-1} + \frac{(-\lambda) H_{ab}^{-1}}{(a-b)\mathcal{Z}[0]} \sum \left[\frac{\partial}{\partial J_{ab}} \left(J_{an} \frac{\partial}{\partial J_{bn}} - J_{nb} \frac{\partial}{\partial J_{na}} \right) \mathcal{Z}[J] \right]_{J=0}$$

  So that we have

$$G_{ab} = H_{ab}^{-1} + \frac{\lambda H_{ab}^{-1}}{(a-b)} (G_a - G_b)$$

  where G_a is the one point function.

- The equation is no longer only on the two-point function, but it is explicitly symmetric.

- We can try the same for the one-point function:

$$G_a = \frac{N}{\mathcal{Z}[0]}\left[e^{-V}\frac{\partial}{\partial J_{aa}}e^{\frac{1}{2}<J,H^{-1}J>}\right]_{J=0}$$

- Now the linear term shows,

$$G_a = \frac{N}{\mathcal{Z}[0]}\left[e^{-V}H^{-1}_{aa}(J_{aa}+m(\lambda,\Lambda))e^{\frac{1}{2}<J,H^{-1}J>}\right]_{J=0}$$

- The Ward identity has a singularity, so in this case we have only

$$G_a = \frac{N}{\mathcal{Z}[0]}\left[H^{-1}_{aa}(-\lambda\sum\frac{\partial}{\partial J_{an}}\frac{\partial}{\partial J_{na}}+m)\mathcal{Z}[J]\right]_{J=0}$$

- Thus

$$G_a = H^{-1}_{aa}\left(m-\lambda\sum G_{an}\right)$$

- We can also take two currents down in the two-point function calculation

$$G_{ab} = H_{ab}^{-1} + H_{ab}^{-2}\left[e^{-V} J_{ab} J_{ba} \mathcal{Z}[J]\right]_{J=0}$$

- This leads to

$$G_{ab} = H_{ab}^{-1} - \lambda H_{ab}^{-2}(G_a + G_b) + \frac{\lambda^2 H_{ab}^{-2}}{a-b}\sum (G_{an} - G_{bn})$$

- But this equation is not linearly independent from the two others.
- Taking G_{a0} and G_{0b} we can write

$$\frac{\lambda H_{ab}^{-2}}{a-b}(aG_a - bG_b) + \frac{H_{ab}^{-2}}{a-b}\left(aH_{a0}^2 G_{a0} - bH_{0b}^2 G_{0b}\right) - H_{ab}^{-1} = \frac{\lambda^2 H_{ab}^{-2}}{a-b}\sum (G_{an} - G_{bn})$$

And so

$$G_{ab} = \frac{\lambda H_{ab}^{-2}}{a-b}(bG_a - aG_b) + \frac{H_{ab}^{-2}}{a-b}\left(aH_{a0}^2 G_{a0} - bH_{0b}^2 G_{0b}\right)$$

- For the three-point funtion we find

$$G_{abc} = -\frac{\lambda H^{-1}_{bc}}{b-c}\left(G_{ab} - G_{ac}\right) = -\frac{\lambda H^{-1}_{ca}}{c-a}\left(G_{bc} - G_{ba}\right) = -\frac{\lambda H^{-1}_{ab}}{a-b}\left(G_{ca} - G_{cb}\right)$$

- Similarly we have the four-point function

$$G_{abcd} = -\frac{\lambda H^{-1}_{cd}}{c-d}\left(G_{abc} - G_{abd}\right)$$

- Inserting the first equation for the two-point in the one-point funtion equation, we have

$$G_a = \frac{m}{2a+\mu^2} - \frac{1}{2a+\mu^2}\sum \frac{1}{a+n+\mu^2} - \frac{\lambda}{2a+\mu^2}\sum \frac{G_a - G_n}{(a+n+\mu^2)(a-n)}$$

- This leads to the singular integral equation of the Carleman type

$$\alpha(a)G(a) + \int_0^{\Lambda} dn \frac{G(n)}{(a-n)(a+n+1)} = f(a),$$

or

$$\alpha(\tilde{a})G(\tilde{a}) + \int_{1/2}^{\Lambda+1/2} d\tilde{n} \frac{G(\tilde{n})}{\tilde{a}^2 - \tilde{n}^2} = f(\tilde{a})$$

- The equation has a non-degenerate kernel. It can be solved by the transformation $n^2 = x$.

- Using the equation for the three-point function we can also have

$$G_{ab} = H_{ab}^{-1} - \sum G_{abn} = H_{ab}^{-1} + \lambda^2 \sum \frac{H_{bn}^{-1}}{b-n} (G_{ab} - Gan)$$

or

$$G_{ab} \left(1 - \lambda^2 \sum \frac{H_{bc}^{-1}}{b-c}\right) = H_{ab}^{-1} - \lambda^2 \sum \frac{H_{bn}^{-1}}{b-n} G_{an}$$

- This allows us to study the mass renormalization μ_R

$$G_{00} \left(1 - \lambda^2 \sum \frac{H_{0n}^{-1}}{b-n}\right) = \mu^{-2} + \lambda^2 \sum \frac{H_{0n}^{-1}}{n} G_{0n}$$

- We have also the renormalization condition $G_a = 0$

$$m = \lambda \sum G_{0n}$$

- Both cnditions are given by G_{0n}

Thank You!