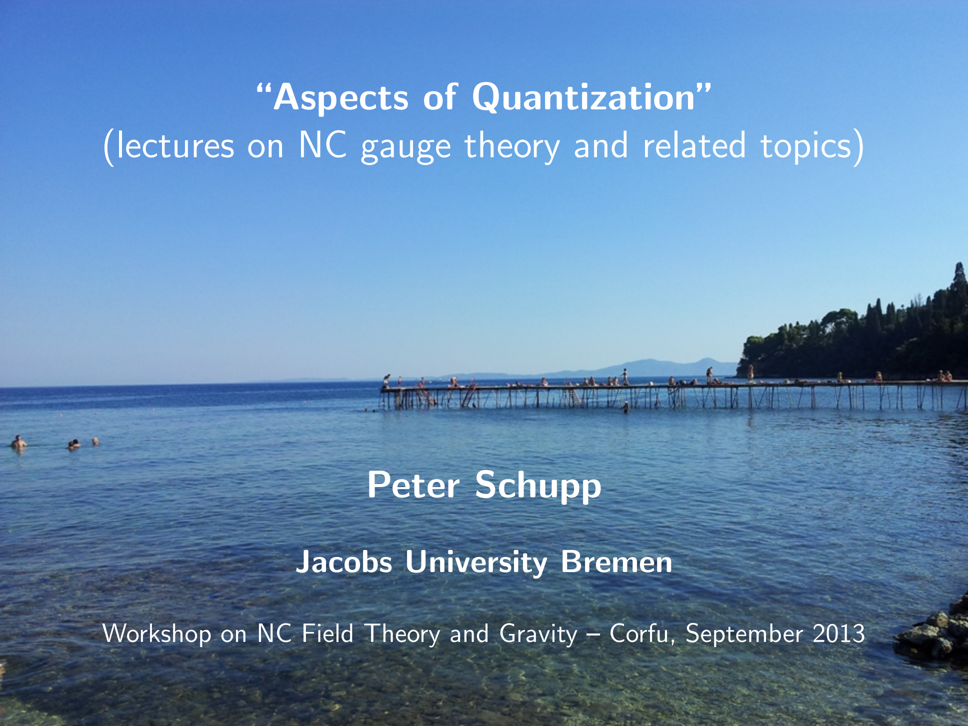


"Aspects of Quantization"

(lectures on NC gauge theory and related topics)

Peter Schupp

Jacobs University Bremen

Workshop on NC Field Theory and Gravity – Corfu, September 2013

Outline

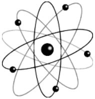
- various quantization schemes
- noncommutative gauge theory
- Seiberg-Witten map
- strings in non-geometric backgrounds
- membrane model and quantization
- generalized geometry
- Nambu-Poisson structures
- effective actions for strings and branes

start 2nd lecture: slide 30, start 3rd lecture: slide 60

Peter Schupp, September 19, 2013
https://www.jacobs-university.de/directory/pschupp
http://www.models-of-gravity.org/

Motivation

Spacetime quantization



Heuristic argument: quantum + gravity

"The gravitational field generated by the concentration of energy required to localize an event in spacetime should not be so strong as to hide the event itself to a distant observer."

→ fundamental length scale, spacetime uncertainty

$$\Delta x \geq \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33}\text{cm}$$

⇒ need to generalize usual notions of smooth Riemannian geometry

Noncommutative geometry: model quantum geometry of spacetime

Noncommutative spacetime

Noncommutative geometry

Idea: consider the algebra of functions on a manifold and make it noncommutative; “points” $\sim$ irreducible representations

- Gelfand–Naimark:
  spacetime manifold $\rightarrow$ noncommutative algebra
- Serre–Swan:
  vector bundles $\rightarrow$ projective modules
- Connes: noncommutative differential geometry
  (Dirac operator, spectral triple, ...)

Noncommutative coordinates

Heuristic model of quantum geometry (e.g. thought of as induced by quantum gravitational effects):

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \qquad \Leftrightarrow \qquad \Delta x^\mu \cdot \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|$$

Macroscopic and microscopic non-commutativity

Noncommutativity in electrodynamics and string theory

- electron in constant magnetic field $\vec{B} = B\hat{e}_z$:

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - e\dot{\vec{x}} \cdot \vec{A} \qquad \text{with} \qquad A_i = -\frac{B}{2}\epsilon_{ij}x^j$$

$$\lim_{m \to 0} \mathcal{L} = e\frac{B}{2}\dot{x}^i\epsilon_{ij}x^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \frac{2i}{eB}\epsilon^{ij}$$

- bosonic open strings in constant B-field

$$S_\Sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\partial_a x^i \partial^a x^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j\right)$$

in low energy limit $g_{ij} \sim (\alpha')^2 \to 0$:

$$S_{\partial\Sigma} = -\frac{i}{2}\int_{\partial\Sigma} B_{ij}x^i\dot{x}^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \left(\frac{i}{B}\right)^{ij}$$

C-S Chu, P-M Ho (1998); V Schomerus (1999); Seiberg, Witten

Weyl quantization ($\theta = \text{const.}$)

consider $\theta = \text{const.}$, symmetric ordering

$$\begin{array}{rcl} \text{commutative functions} & \longrightarrow & \text{NC operators} \\ x^\mu x^\nu = x^\nu x^\mu & & [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \\ x^\mu & \longleftrightarrow & \hat{x}^\mu \\ x^\mu x^\nu & \longleftrightarrow & \frac{1}{2}(\hat{x}^\mu \hat{x}^\nu + \hat{x}^\nu \hat{x}^\mu) \\ \cdots & & \cdots \\ f(x) = \int d^n k \; \tilde{f}(k) e^{ix\cdot k} & \longleftrightarrow & \widehat{f(x)} = \int d^n k \; \tilde{f}(k) e^{i\hat{x}\cdot k} \end{array}$$

evaluate product of operators using BCH formula

$$\begin{aligned} \widehat{f(x)}\widehat{g(x)} &= \int d^n k d^n k' \; \tilde{f}(k)\tilde{g}(k') e^{i\hat{x}\cdot k} e^{i\hat{x}\cdot k'} \\ &= \int d^n k d^n k' \; \tilde{f}(k)\tilde{g}(k') e^{i\hat{x}\cdot(k+k') - \frac{1}{2} k_\mu k'_\nu [\hat{x}^\mu, \hat{x}^\nu]} =: \widehat{f \star g} \end{aligned}$$

Weyl quantization ($\theta = \text{const.}$)

Moyal-Weyl star product

$$(f \star g)(x) = \cdot\left[e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}(f\otimes g)\right]$$
$$\equiv \sum \frac{1}{m!}\left(\frac{i}{2}\right)^m \theta^{\mu_1\nu_1}\ldots\theta^{\mu_m\nu_m}(\partial_{\mu_1}\ldots\partial_{\mu_m}f)(\partial_{\nu_1}\ldots\partial_{\nu_m}g)$$

partials commute, $[\partial_\mu, \partial_\nu] = 0 \;\Rightarrow$ star product $\star$ is associative

BCH quantization: works also for θ linear or quadratic in x.

Integral formula, non-local star product:

$$(f \star g)(x) = f(x + \frac{i}{2}\theta\cdot\partial)g(x)$$
$$\equiv \int d^n y d^n k \; f(x + \frac{1}{2}\theta\cdot k)g(y)e^{ik(y-x)}$$

translation invariance of integral $\;\Rightarrow$ star product $\star$ is associative

Twist quantization

Drinfel'd twist for Hopf algebra $H(\Delta, S, \epsilon, \cdot)$

$$F \equiv \sum F^{(1)} \otimes F^{(2)} \in H \otimes H$$

with $(\epsilon \otimes id)F = (id \otimes \epsilon)F = 1$ and cocycle condition

$$(F \otimes 1)\Delta_1 F = (1 \otimes F)\Delta_2 F$$

maps H to a new Hopf algebra $H_F(\Delta_F, S_F, \epsilon, \cdot)$, with

$$\Delta_F = F \circ \Delta \circ F^{-1} \ , \quad S_F = \gamma \circ S \circ \gamma^{-1} \ ,$$

where $\gamma = \sum F^{(1)} S F^{(2)} \cdot$.

An H-module algebra A is deformed (quantized) as:

$$f \star g = \sum \bar{F}^{(1)} f \cdot \bar{F}^{(2)} g \ , \qquad F^{-1} \equiv \sum \bar{F}^{(1)} \otimes \bar{F}^{(2)}$$

For the Moyal-Weyl star product: $F = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$.

Deformation quantization $\theta(x) \rightsquigarrow \star$

Let A be the algebra of functions on a finite-dimensional C^∞-manifold. A star product $\star$ is an associative product on $A[[\hbar]]$,

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \dots \ ,$$

with a formal deformation parameter $\hbar$ and bi-differential operators B_n.

There is a natural gauge symmetry

$$\star \mapsto \star' \ , \quad f \star' g = D^{-1}(Df \star Dg) \ ,$$

with $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$.

Up to gauge equivalence

$$f \star g = f \cdot g + \frac{i}{2} \sum \theta^{ij} \, \partial_i f \cdot \partial_j g - \frac{\hbar^2}{4} \sum \theta^{ij} \theta^{kl} \, \partial_i \partial_k f \cdot \partial_j \partial_l g \\ - \frac{\hbar^2}{6} \left(\sum \theta^{ij} \partial_j \theta^{kl} \, (\partial_i \partial_k f \cdot \partial_l g - \partial_k f \cdot \partial_i \partial_l g) \right) + \dots \ ,$$

where $\theta = \theta^{ij} \partial_i \otimes \partial_j$ is a Poisson bi-vector.

Deformation quantization

Kontsevich formality and star product

U_n maps n k_i-multivector fields to a $(2 - 2n + \sum k_i)$-differential operator

$$U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) = \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n) \ ,$$

where the sum is over all possible diagrams with weight

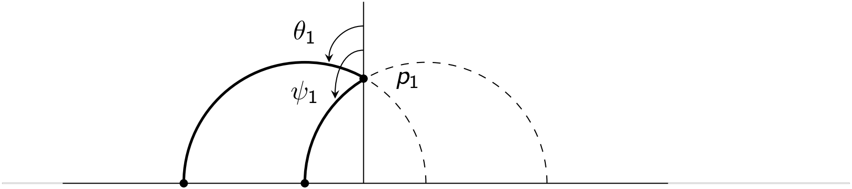
$$w_\Gamma = \frac{1}{(2\pi)^{\sum k_i}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_i^1} \wedge \cdots \wedge \mathrm{d}\phi^h_{e_i^{k_i}} \right) \ .$$

The star product for a given bivector θ is:

$$f \star g = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} \, U_n(\Theta, \ldots, \Theta)(f, g)$$

Deformation quantization

Example constant θ:
The graphs and hence the integrals factorize. The basic graph



yields the weight

$$w_{\Gamma_1} = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\phi_1 = \frac{1}{(2\pi)^2} \left[\frac{1}{2} (\psi_1)^2 \right]_0^{2\pi} = \frac{1}{2}$$

and the star product turns out to be the Moyal-Weyl one:

$$f \star g = \sum \frac{(i\hbar)^n}{n!} \left(\frac{1}{2} \right)^n \theta^{\mu_1 \nu_1} \dots \theta^{\mu_n \nu_n} (\partial_{\mu_1} \dots \partial_{\mu_n} f)(\partial_{\nu_1} \dots \partial_{\nu_n} g)$$

Deformation quantization

Formality condition

The U_n define a quasi-isomorphisms of L_∞-DGL algebras and satisfy

$$\mathrm{d}.U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n)+\frac{1}{2}\sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\ldots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}}\varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J})\left[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}})\,,\,U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\right]_{\mathrm{G}}$$
$$=\sum_{i<j}(-1)^{\alpha_{ij}}\,U_{n-1}\big([\mathcal{X}_i,\mathcal{X}_j]_{\mathrm{S}},\mathcal{X}_1,\ldots,\widehat{\mathcal{X}}_i,\ldots,\widehat{\mathcal{X}}_j,\ldots,\mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

Kontsevich (1997)

This implies in particular $\mathrm{d}_\star\Phi(\Theta)=\mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\Theta\Theta)$, i.e.

$$\star\ \text{associative}\quad\Leftrightarrow\quad\theta\ \text{Poisson}$$

Hilbert space representations

Note that $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ with constant θ, are ordinary canonical commutation relations in disguise.

To study representations by self-adjoint operators acting on a Hilbert space, we should consider $U^\mu(t) := \exp(it\hat{x}^\mu)$, the Heisenberg group, Weyl braiding relations

$$U^\mu(t)V^\nu(t') = e^{itt'\theta^{\mu\nu}}U^\nu(t')U^\mu(t) \ ,$$

representations by Sylvester's clock and shift matrices, the Stone-von Neumann theorem etc.

Examples

Noncommutative BTZ black hole

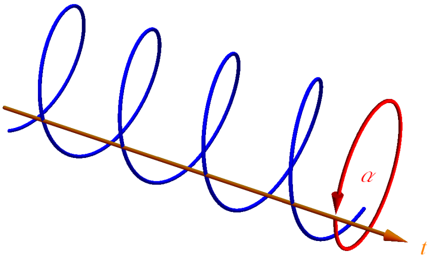
2+1 dimensions: ϕ, ρ, t; angle-time noncommutativity $[t, \phi] = i\tau$

$$[e^{i\phi}, t] = \tau e^{i\phi}$$

Irreducible representations (labeled by $\alpha \in [0, \tau)$)

$$t|n, \alpha\rangle = (n\tau + \alpha)|n, \alpha\rangle$$

Dolan, Gupta, Stern



Examples

Non-commutative Gravity   [,] ≠ 0

- general relativity on noncommutative spacetime
- theoretical laboratory for physics beyond QFT/GR

problem: fundamental length incompatible with spacetime symmetries

⇒ The symmetry (Hopf algebra) must be twisted, e.g.:

$$\mathcal{F} = \exp\left(-\frac{i}{2}\theta^{ab}\; V_a \otimes V_b\right) \;, \qquad [V_a, V_b] = 0$$

$$\Delta_\star(f) = \mathcal{F}\Delta(f)\mathcal{F}^{-1} \qquad f \star g = \bar{\mathcal{F}}(f \otimes g)$$

twisted tensor calculus, deformed Einstein equations

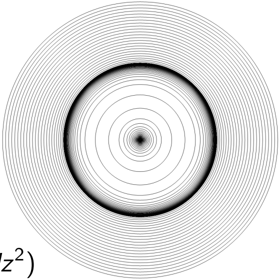
Aschieri, Blohmann, Dimitrijevic, Meyer, PS, Wess (2005)

Examples

Exact NC black hole solution with rotational symmetry

star product (twist): $[x_i \overset{\star}{,} x_j] = 2i\lambda\epsilon_{ijk}x_k$,

$$V \star W = VW + \sum_{n=1}^{\infty} B(n, \frac{\rho}{\lambda})\mathcal{L}^n_{\xi_+} V \, \mathcal{L}^n_{\xi_-} W$$



metric in isotropic coordinates

$$ds^2 = -\left(1 - \frac{a}{\rho}\right) dt^2 + \frac{r^2}{\rho^2}\left(dx^2 + dy^2 + dz^2\right)$$

with $r = (\rho + a/4)^2/\rho$, $a = 2M$, $\rho^2 = x^2 + y^2 + z^2$

$\Rightarrow$ quantized, quasi 2-dimensional "onion"-spacetime:

$$\boxed{\rho = 2j\lambda = n\lambda; \quad n = 0, 1, 2, \ldots}$$

PS, Solodukhin (2008)

Quantization and coherent states

The "black hole" star product, can be computed using coherent states:

Spin coherent state

$$|\Omega\rangle = \mathcal{R}_\Omega |j, j\rangle, \quad \mathcal{R}_\Omega \in SU(2)/U(1); \qquad (2j+1)\int \frac{d\Omega}{4\pi} |\Omega\rangle\langle\Omega| = 1_j$$

Star product

For $A(\Omega) := \langle\Omega|A|\Omega\rangle$ and $B(\Omega) := \langle\Omega|B|\Omega\rangle$ define:

$$A(\Omega) \star B(\Omega) = \langle\Omega|AB|\Omega\rangle$$

Coherent states and entropy

Von Neumann entropy

$$S_Q(\rho) = -\operatorname{tr} \rho \ln \rho = -(2j+1) \int \frac{d\Omega}{4\pi} \rho(\Omega) \star \ln_\star \rho(\Omega)$$

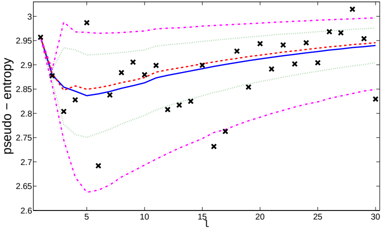
Now "switch off" (or ignore) noncommutativity $\Rightarrow$

Wehrl entropy

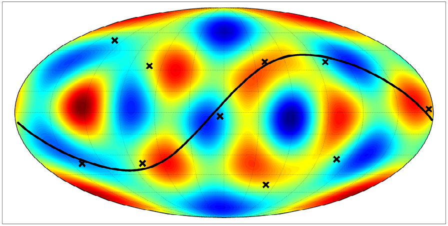
$$\begin{aligned} S_W(\rho) &= -(2j+1) \int \frac{d\Omega}{4\pi} \rho(\Omega) \ln \rho(\Omega) \\ &\geq -(2j+1) \int \frac{d\Omega}{4\pi} |\langle \Omega | \Psi \rangle|^2 \ln |\langle \Omega | \Psi \rangle|^2 \\ &> 0 \quad \text{even for pure states} \end{aligned}$$

Application: anisotropy of CMB

Coherent state analysis of cosmic microwave background



CMB at $j = 5$ with multi-pole vectors ($\sim$ coherent states)



Example of quantization guided by physics

Non-associativity in electrodynamics with magnetic sources

A charged particle (charge e, mass m) experiences a magnetic field $\vec{B}$ (with sources) only via the Lorentz force $\dot{\vec{p}} = \frac{e}{m}\vec{p} \times \vec{B}$.

The Hamiltonian $H = \frac{1}{2m}\vec{p}^2$ is purely kinetic, but $\dot{\vec{p}} = i[H, \vec{p}]$ must imply the Lorentz force $\Rightarrow$ momenta cannot commute

$$[r^i, r^j] = 0 \ , \quad [r^i, p^j] = i\hbar\delta^{ij} \quad [p^i, p^j] = ie\epsilon^{ijk}B^k$$

$\Rightarrow$ translations are generated by $U(\vec{a}) = \exp(i\vec{a} \cdot \vec{p})$

$$U(\vec{a}_1)U(\vec{a}_2) = e^{-ie\Phi_{12}}U(\vec{a}_1+\vec{a}_2) \ , \quad \Phi_{12} = \text{flux through triangle } (\vec{a}_1, \vec{a}_2)$$

(non)associativity:

$$[U(\vec{a}_1)U(\vec{a}_2)]U(\vec{a}_3) = e^{-ie\Phi_{123}}U(\vec{a}_1)[U(\vec{a}_2)U(\vec{a}_3)] \ ,$$

where Φ_{123} is the flux through the tetrahedron $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$.

infinitesimally:

$$[p^1, [p^2, p^3]] + [p^2, [p^3, p^1]] + [p^3, [p^1, p^2]] = e\nabla \cdot \vec{B}$$

$\vec{p}$ can be realized as a linear operator $-i\nabla - e\vec{A}$ only in the associative regime, i.e.:

- for $\nabla \cdot B = 0$: no sources, no flux, associativity;
- for $\nabla \cdot B \neq 0$: non-associativity, unless $\Phi_{123}/(2\pi)$ is an integer.[1]

In the latter case $\nabla \cdot \vec{B}$ must consist of delta functions, so that the total flux does not change continuously when the $\vec{a}$'s change $\Rightarrow$ monopoles furthermore, the Dirac quantization condition must be satisfied

Jackiw (1985)

[1] Taking the into account that the electron is spin 1/2 fermion with double-valued wave function, this becomes $\Phi_{123}/\pi \in \mathbb{Z}$.

Covariant coordinates

Noncommutative gauge transformation of a field:

$$\delta\widehat{\Psi} = i\hat{\Lambda} \star \Psi$$

Note:

$$\delta(x^\mu \star \widehat{\Psi}) = ix^\mu \star \hat{\Lambda} \star \widehat{\Psi} \neq i\hat{\Lambda} \star x^\mu \star \widehat{\Psi}$$

$\Rightarrow$ introduce covariant coordinates $X^\mu = \mathcal{D}_A(x^\mu)$ and more generally covariant functions $\mathcal{D}_A(f(x))$, s.t.

$$\delta\mathcal{D}_A(f(x)) = i[\hat{\Lambda} \stackrel{\star}{,} \mathcal{D}_A(f(x))]$$

Noncommutative gauge theory

Covariant coordinate, NC gauge potential:

$$X^\mu = \mathcal{D}_A(x^\mu) = x^\mu + \theta^{\mu\nu}\hat{A}_\nu$$

NC (abelian) gauge transformation:

$$\delta X^\mu = i[\hat{\Lambda} \stackrel{\star}{,} X^\mu] \quad \rightarrow \quad \delta\hat{A}_\nu = \partial_\nu\hat{\Lambda} + i[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\nu]$$

NC field strength:

$$[X^\mu \stackrel{\star}{,} X^\nu] \quad \rightarrow \quad \widehat{F}_{\mu\nu} = \partial_\mu\widehat{A}_\nu - \partial_\nu\widehat{A}_\mu - i[\widehat{A}_\mu \stackrel{\star}{,} \widehat{A}_\nu], \quad \hat{\delta}\widehat{F}_{\mu\nu} = i[\hat{\Lambda} \stackrel{\star}{,} \widehat{F}_{\mu\nu}]$$

Covariant derivative:

$$X^\mu \star \widehat{\Psi} - \widehat{\Psi} \star x^\mu \quad \rightarrow \quad \widehat{D}_\mu\widehat{\Psi} = \partial_\mu\widehat{\Psi} - i\widehat{A}_\mu \star \widehat{\Psi}$$

$$\delta\widehat{\Psi} = i\hat{\Lambda} \star \widehat{\Psi} \quad \rightarrow \quad \delta(\widehat{D}_\mu\widehat{\Psi}) = i\hat{\Lambda} \star \widehat{D}_\mu\widehat{\Psi}$$

Madore, Schraml, PS, Wess (2000)

Noncommutative gauge theory

NC generalization of the Maxwell-Dirac action

$$\widehat{S} = \int d^4x \left(-\frac{1}{4}\, Tr(\widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}) + \overline{\widehat{\Psi}} \star i\widehat{\not{D}}\widehat{\Psi} \right)$$

written in units with coupling constant $e \equiv 1$ and

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu \stackrel{\star}{,} \widehat{A}_\nu]\,.$$

The action is invariant under NC gauge transformations, because of the $\star$-trace property of the integral

$$\int d^4x\, f \star g = \int d^4x\, g \star f = \int d^4x\, f\, g\,.$$

Charge quantization

Physical fields and gauge parameters

$$\hat{A}_\mu = Q\,\hat{a}_\mu(x)\,, \qquad \hat{\Lambda} = Q\,\hat{\lambda}(x)$$

with $U(1)$ generator Q (charge operator)

Star commutator

$$[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\mu] = \frac{1}{2}\{\hat{\lambda}(x) \stackrel{\star}{,} \hat{a}_\mu(x)\}\underbrace{[Q, Q]}_{=0} + \frac{1}{2}[\hat{\lambda}(x) \stackrel{\star}{,} \hat{a}_\mu(x)]\underbrace{\{Q, Q\}}_{=2Q^2}$$

The Lie algebra does not close. Two options:

- $Q^2 = Q$, i.e. $Q = 1$ or $Q = 0$, or
- Enveloping-algebra valued fields and gauge parameters

Charge quantization

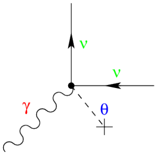
Covariant couplings

The only covariant couplings of the NC photon to charged matter are through the covariant derivatives

$$\partial_\mu \widehat{\Psi} - i\widehat{A}_\mu \star \widehat{\Psi}\,, \quad \partial_\mu \widehat{\Psi} + i\widehat{\Psi} \star \widehat{A}_\mu\,, \quad \partial_\mu \widehat{\Psi} - i[\widehat{A}_\mu \overset{\star}{,} \widehat{\Psi}]$$

corresponding to charge $+1$, -1, and zero respectively.

- "left" and "right" charges are distinguished in the NC setting, their sum is the usual commutative charge
- Neutral particles can couple to electromagnetic fields via a star-commutator.



Seiberg-Witten map

Enveloping algebra valued fields and gauge parameters

Star-commutator in NC non-abelian setting:

$$[\Lambda \stackrel{\star}{,} \Lambda'] = \frac{1}{2}\{\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)\}[T^a, T^b] + \frac{1}{2}[\Lambda_a(x) \stackrel{\star}{,} \Lambda'_b(x)]\{T^a, T^b\}$$

$\Rightarrow \hat{\Lambda}$ is valued in the enveloping algebra of $U(\mathrm{Lie}\, G)$:

$$\hat{\Lambda} = \Lambda_a(x) T^a + \Lambda_{ab}(x) : T^a T^b : + \Lambda_{abc}(x) : T^a T^b T^c : + \ldots$$

No restriction on gauge group or representation (charge) anymore.
($\rightarrow$ NC Standard Model, NC GUTs, ... can be constructed.)

Degrees of freedom?

Star product and Seiberg-Witten maps

Star product:

$$f \star_{[\theta]} g = fg + \frac{1}{2}\theta^{\mu\nu}\partial_\mu f\,\partial_\nu g + \dots .$$

Similarly, expansion via Seiberg-Witten maps:

$$\begin{aligned}
\widehat{A}_\mu[A,\theta] &= A_\mu + \frac{1}{4}\theta^{\xi\nu}\left\{A_\nu, \partial_\xi A_\mu + F_{\xi\mu}\right\} + \dots \\
\widehat{\Psi}[\Psi, A,\theta] &= \Psi + \frac{1}{2}\theta^{\mu\nu}A_\nu\partial_\mu\Psi + \frac{1}{4}\theta^{\mu\nu}\partial_\mu A_\nu\Psi + \dots \\
\widehat{\Lambda}[\Lambda, A,\theta] &= \Lambda + \frac{1}{4}\theta^{\xi\nu}\left\{A_\nu, \partial_\xi\Lambda\right\} + \dots
\end{aligned}$$

Cocycle condition $[\widehat{\Lambda} \stackrel{\star}{,} \widehat{\Lambda}'] + i\delta_\Lambda\widehat{\Lambda}' - i\delta_{\Lambda'}\widehat{\Lambda} = \widehat{[\Lambda, \Lambda']}$

Noncommutative gauge theory

Finite gauge transformations

classical gauge transformation: $\psi \mapsto \psi_g = g\psi$ and $a \mapsto a_g = a + gdg^{-1}$
gauge equivalence $\Rightarrow$

$$\Psi_{[\psi_g, a_g]} = G_{[g,a]} \star \Psi_{[\psi,a]} \ , \quad \mathcal{D}_{[a_g]}(f) = G_{[g,a]} \star \mathcal{D}_{[a]}(f) \star (G_{[g,a]})^{-1}$$

$$G_{[g_1, a_{g_2}]} \star G_{[g_2,a]} = G_{[g_1 \cdot g_2, a]} \quad \text{(noncommutative group law)}$$

NC gauge theory and equivalent star products

NC gauge theory = gauge theory of noncommutativity:

$$\mathcal{D}_{[a]}(f \star' g) = \mathcal{D}_{[a]} f \star \mathcal{D}_{[a]} g$$

Star products $\star$, $\star'$: locally equivalent, globally Morita equivalent.

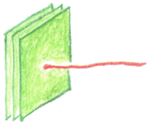
Jurco, PS, Wess, *Noncommutative line bundle and Morita equivalence*, Lett.Math.Phys. 61 (2002) 171-186

Outline 2nd lecture

- string theory and noncommutative geometry
- Seiberg-Witten map (exact solution in closed form)
- AKSZ sigma models in 1+1 and 1+2 dimensions
- non-geometric backgrounds and their quantization
- non-associative dynamical star product

$$\theta(x) \neq \text{const.}$$

Strings and Noncommutative Geometry



Dynamics of open strings ending on D-branes:
effective description by (non)abelian gauge theory

$$\langle x^i(\tau)x^j(\tau')\rangle = -G^{ij}\ln(\tau-\tau')^2 + \tfrac{i}{2}\theta^{ij}\,\mathrm{sgn}(\tau-\tau')$$

String endpoints become non-commutative in a B-field background with star product $\star$ depending on background fields via the closed-open string relations:

$$\frac{1}{g+B} = \frac{1}{G+\Phi} + \theta \qquad \text{for } \Phi = -B \text{ (or } \alpha' \to 0) : \quad \theta = B^{-1}$$

Ordinary versus non-commutative gauge theory

switch on fluctuations $B \mapsto \mathcal{F} = B + F$
In closed string background 2-form B-field

$$\to \quad \begin{cases} \text{non-commutative gauge theory} & \text{(e.g. point-splitting)} \\ \text{ordinary gauge theory} & \text{(e.g. Pauli-Villars)} \end{cases}$$

depending on regularization scheme $\Rightarrow$ SW map

Seiberg-Witten map

Recall: A Seiberg-Witten map is a field redefinition

$$\widehat{A}_\mu[A,\theta] = A_\mu + \frac{1}{4}\theta^{\xi\nu}\left\{A_\nu, \partial_\xi A_\mu + F_{\xi\mu}\right\} + \dots \ ,$$

such that

$$\delta A_\mu = \partial_\mu \Lambda \quad \Leftrightarrow \quad \delta \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda} \stackrel{\star}{,} \hat{A}_\mu] \ .$$

Furthermore

$$X^\nu = \mathcal{D}(x^\nu) = x^\nu + \theta^{\nu\mu}\hat{A}_\mu[A,\theta] \qquad \text{and} \qquad \mathcal{D}(f \star' g) = \mathcal{D}f \star \mathcal{D}g \ .$$

Seiberg-Witten map

Seiberg-Witten map from equivalence of star products

The map from ordinary to NC gauge theory is related to the equivalence map $\mathcal{D}$ of star products $\star$, $\star'$ and is a quantum analog of Moser's lemma.

Let $F = dA$ and ρ the flow generated by the vector field $A_\Theta = \Theta(A, -)$:

$$\begin{array}{rccc}
B: & & \Theta & \xrightarrow{\text{quantization}} & \star \\
\text{Moser} \big\downarrow \rho & & \big\downarrow \rho & & \big\downarrow \mathcal{D} \\
B + F: & & \Theta' & \xrightarrow{\text{quantization}} & \star'
\end{array}$$

where $\Theta' = \Theta(1 + \hbar F \Theta)^{-1}$ and $\mathcal{D}(f \star' g) = \mathcal{D}f \star \mathcal{D}g$.

The noncommutative gauge field $\hat{A}$ is obtained from $\mathcal{D}x =: x + \hat{A}$, such that ordinary gauge transform of $A \;\Rightarrow$ NC gauge transform of $\hat{A}$.

$\rightarrow$ explicit expression for the SW map for arbitrary $\Theta(x)$
$\rightarrow$ can be globalized (and extended to gerbes)

Jurco, PS, Wess (2000-2002)

Seiberg-Witten map

Moser's lemma on "nearby symplectic structures"

B: closed ($dB = 0$), non-degenerate ($\theta := B^{-1}$) 2-form
$B' = B + F$, F exact ($F = dA$)

$B_t = B + tF$, non-degenerate, $t \in [0, 1]$.

$\Rightarrow$ B' is obtained from B by a change of coordinates.

Proof: Let $\xi_t = \theta_t^{ij} A_j \partial_i$, i.e $i_{\xi_t} B_t = -A$.

$$\Rightarrow \mathcal{L}_{\xi_t} B_t = i_{\xi_t} dB + d i_{\xi_t} B = 0 - dA = -F = -\partial_t B_t \ .$$

We now integrate the flow generated by $\mathcal{L}_{\xi_t}$ from $t = 0$ to $t = 1$ and obtain a map ρ that depends on A and relates B' to B.

While B' is gauge invariant, the map ρ transforms by a canonical transformation, which is a (semi-classical) NC gauge transformation.

Semi-classical Seiberg-Witten map

θ: Poisson bi-vector (can be degenerate) ; $F = dA$
$\theta' = \theta - \theta \cdot F \cdot \theta + \theta \cdot F \cdot \theta \cdot F \cdot \theta - + \dots$

$\theta_t = \theta \cdot (1 + tF \cdot \theta)^{-1}$, Poisson, $t \in [0, 1]$; $\xi_t = -A \cdot \theta_t \cdot \partial$.

$$\Rightarrow \partial_t \theta_t = -\mathcal{L}_{\xi_t} \theta_t + k[\theta, \theta]_S \cdot A = -\mathcal{L}_{\xi_t} + 0 \ .$$

$$\rho^*(\theta') = \theta \ , \text{ with } \rho^* = \exp(\mathcal{L}_{\xi_t} + \partial_t) \exp(-\partial_t)\big|_{t=0}$$

Gauge transformation: $\delta A = d\Lambda$ implies $\delta \rho^*_A(f) = \{\rho^*_A(f), \tilde{\Lambda}\}$,
where $\tilde{\Lambda} = \sum \frac{1}{n!}(\xi_t + \partial_t)^{n+1}(\Lambda)\big|_{t=0}$.

Seiberg-Witten map

Quantized Seiberg-Witten map

Formality: vector field $\mapsto$ differential operator:

$$\xi = \xi^i(x)\partial_i \quad \mapsto \quad \Xi = \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\xi, \theta, \ldots, \theta)$$

$$\Xi(f \star g) = \Xi f \star g + g \star \Xi g + f[\mathcal{L}_\xi \star] g$$

The differential operator Ξ_t generates deformed diffeomorphisms that can be integrated to a flow $\mathcal{D}$, which is the SW map (exact, to all orders):

Let $\star_t = \sum \frac{i\hbar}{n!} U_n(\theta_t, \ldots \theta_t)$, $\star' = \star_1$, then $\partial_t(\star_t) = -[\Xi_t, \star_t]_G$

$$\mathcal{D}(\star') = \star \ , \ \text{with } \mathcal{D} = \exp(\Xi_t + \partial_t) \exp(-\partial_t)\big|_{t=0}$$

Gauge transformation: $\delta A = d\Lambda$ implies $\delta \mathcal{D}_A(f) = i[\hat{\Lambda}, \mathcal{D}_A(f)]$, where $\hat{\Lambda} = \sum \frac{1}{n!}(\Xi_t + \partial_t)^{n+1}(\Lambda)\big|_{t=0}$.

AKSZ sigma-models

AKSZ construction: action functionals in BV formalism of sigma model QFT's for symplectic Lie n-algebroids E

Alexandrov, Kontsevich, Schwarz, Zaboronsky (1995/97)

Poisson sigma model

2-dimensional topological field theory, $E = T^*M$

$$S^{(1)}_{\mathrm{AKSZ}} = \int_{\Sigma_2} \Big(\xi_i \wedge \mathrm{d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) \ ,$$

with $\Theta = \frac{1}{2}\,\Theta^{ij}(x)\,\partial_i \wedge \partial_j$, $\xi = (\xi_i) \in \Omega^1(\Sigma_2, X^*T^*M)$

perturbative expansion $\Rightarrow$ Kontsevich formality maps

(valid on-shell ($[\Theta,\Theta]_S = 0$) as well as off-shell, e.g. twisted Poisson)

Courant sigma model

standard Courant algebroid:
$C = TM \oplus T^*M$ with natural frame (ϱ_i, χ^i), metric $\langle \varrho_i, \chi^j \rangle = \delta_i{}^j$

TFT with 3-dimensional membrane world volume Σ_3

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \frac{1}{2}\, h_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - P_I{}^i(X)\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big)$$

with embeddings $X : \Sigma_3 \to M$, 1-form α, aux. 2-form ϕ, fibre metric h, anchor matrix P, 3-form T (e.g. H-flux, f-flux, Q-flux, R-flux).

Flux compactification

Compactification relates string theory to 3+1 dimensional observably phenomenology and cosmology. Fluxes stabilize moduli and can lead to generalized geometric structures; patching by string symmetries.

Non-geometric flux backgrounds

T-dualizing a 3-torus with 3-form H-flux gives rise to geometric and

non-geometric fluxes $H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc}$

Hull (2005), Shelton, Taylor, Wecht (2005)

Q-flux: T-duality transitions between local trivializations $\rightarrow$ T-folds
R-flux: metric and B-field not even locally defined; non-geometric strings

$\rightarrow$ non-commutative non-associative structures

Lüst (2010), Blumenhagen, Plauschinn (2010)
Blumenhagen, Deser, Lüst, Plauschinn, Rennecke (2011)
Mylonas, PS, Szabo (2012)

H-space sigma-model

relevant for geometric flux compactifications: $C = TM \oplus T^*M$ twisted by 3-form flux $H = \frac{1}{6}\, H_{ijk}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$

H-twisted Courant–Dorfman bracket

$$\big[(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big]_H := \big([Y_1,Y_2]_{TM}\,,\,\mathcal{L}_{Y_1}\alpha_2-\mathcal{L}_{Y_2}\alpha_1 \\ -\tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1)-\alpha_1(Y_2)\big)+H(Y_1,Y_2,-)\big)$$

metric: natural dual pairing

$$\big\langle(Y_1,\alpha_1)\,,\,(Y_2,\alpha_2)\big\rangle = \alpha_2(Y_1)+\alpha_1(Y_2)$$

anchor map: projection $\rho : C \to TM$
non-trivial bracket and 3-bracket

$$[\varrho_i,\varrho_j]_H = H_{ijk}\,\chi^k\ , \qquad [\varrho_i,\varrho_j,\varrho_k]_H = H_{ijk}$$

H-space sigma-model action

$$S^{(2)}_{\rm WZ} = \int_{\Sigma_3} \Big(\phi_i \wedge {\rm d}X^i + \alpha^i \wedge {\rm d}\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6}\, H_{ijk}(X)\, \alpha^i \wedge \alpha^j \wedge \alpha^k \Big) \ .$$

where $(\alpha^I) = (\alpha^1, \dots, \alpha^{2d}) \equiv (\alpha^1, \dots, \alpha^d, \xi_1, \dots, \xi_d)$

If $\Sigma_2 := \partial\Sigma_3 \neq \emptyset$, we can add a boundary term $\Rightarrow$ boundary/bulk open topological membrane action

$$\widetilde{S}^{(2)}_{\rm WZ} = S^{(2)}_{\rm WZ} + \int_{\Sigma_2} \frac{1}{2}\, \Theta^{ij}(X)\, \xi_i \wedge \xi_j \ .$$

(other boundary terms are possible, but will not be considered here)

H-space sigma-model

H-twisted Poisson sigma-model

Integrating out the two-form fields ϕ_i yields the AKSZ action

$$\widetilde{S}^{(1)}_{\rm AKSZ} = \int_{\Sigma_2} \Big(\xi_i \wedge {\rm d}X^i + \frac{1}{2}\,\Theta^{ij}(X)\,\xi_i \wedge \xi_j\Big) + \int_{\Sigma_3} \frac{1}{6}\,H_{ijk}(X)\,{\rm d}X^i \wedge {\rm d}X^j \wedge {\rm d}X^k \ ,$$

which is the action of the H-twisted Poisson sigma-model with target space M. Consistency of the equations of motion require Θ to be H-twisted Poisson, i.e.

$$[\Theta,\Theta]_{\rm S} = \textstyle\bigwedge^3 \Theta^\sharp(H) \neq 0$$

$\Rightarrow$ the Jacobi identity for the bracket is violated.

R-space sigma-model

From H to Q to R

Closed strings in Q-space via two T-duality transformations on 3-torus $\mathbb{T}^3$; locally filtration of $\mathbb{T}^2$ over S^1, globally not well-defined (T-fold). Closed string world sheet $\mathcal{C} = \mathbb{R} \times S^1$, coordinates (σ^0, σ^1), winding number $\tilde{p}^3$, twisted boundary conditions at σ'^1.

Closed string non-commutativity expressed via Poisson brackets:

$$\{x^i, x^j\}_Q = Q^{ij}{}_k\, \tilde{p}^{\,k} \qquad \text{and} \qquad \{x^i, \tilde{p}^{\,j}\}_Q = 0 = \{\tilde{p}^{\,i}, \tilde{p}^{\,j}\}_Q$$

Another T-duality transformation sends $Q^{ij}{}_k \mapsto R^{ijk}$, $\tilde{p}^k \mapsto p_k$ and the Poisson brackets to the twisted Poisson structure

$$\{x^i, x^j\}_\Theta = R^{ijk}\, p_k\ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{p_i, p_j\}_\Theta = 0\ .$$

Lüst (2010,2012)

R-space sigma-model

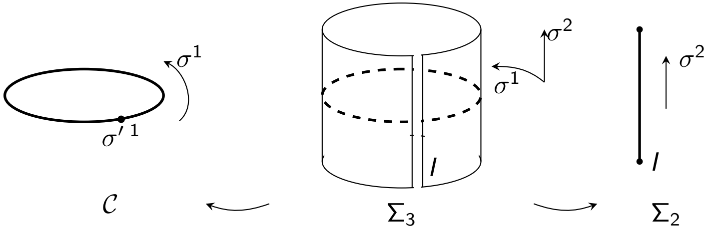
The hidden open string

CFT computation: insert twist field at $\sigma'^1 \in S^1$ $\rightarrow$ generates branch cut

There are indications that the appropriate R-space theory is a membrane sigma model, not a string theory:

- open strings do not decouple from gravity in R-space
- membrane theory geometrizes the non-geometric R-flux background

$\Rightarrow$ extend world sheet $\mathcal{C}$ to membrane world volume $\Sigma_3 = \mathbb{R} \times (S^1 \times \mathbb{R})$; resulting branch surface can be interpreted as open string world sheet:



closed $\leftrightarrow$ open string duality

R-space sigma-model

General Courant sigma-model with standard Courant algebroid $C = TM \oplus T^*M$, twisted by a trivector flux $R = \frac{1}{6}\, R^{ijk}(x)\, \partial_i \wedge \partial_j \wedge \partial_k$.

Roytenberg's R-twisted Courant-Dorfman bracket

$$\big[(Y_1, \alpha_1)\,,\, (Y_2, \alpha_2)\big]_R := \big([Y_1, Y_2]_{TM} + R(\alpha_1, \alpha_2, -)\,,\\ \mathcal{L}_{Y_1}\alpha_2 - \mathcal{L}_{Y_2}\alpha_1 - \tfrac{1}{2}\,\mathrm{d}\big(\alpha_2(Y_1) - \alpha_1(Y_2)\big)\big)$$

non-trivial bracket and 3-bracket

$$[\chi^i, \chi^j]_R = R^{ijk}\, \varrho_k \ , \qquad [\chi^i, \chi^j, \chi^k]_R = R^{ijk} \ .$$

R-space sigma-model

R-space sigma-model action

$$S_R^{(2)} = \int_{\Sigma_3} \Big(\phi_i \wedge \big(\mathrm{d}X^i - \alpha^i \big) + \alpha^i \wedge \mathrm{d}\xi_i + \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k \Big) + \frac{1}{2} \int_{\Sigma_2} g^{ij}(X)\, \xi_i \wedge * \xi_j \ ,$$

where we have added a non-topological term involving g^{ij}, to ensure consistency of $R^{ijk} \neq 0$.

Integrating out the 2-form field ϕ yields:

$$\boxed{S_R^{(2)} = \int_{\Sigma_2} \xi_i \wedge \mathrm{d}X^i + \int_{\Sigma_3} \frac{1}{6}\, R^{ijk}(X)\, \xi_i \wedge \xi_j \wedge \xi_k + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}(X)\, \xi_i \wedge * \xi_j \ .}$$

assume now constant R^{ijk} and g^{ij} and consider e.o.m. for X ...

R-space sigma-model

$\Rightarrow \xi_i = dP_i$ and the action reduces to a pure boundary action:

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\mathrm{d}P_i \wedge \mathrm{d}X^i + \frac{1}{2}\, R^{ijk}\, P_i\, \mathrm{d}P_j \wedge \mathrm{d}P_k \Big) + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}\, \mathrm{d}P_i \wedge *\mathrm{d}P_j \ ,$$

which can be rewritten as

$$S_R^{(2)} = \int_{\Sigma_2} -\frac{1}{2}\, \Theta^{-1}_{IJ}(X)\, \mathrm{d}X^I \wedge \mathrm{d}X^J + \int_{\Sigma_2} \frac{1}{2}\, g_{IJ}\, \mathrm{d}X^I \wedge *\mathrm{d}X^J \ ,$$

with

$$\Theta^{-1} = \big(\Theta^{-1}_{IJ}\big) = \begin{pmatrix} 0 & -\delta_i{}^j \\ \delta^i{}_j & R^{ijk}\, p_k \end{pmatrix} \ , \qquad \big(g_{IJ}\big) = \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix}$$

and $X = (X^I) = (X^1, \dots, X^{2d}) := (X^1, \dots, X^d, P_1, \dots, P_d)$.

$\Rightarrow$ effective target space $=$ phase space

The "closed string metric" g_{IJ} acts only on momentum space.

R-space sigma-model

Linearized action

Generalized Poisson sigma-model

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_I \wedge \mathrm{d}X^I + \frac{1}{2}\, \Theta^{IJ}(X)\, \eta_I \wedge \eta_J \Big) + \int_{\Sigma_2} \frac{1}{2}\, G^{IJ}\, \eta_I \wedge *\eta_J \ ,$$

with auxiliary fields η_I and

$$\Theta = \big(\Theta^{IJ}\big) = \begin{pmatrix} R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \ , \qquad \big(G^{IJ}\big) = \begin{pmatrix} g^{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

obeying the usual closed-open string relations, w.r.t. Θ^{-1} and g.

In phase-space component form:

$$S_R^{(2)} = \int_{\Sigma_2} \Big(\eta_i \wedge \mathrm{d}X^i + \pi^i \wedge \mathrm{d}P_i + \frac{1}{2}\, R^{ijk}\, P_k\, \eta_i \wedge \eta_j + \eta_i \wedge \pi^i \Big) + \int_{\Sigma_2} \frac{1}{2}\, g^{ij}\, \eta_i \wedge *\eta_j \ ,$$

with $(\eta_I) = (\eta_1, \dots, \eta_{2d}) \equiv (\eta_1, \dots, \eta_d, \pi^1, \dots, \pi^d)$.

R-space sigma-model

Non-commutative, non-associative phase space

Θ is an H-twisted Poisson bi-vector: $[\Theta,\Theta]_{\mathrm{S}} = \bigwedge^3 \Theta^\sharp(H)$, where

$$H = \tfrac{1}{6}\, R^{ijk}\, \mathrm{d}p_i \wedge \mathrm{d}p_j \wedge \mathrm{d}p_k = dB \ , \ \text{and} \ \ B = \tfrac{1}{6}\, R^{ijk}\, p_k\, \mathrm{d}p_i \wedge \mathrm{d}p_j \ .$$

Twisted Poisson brackets

$$\{x^i, x^j\}_\Theta = R^{ijk}\, p_k \ , \qquad \{x^i, p_j\}_\Theta = \delta^i{}_j \qquad \text{and} \qquad \{p_i, p_j\}_\Theta = 0 \ .$$

Corresponding Jacobiator:

$$\{x^i, x^j, x^k\}_\Theta = R^{ijk} \ ,$$

where $\{x^I, x^J, x^K\}_\Theta := [\Theta,\Theta]_{\mathrm{S}}(x^I, x^J, x^K) = \Pi^{IJK}$ and

$$\big(\Pi^{IJK}\big) = \frac{1}{3}\,\big(\Theta^{KL}\,\partial_L\Theta^{IJ} + \Theta^{IL}\,\partial_L\Theta^{JK} + \Theta^{JL}\,\partial_L\Theta^{KI}\big) = \begin{pmatrix} R^{ijk} & 0 \\ 0 & 0 \end{pmatrix} \ .$$

Path integral quantization

Mapping the open string endpoints to finite values and imposing natural boundary conditions, we are let to the following schematic functional integrals that reproduce Kontsevich's graphical expansion of global deformation quantization. For multivector fields $\mathcal{X}_r$ of degree k_r:

$$U_n(\mathcal{X}_1,\dots,\mathcal{X}_n)(f_1,\dots,f_m)(x)=\int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_R^{(2)}}\, S_{\mathcal{X}_1}\cdots S_{\mathcal{X}_n}\,\mathcal{O}_x(f_1,\dots,f_m)\,,$$

where $m=2-2n+\sum_r k_r$, $S_{\mathcal{X}_r}=\frac{\mathrm{i}}{\hbar}\int_{\Sigma_2}\frac{1}{k_r!}\mathcal{X}_r^{I_1\dots I_r}(X)\,\eta_{I_1}\cdots\eta_{I_r}$, and

$$\mathcal{O}_x(f_1,\dots,f_m)=\int_{X(\infty)=x}\Big[f_1\big(X(q_1)\big)\cdots f_m\big(X(q_m)\big)\Big]^{(m-2)}\,,$$

with $1=q_1>q_2>\cdots>q_m=0$ and ∞ distinct points on the boundary of the disk $\partial\Sigma_2$; the path integrals are weighted with the full gauge-fixed action and the integrations taken over all fields including ghosts.

Cattaneo, Felder (2000)

Quantization

Kontsevich formality maps

U_n maps n multivector fields to a differential operator

$$U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) = \sum_{\Gamma \in G_n} w_\Gamma \, D_\Gamma(\mathcal{X}_1, \ldots, \mathcal{X}_n) \ ,$$

where the sum is over all possible diagrams with weight

$$w_\Gamma = \frac{1}{(2\pi)^{2n+m-2}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_i^1} \wedge \cdots \wedge \mathrm{d}\phi^h_{e_i^{k_i}} \right) \ .$$

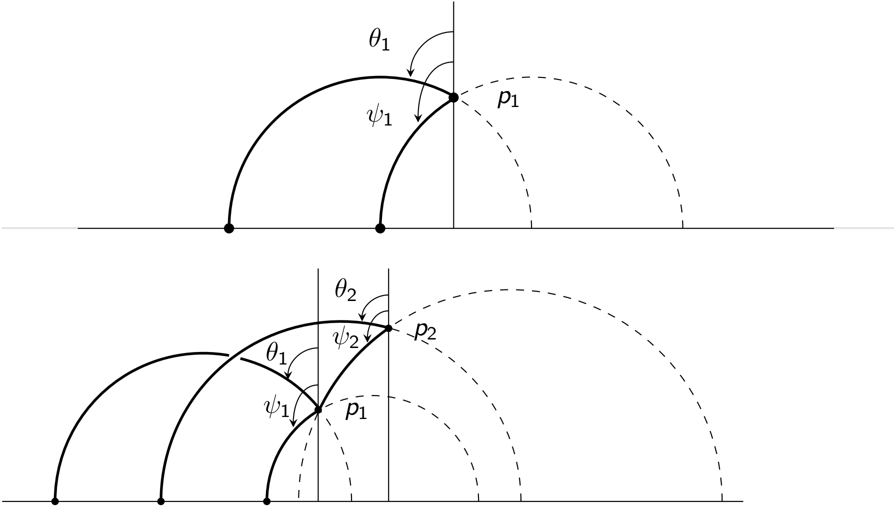
The star product and the 3-bracket are given by

$$f \star g = \sum_{n=0}^{\infty} \frac{(\,\mathrm{i}\,\hbar)^n}{n!} \, U_n(\Theta, \ldots, \Theta)(f, g) =: \Phi(\Theta)(f, g) \ ,$$

$$[f, g, h]_\star = \sum_{n=0}^{\infty} \frac{(\,\mathrm{i}\,\hbar)^n}{n!} \, U_{n+1}(\Pi, \Theta, \ldots, \Theta)(f, g, h) =: \Phi(\Pi)(f, g, h) \ .$$

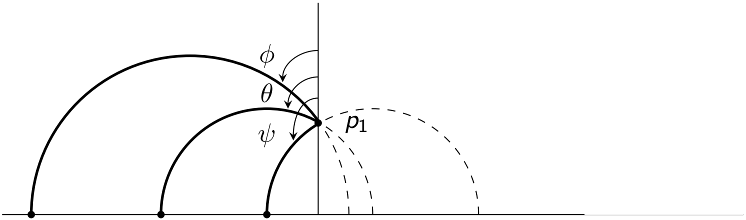
Quantization

Relevant diagrams involve the bivector $\Theta = \frac{1}{2}\Theta^{IJ}\partial_I \wedge \partial_J$ ...



Quantization

...and the trivector $\Pi = \frac{1}{6}\Pi^{IJK}\partial_I \wedge \partial_J \wedge \partial_K = d_\Theta \Theta = [\Theta, \Theta]_S$:



For constant Π all other diagrams factorize and their weights can be expressed in terms of these three diagrams (up to permutations).

Quantization

Formality condition

The U_n define L_∞-morphisms and satisfy

$$\mathrm{d}.U_n(\mathcal{X}_1,\dots,\mathcal{X}_n)+\frac{1}{2}\sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\dots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}}\varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J})\,\big[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}})\,,\,U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\big]_{\mathrm{G}}$$

$$=\ \sum_{i<j}(-1)^{\alpha_{ij}}\,U_{n-1}\big([\mathcal{X}_i,\mathcal{X}_j]_{\mathrm{S}},\mathcal{X}_1,\dots,\widehat{\mathcal{X}_i},\dots,\widehat{\mathcal{X}_j},\dots,\mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

Kontsevich (1997)

This implies in particular

$$\mathrm{d}_\star\Phi(\Theta)=\ \mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\Theta\Theta)\ ,$$

which explicitly quantifies the lack of associativity of the star product:

$$(f\star g)\star h-f\star(g\star h)=\tfrac{\hbar}{2\,\mathrm{i}}[f,g,h]_\star=\tfrac{\hbar}{2\,\mathrm{i}}\Phi(\Pi)(f,g,h)\ .$$

Quantization

The formality condition implies derivation properties:

- For a function h, the Hamiltonian vector field $\mathrm{d}_\Theta h = \{-, h\}$ is mapped to the inner derivation $\mathrm{d}_\star \underline{h} = \frac{\mathrm{i}}{\hbar} [\, \underline{h}\, , -]_\star = i\hbar \Phi(\mathrm{d}_\Theta h)$, where $\underline{h} = \Phi(h) \equiv \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} U_{n+1}(h, \Theta, \ldots, \Theta)$.
- A Poisson structure preserving vector field $\mathcal{X}$ ($\mathrm{d}_\Theta \mathcal{X} = 0$) is mapped to a differential operator $\underline{\mathcal{X}} = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!} U_{n+1}(\mathcal{X}, \Theta, \ldots, \Theta)$ satisfying $\mathcal{X}(f \star g) = \mathcal{X}(f) \star g + f \star \mathcal{X}(g)$.
- The formality condition $\mathrm{d}_\star \Phi(\Pi) = i\hbar \Phi(\mathrm{d}_\Theta \Pi)$ and higher derivation properties encode quantum analogs of the derivation property and fundamental identity for a Nambu-Poisson structure.
- In particular, in the present case, where $\mathrm{d}_\Theta \Pi = 0$:

$$[f \star g, h, k]_\star - [f, g \star h, k]_\star + [f, g, h \star k]_\star = f \star [g, h, k]_\star + [f, g, h]_\star \star k\ .$$

Quantization

Explicit formulas

- Dynamical non-associative star product: $f\star g\equiv f\star_p g$, with

$$f\star_p g=\cdot\Big[e^{\frac{\mathrm{i}\,\hbar}{2}\,R^{ijk}\,p_k\,\partial_i\otimes\partial_j}\;e^{\frac{\mathrm{i}\,\hbar}{2}\,\left(\partial_i\otimes\tilde\partial^i-\tilde\partial^i\otimes\partial_i\right)}(f\otimes g)\Big]$$

- Replacing the dynamical variable p with a constant $\tilde p$ we obtain an associative Moyal-Weyl type star product $\tilde\star:=\star_{\tilde p}$.
- Triple products and 3-bracket:

$$(f\star g)\star h=\Big[\,\tilde\star\Big(\exp\left(\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\right)(f\otimes g\otimes h)\Big)\Big]_{\tilde p\to p}$$

$$[f,g,h]_\star=\frac{4\,\mathrm{i}}{\hbar}\,\Big[\,\tilde\star\Big(\sinh\left(\tfrac{\hbar^2}{4}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\right)(f\otimes g\otimes h)\Big)\Big]_{\tilde p\to p}$$

- Trace property: $\int[f,g,h]_\star=0$

Mylonas, PS, Szabo (2012)

Seiberg-Witten maps

Twisted Poisson structure, NC gerbes

Poisson structure twisted by closed 3-form H: $[\Theta,\Theta]_S = \bigwedge^3 \Theta^\sharp H$

For covering by contractible open patches labeled by $\alpha,\beta,\gamma,\ldots$:

$$H|_\alpha = dB_\alpha \ , \qquad (B_\beta - B_\alpha)|_{\alpha\cap\beta} = F_{\alpha\beta} = dA_{\alpha\beta}$$

Θ can be locally untwisted by B_α: $\Theta_\alpha := \Theta(1-\hbar B_\alpha\Theta)^{-1}$.

quantization of Θ $\rightarrow$ nonassociative $\star$
quantization of Θ_α, Θ_β $\rightarrow$ associative $\star_\alpha$, $\star_\beta$ related by $\mathcal{D}_{\alpha\beta}$

for more details: Aschieri, Bakovic, Jurco, PS (2010)

SW maps for R-twisted Poisson structures

trivial gerbe $\rightarrow$ replace patch label α by the (constant) vector $\tilde{p}$:

$$\Theta = \begin{pmatrix} \hbar\, R^{ijk}\, p_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad \Theta_{\tilde{p}} = \begin{pmatrix} \hbar\, R^{ijk}\, \tilde{p}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \quad B_{\tilde{p}} = \begin{pmatrix} 0 & 0 \\ 0 & R^{ijk}\,(p_k - \tilde{p}_k) \end{pmatrix}$$

Θ: twisted Poisson $\qquad$ $\Theta_{\tilde{p}}$: Poisson $\qquad$ $H = dB_{\tilde{p}} = \frac{1}{2} R^{ijk}\, dp_i\, dp_j\, dp_k$

Seiberg-Witten maps

Gauge potential: $A = A_I \mathrm{d}x^I = a_i(x,p)\mathrm{d}x^i + \tilde{a}^i(x,p)\mathrm{d}p_i$

Maps between *associative* $\tilde{\star}$ and $\tilde{\star}'$ are generated by
$A_{\tilde{p}\tilde{p}'} = R^{ijk} p_i (\tilde{p}_k - \tilde{p}'_k)\mathrm{d}p_j$ with $F_{\tilde{p}\tilde{p}'} = R^{ijk}(\tilde{p}_k - \tilde{p}'_k)\mathrm{d}p_i \mathrm{d}p_j$.

Special case $\tilde{p} = 0$: canonical Moyal-Weyl star product $\star_0$.

Generalization of SW maps for non-associative structures

A construction directly based on twisted Θ is spoiled by $[\Theta, \Theta]_S$-terms.
These can be avoided in the present case by choosing $a_i(x,p) = 0$!

- general coordinate transformations generated by $\Theta(A, -) = \tilde{a}^i(x,p)\partial_i$
- Nambu-Poisson maps: choose $A = R(a_2, -)$ for *any* 2-form a_2;
  $\rightarrow$ higher "Nambu-Poisson" gauge theory.
- map from associative to nonassociative: $\mathcal{D}_{\tilde{p}}$ generated by $A_{\tilde{p}} = \frac{1}{2} R^{ijk} p_i \tilde{p}_k \mathrm{d}p_j$ can be explicitly computed and satisfies

$$f \star g = \left[\mathcal{D}_{\tilde{p}} f \star_0 \mathcal{D}_{\tilde{p}} g\right]_{\tilde{p}\rightarrow p}$$

Nambu-Poisson and Membranes

Remarks on Nambu-Poisson structures

- The trivector $\Pi = \frac{1}{6} R^{ijk} \partial_i \wedge \partial_j \wedge \partial_k$ is an example of a Nambu-Poisson tensor.
- Nambu mechanics: multi-Hamiltonian dynamics with generalized Poisson brackets; e.g. Euler's equations for the spinning top :

$$\frac{d}{dt} L_i = \{L_i, \frac{\vec{L}^2}{2}, T\} \qquad \text{with} \quad \{f, g, h\} \propto \epsilon^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

- more generally:

$$\{f, h_1, \ldots, h_p\} = \Pi^{i\, j_1 \ldots j_p}(x)\, \partial_i f\, \partial_{j_1} h_1 \, \cdots\, \partial_{j_p} h_p$$

$$\begin{aligned}\{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} &= \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \ldots \\ &\quad \ldots + \{f_0, \ldots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\}\end{aligned}$$

- Our construction may be useful to quantize these objects.

Outline 3rd lecture

- generalized geometry and NC gauge theory
- effective string actions
- Nambu-Poisson structures and NP sigma model
- effective brane actions

$$p = 1 \quad \rightsquigarrow \quad p > 1$$

Generalized geometry

Generalize geometry to accommodate string symmetries.
Replace Lie algebroid TM by a Courant algebroid E

- $TM \oplus T^*M$ (type I/II without RR fluxes)
- $TM \oplus T^*M \oplus G$ (type I + YM)
- $TM \oplus \Lambda^2 T^*M \oplus \Lambda^2 TM \oplus T^*M$ (M-theory)

Leibnitz algebroid $(E, [\ ,\ ], \rho)$:
vector bundle $E \to M$ with bracket on $\Gamma(E)$ and anchor map morphism of vector bundles $\rho : E \to TM$, s.t.: $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$,

$$\rho[A, B] = [\rho A, \rho B] \ , \qquad [A, fB] = f[A, B] + [\rho A, f]B$$

Courant algebroid: add field of bilinear form $\langle\ ,\ \rangle$

Exact Courant algebroid $0 \to T^*M \to E \to TM \to 0$: $E \cong TM \oplus T^*M$.

Generalized geometry

Generalized geometry (Hitchin, Gualtieri, ...): replace structures on TM ($[\ ,\ ]$, i_V, $\mathcal{L}_V$, d, ...) by structures on E.

- sections $V + \xi \in \Gamma(TM \oplus T^*M)$
- bilinear form $\langle V + \xi, W + \eta \rangle = i_V \eta + i_W \xi$
- (Dorfman) bracket $[V + \xi, W + \eta] = [V, W] + \mathcal{L}_V \eta - i_W d\xi$
- Clifford algebra $\{\gamma_{V+\xi}, \gamma_{W+\eta}\} = 2\langle V + \xi, W + \eta \rangle$

Symmetries of $\langle\ ,\ \rangle$: $O(d, d)$

e.g. $e^B(V + \xi) = V + \xi + i_V B$ preserves bracket up to $i_V i_W dB$
$\Rightarrow$ symmetries of bracket: $\mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{closed}}(M)$.

Twisted Dorfman bracket $[\ ,\ ]_H = [\ ,\ ] + i_V i_W H$ for $H \in \Omega^3_{\mathrm{closed}}(M)$,
then: $e^B : [\ ,\ ]_H \mapsto [\ ,\ ]_{H+dB}$; twisted differential: $d_H = d + H\wedge$.

Generalized geometry

$E = TM \oplus T^*M$, $V + \xi \in E$, $W + \eta \in E$

bilinear form $\langle V + \xi, W + \eta \rangle = i_V \eta + i_W \xi$ in matrix form: $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

signature (n, n) $\Rightarrow$ symmetries: $O(n, n)$.
Examples of $O(n, n)$ transformation:

- B(2-form)-transform: $e^B(V + \xi) = V + \xi + B(V)$, matrix: $\begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$
- θ(bivector)-transform: $e^\theta(V + \xi) = V + \xi + \theta(\xi)$, matrix: $\begin{pmatrix} I & \theta \\ 0 & I \end{pmatrix}$
- $O_N(V + \xi) = N(V) + N^{-T}(\xi)$, smooth; matrix: $\begin{pmatrix} N & 0 \\ 0 & N^{-T} \end{pmatrix}$

Any $\mathcal{O} \in O(n, n)$ can be written as $\mathcal{O} = e^{-B} O_N e^{-\theta}$.

Generalized geometry

consider an idempotent linear map $\tau : \Gamma(E) \to \Gamma(E)$, $\tau^2 = 1$

eigenvalues $\pm 1 \rightsquigarrow$ splitting $E = V_+ \oplus V_-$ with eigenbundle:

$$V_+ = \{V + A(V) \mid V \in TM\} = \{A^{-1}(\xi) + \xi \mid \xi \in T^*M\} \qquad A = g + B$$

$$V_- = \{V + \tilde{A}(V) \mid V \in TM\} = \{\tilde{A}^{-1}(\xi) + \xi \mid \xi \in T^*M\} \qquad \tilde{A} = -g + B$$

in matrix form: $\tau \begin{pmatrix} V \\ \xi \end{pmatrix} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \begin{pmatrix} V \\ \xi \end{pmatrix}$

positive definite metric via τ: $(e_1, e_2)_\tau := \langle \tau e_1, e_2 \rangle = \langle e_1, \tau e_2 \rangle$
$\Rightarrow$ generalized metric, factorized using Schur decomposition,

$$\mathbb{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix}$$

Generalized geometry and NC gauge theory

A θ-transform will yield a new generalized metric $\mathbb{H} = e^{\theta}\mathbb{G}$, which can again be factorized:

$$\mathbb{H} = \begin{pmatrix} I & \Phi \\ 0 & I \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Phi & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\theta & I \end{pmatrix} \mathbb{G} \begin{pmatrix} I & \theta \\ 0 & I \end{pmatrix} .$$

In terms of the eigenbundle:

$$\begin{aligned} e^{\theta} V_+ &= \{(g+B)^{-1}(\xi) - \theta(\xi) + \xi \mid \xi \in T^*M\} \\ &= \{(G+\Phi)^{-1}(\xi) + \xi \mid \xi \in T^*M\} \end{aligned}$$

$\Rightarrow$ closed-open relations (!)

$$\frac{1}{g+B} = \frac{1}{G+\Phi} + \theta$$

Generalized geometry and NC gauge theory

Add fluctuations $B \mapsto B' = B + F \;\Rightarrow \mathbb{G} \mapsto \mathbb{G}' = e^{-F}\mathbb{G}$

and similarly $\Phi \mapsto \Phi' = \Phi + F'$, $\Rightarrow \mathbb{H} \mapsto \mathbb{H}' = e^{-F'}\mathbb{H}$.

$\mathbb{H}'$ and $\mathbb{G}'$ are related by $O_N e^{-\theta'}$,
where $N = 1 + \theta F$, $F' = FN^{-1}$, $\theta' = N^{-1}\theta$.

We find an interesting determinant identity ("miraculous identity")

$$\det(g-(B+F)g^{-1}(B+F)) = det(N^2)\det(G-(\Phi+F')G^{-1}(\Phi+F'))$$

and from the transformation of the eigenbundle:

$$\frac{1}{g+B+F} = \frac{1}{N^T(G+\Phi+F')N^{-1}} + \theta'$$

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Generalized geometry and NC gauge theory

Based on the "miraculous identity" we can make ansätze for effective open string actions: A commutative version and a non-commutative version. The latter requires the use of the semi-classical SW map and its Jacobian needs to match an appropriate power of the factor det N.

This fixes the actions to be

$$\int \frac{d^n x}{g_s} \det{}^{\frac{1}{2}}(g + B + F) = \int \frac{d^n x}{G_s} \det{}^{\frac{1}{2}} N \det{}^{\frac{1}{2}}(G + \Phi + F')$$

and after a covariantizing change of integration variables (SW map):

$$\int \frac{d^n x}{g_s} \det{}^{\frac{1}{2}}(g + B + F) = \int \frac{d^n x}{\hat{G}_s} \frac{\det^{\frac{1}{2}} \hat{\theta}}{\det^{\frac{1}{2}} \theta} \det{}^{\frac{1}{2}}(\hat{G} + \hat{\Phi} + \hat{F}')$$

Effective Actions

Open string effective action

$$\mathcal{S}_{\mathrm{DBI}} = \int d^n x \frac{1}{g_s} \det{}^{\frac{1}{2}}(g + B + F) = \int d^n x \frac{1}{\hat{G}_s} \det{}^{\frac{1}{2}}(\hat{G} + \hat{\Phi} + \hat{F})$$

commutative $\leftrightarrow$ non-commutative duality

Expand to second order, ignore (cosmological) constants $\Rightarrow$

$$\mathcal{S}_{\mathrm{DBI}} = \int d^n x \frac{|-g|^{\frac{1}{2}}}{4 g_s} g^{ij} g^{kl} (B+F)_{ik} (B+F)_{jl} \qquad \text{(Maxwell/Yang-Mills)}$$

$$\mathcal{S}_{\mathrm{DBI}}^{NC} = \int d^n x \frac{|\theta|^{-\frac{1}{2}}}{4 \hat{g}_s} \hat{g}_{ij} \hat{g}_{kl} \{\hat{X}^i, \hat{X}^k\} \{\hat{X}^j, \hat{X}^l\} \qquad \text{(Matrix Model)}$$

Covariant coordinates: $\hat{X}^i = x^i + \hat{A}^i$

Commutative $\leftrightarrow$ non-commutative duality fixes form of action

Massless bosonic modes

- open strings: A_μ, ϕ^i $\rightarrow$ gauge and scalar fields
- closed strings: $g_{\mu\nu}$, $B_{\mu\nu}$, Φ $\rightarrow$ background geometry, gravity

Closed string effective action

Weyl invariance (at 1 loop) requires vanishing beta functions:

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$$

$$\Downarrow$$

equations of motion for $g_{\mu\nu}$, $B_{\mu\nu}$, Φ

$$\Uparrow$$

closed string effective action

$$\int d^D x |-g|^{\frac{1}{2}} \left(R - \frac{1}{12} e^{-\Phi/3} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \Phi \partial^\mu \Phi + \dots \right)$$

Noncommutative version of this?

Nambu-Poisson structures

Nambu mechanics

multi-Hamiltonian dynamics with generalized Poisson brackets

e.g. Euler's equations for spinning top

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0 \qquad \text{etc.}$$

$$\Rightarrow \dot{L}_i = \epsilon_{ijk} L_j L_k / I_j = \{L_i, T, \frac{1}{2}\vec{L}^2\}$$

with $\vec{L} = I \cdot \vec{\omega}$, $T = \frac{1}{2}\vec{L} \cdot \vec{\omega}$ and Nambu-Poisson bracket

$$\{f, g, h\} = \det\left[\frac{\partial(f, g, h)}{\partial(L_1, L_2, L_3)}\right] = \epsilon^{ijk}\,\partial_i f\,\partial_j g\,\partial_k h$$

Nambu-Poisson structures

Nambu-Poisson (NP) bracket

more generally: skew-symmetric, multi-linear, derivation

$$\{f, h_1, \ldots, h_p\} = \Pi^{i\, j_1 \ldots j_p}(x)\, \partial_i f\, \partial_{j_1} h_1 \cdots \partial_{j_p} h_p$$

$+$ Fundamental Identity (FI)

$$\begin{aligned}\{\{f_0, \cdots, f_p\}, h_1, \cdots, h_p\} &= \{\{f_0, h_1, \cdots, h_p\}, f_1, \cdots, f_p\} + \ldots \\ &\ldots + \{f_0, \ldots, f_{p-1}, \{f_p, h_1, \cdots, h_p\}\}\end{aligned}$$

Alternative viewpoint

- Nambu tensor $\Pi \in TM \otimes \Lambda^p TM$ maps a time-evolution p-form η "Nambuian" ($\eta = dH$ for $p = 1$) to a time-evolution vector field

$$\Pi(\eta) = \frac{1}{p!}\Pi^{i\,j_1\ldots j_p}\,\eta_{j_1\ldots j_p}\partial_i \equiv \Pi^{iJ}\eta_J\partial_i \in TM$$

  with $J = (j_1, \ldots, j_p)$: ordered multi-index

- Canonical transformation property

$$d\eta = 0 \quad \Rightarrow \quad \mathcal{L}_{\Pi(\eta)}\,\Pi = 0$$

- Conservation law property

$$\eta = dh_1 \wedge \ldots \wedge dh_p \quad \Rightarrow \quad \mathcal{L}_{\Pi(\eta)}\,\eta = 0$$

Nambu-Poisson structures

For $p = 1$:
ordinary Poisson structure, differential constraint (Jacobi identity)

For $p > 1$:
Nambu-Poisson structure, differential & algebraic constraint

$\Leftrightarrow$ Π factorizes into wedge product of vector fields

$$\Pi = V_0 \wedge V_1 \wedge \ldots \wedge V_p = |\Pi(x)|^{\frac{1}{p+1}} \, e_0 \wedge \ldots \wedge e_p$$

- foliation into $(p+1)$-dimensional submanifolds
- $|\Pi(x)|^{\frac{1}{p+1}}$ is a scalar density of weight -1
- $|\Pi(x)|$ is the generalized determinant of the rectangular matrix Π^{iJ}

Poisson σ-model

Nonlinear gauge theory/Poisson σ-model (Ikeda; Schaller, Strobel)

$$S[A,X] = \int_\Sigma \left(A_i \wedge dX^i - \frac{1}{2}\Pi^{ij} A_i \wedge A_j \right) \qquad \Pi = \frac{1}{2}\Pi^{ij}(X)\partial_i \wedge \partial_j$$

$X : \Sigma \to M$ (Σ: 2D world sheet, M: target space)

$A(\sigma) =$ 1-form on Σ with values in $T^*_{X(\sigma)}M$

equations of motion

$$dX^i - \Pi^{ij} A_j = 0 \qquad dA_i + \frac{1}{2}\partial_i \Pi^{kl} A_k \wedge A_l = 0$$

consistency of eom requires

$$[\Pi,\Pi]_S^{ijk} = \frac{1}{3}\left(\Pi^{il}\partial_l \Pi^{jk} + \mathrm{cycl}\right) = 0 \quad \Rightarrow (M,\Pi) \text{ must be Poisson}$$

Poisson σ-model

Generalized (non-topological) Poisson σ-model

$$S = \int_\Sigma \left(A_i \wedge dX^i - \frac{1}{2}\Pi^{ij} A_i \wedge A_j - \frac{1}{2}(G^{-1})^{ij} A_i \wedge *A_j \right)$$

$A_i = A_{i\alpha}(\sigma) d\sigma^\alpha$ are auxiliary fields $\to$ integrate out

$$S' = -\int_\Sigma \frac{1}{2} \left(g_{ij} dX^i \wedge *dX^j + B_{ij} dX^i \wedge dX^j \right)$$

$\Rightarrow$ closed-open string relations

$$\frac{1}{g+B} = G^{-1} + \Pi \quad \Rightarrow \quad G = g - Bg^{-1}B\,, \quad \theta = -G^{-1}Bg^{-1}$$

Nambu σ-model

Let $\eta_i = \eta_i(\sigma) d\sigma^1 := -A_{i1}(\sigma) d\sigma^1$ and $\tilde{\eta}_j = \tilde{\eta}_j(\sigma) d\sigma^0 := A_{j0}(\sigma) d\sigma^0$

Generalized Poisson σ-model

$$S = \int_{\Sigma_{1+1}} \left(dX^i \wedge \eta_i + \tilde{\eta}_j \wedge dX^j - \Pi^{ij} \tilde{\eta}_j \wedge \eta_i - \frac{1}{2} G^{ij} \eta_i \wedge * \eta_j - \frac{1}{2} G^{ij} \tilde{\eta}_i \wedge * \tilde{\eta}_j \right)$$

p-brane version $\rightarrow$ Nambu σ model

$$S = \int_{\Sigma_{1+p}} \left(dX^i \wedge \eta_i + \tilde{\eta}_J \wedge d^p X^J - \Pi^{iJ} \tilde{\eta}_J \wedge \eta_i - \frac{1}{2} G^{ij} \eta_i \wedge * \eta_j - \frac{1}{2} \tilde{G}^{IJ} \tilde{\eta}_I \wedge * \tilde{\eta}_J \right)$$

Nambu sigma model

Notation

$$X^i(\sigma) \qquad \text{embedding fn's (scalar fields)}$$

$$I, J \qquad \text{ordered } p\text{-tuple multi-indices}$$

$$I = (i_1, \ldots, i_p) \qquad 0 \leq i_1 < \ldots < i_p \leq D-1$$

$$\widetilde{\partial X}^I \equiv \sum_{a_1,\ldots,a_p=1}^{p} \epsilon^{a_1\ldots a_p} \partial_{a_1} X^{i_1} \cdots \partial_{a_p} X^{i_p}$$

$$\alpha, \beta = 0, 1, \ldots, p \qquad \text{world volume indices}$$

$$a, b = 1, \ldots, p \qquad \text{spatial components}$$

A tilde distinguishes fields that carry multi indices.

Nambu sigma model (in components)

$$S[\eta, \tilde{\eta}, X] = \int d^{p+1}\sigma \, \Big[-\frac{1}{2}(G^{-1})^{ij}\eta_i\eta_j + \frac{1}{2}(\tilde{G}^{-1})^{IJ}\tilde{\eta}_I\tilde{\eta}_J + \eta_i\partial_0 X^i + \tilde{\eta}_I\widetilde{\partial X}^I - \Pi^{iJ}\eta_i\tilde{\eta}_J \Big]$$

Nambu gauge theory

Nambu-Poisson map

add fluctuations: p-form gauge potential A with field strength $F = dA$

gauge action of F on Π:

$$\Pi \mapsto \Pi^F = (I - \Pi F^T)^{-1}\Pi = (1 - \langle \Pi, F\rangle)^{-1}\Pi$$

with inner product $\langle \Pi, F\rangle \equiv \operatorname{tr} \Pi F^T$

Nambu-Poisson map $\rho_{[A]}$ (change of coordinates) relates Π and Π^F

gauge tr. $\delta A = d\lambda \;\Rightarrow \delta\rho_{[A]}$ generated by $X_{[\lambda,A]} = \Pi^{iJ}(d\hat{\lambda}_{[\lambda,A]})_J\partial_i$

$$\hat{\lambda}_{[\lambda,A]} = \sum_k \frac{(-\mathcal{L}_{\Pi^{tF}(A)} + \partial_t)^k(\lambda)}{(k+1)!}\Big|_{t=0}.$$

Nambu gauge theory

Covariant functions and coordinates:

$$\hat{f} = \rho_{[A]}(f) \quad \rightsquigarrow \quad \delta\hat{f} = \mathcal{L}_{\Pi(d\hat{\lambda})}\hat{f} = \sum\{\hat{f}, \hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(p)}\}$$

$$\hat{x}^i = \rho_{[A]}(x^i) = x^i + \hat{A}^i \quad \rightsquigarrow \quad \delta\hat{A}^i = \sum\{\hat{x}^i + \hat{A}^i, \hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(p)}\}$$

$(d\hat{\lambda} \equiv \sum d\hat{\lambda}^{(1)} \wedge \ldots \wedge d\hat{\lambda}^{(p)})$

Jacobian of $\rho_{[A]} : x^i \mapsto \hat{x}^i$

Using the decomposability of Π for $p > 1$ and fact that the degenerate matrix $F\Pi^T$ acts non-trivially only on a $(p+1)$-dimensional subspace (via multiplication by $\langle \Pi, F \rangle$):

$$\det(1 - F\Pi^T) = (1 - \langle \Pi, F \rangle)^{p+1} = \frac{|\Pi(\hat{x})|}{|\Pi(x)|} \cdot \left|\frac{\partial x}{\partial \hat{x}}\right|^{p+1} .$$

Membrane actions

Nambu-Goto *p*-brane action

$$S[X] = T_p \int_\Sigma d^{p+1}\sigma \, \sqrt{\det(g_{ij}\partial_\alpha X^i \partial_\beta X^j)}$$

classically equivalent: *p*-brane sigma model action

$$S[X,h] = \frac{T'_p}{2} \int_\Sigma d^{p+1}\sigma \, \sqrt{\det h} \left[g_{ij} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j - (p-1)\lambda \right]$$

where $T'_p = \lambda^{\frac{p-1}{2}} T_p$ and $\lambda > 0$

Membrane actions

gauge fix

$$h_{a,0} = h_{0,b} = 0 \text{ and } h_{00} = \lambda^{p-1} \det(h_{ab})$$

(valid globally for Σ of form $\Sigma_p \times \mathbb{R}$, $\Sigma_p \times I$ or $\Sigma_p \times S^1$)

eliminate h_{ab} $\Rightarrow$

$$S_{\text{gf}}[X] = \frac{T_p}{2} \int d^{p+1}\sigma \left[g_{ij}\partial_0 X^i \partial_0 X^j + \det(g_{ij}\partial_a X^i \partial_b X^j) \right]$$

introduce multi-index notation

$$\tilde{g}_{IJ} \equiv \sum_{\pi \in \mathfrak{S}_p} \operatorname{sgn}(\pi) g_{i_{\pi(1)} j_1} \cdots g_{i_{\pi(p)} j_p}$$

Membrane actions

gauge-fixed p-brane action in multi-index notation

$$S_{\mathrm{gf}}[X] = \frac{T_p}{2} \int d^{p+1}\sigma \left[g_{ij} \partial_0 X^i \partial_0 X^j + \tilde{g}_{IJ} \widetilde{\partial X}^I \widetilde{\partial X}^J \right]$$

add background C_{p+1}-field

$$\frac{1}{(p+1)!} C_{ij_1 \ldots j_p} dx^i dx^{j_1} \ldots dx^{j_p}$$

with field strength $H = dC$ $\rightarrow$ membrane σ model

$$S[X] = \int d^{p+1}\sigma \Big[g_{ij} \partial_0 X^i \partial_0 X^j + \tilde{g}_{IJ} \widetilde{\partial X}^I \widetilde{\partial X}^J - i \int d^{p+1}\sigma \sum_{i,J} C_{iJ} \partial_0 X^i \widetilde{\partial X}^J \Big]$$

Membrane versus Nambu sigma model

Closed-open membrane relations

$$
\begin{aligned}
g + C\tilde{g}^{-1}C^T &= G + \Phi\tilde{G}^{-1}\Phi^T \\
\tilde{g} + C^T g^{-1} C &= \tilde{G} + \Phi^T G^{-1}\Phi \\
g^{-1}C &= G^{-1}\Phi - \Pi(\tilde{G} + \Phi^T G^{-1}\Phi) \\
C\tilde{g}^{-1} &= \Phi\tilde{G}^{-1} - (G + \Phi\tilde{G}^{-1}\Phi^T)\Pi
\end{aligned}
$$

$$
\frac{1}{g + C} = \frac{1}{G + \Phi} + \Pi \qquad (\text{for } p = 1)
$$

Nambu-Dirac-Born-Infeld action

(B Jurco & PS 2012)

commutative $\leftrightarrow$ non-commutative duality implies

$$S_{p\text{-DBI}} = \int d^n x \frac{1}{g_m} \det{}^{\frac{p}{2(p+1)}} [g] \det{}^{\frac{1}{2(p+1)}} \left[g + (C+F)\tilde{g}^{-1}(C+F)^T \right]$$

$$= \int d^n x \frac{1}{G_m} \det{}^{\frac{p}{2(p+1)}} \left[\widehat{G} \right] \det{}^{\frac{1}{2(p+1)}} \left[\hat{G} + (\hat{\Phi} + \hat{F}) \widehat{\tilde{G}}^{-1} (\hat{\Phi} + \hat{F})^T \right]$$

This action interpolates between early proposals based on supersymmetry and more recent work featuring higher geometric structures.

Expansion of action

ignore a cosmological constant term and let $\mathcal{F} = C + F$

$$\mathcal{S}_{p\text{-DBI}} = \int d^n x \frac{1}{2(p+1)g_m} \det{}^{\frac{1}{2}}(g)\,\mathrm{tr}\left[g^{-1}\mathcal{F}\tilde{g}^{-1}\mathcal{F}^T\right] + \dots$$

the coupling constant g_m is dimensionless for:

- strings on D3 with 2-form field strength (Maxwell/Yang-Mills)
- 2-branes on 5-brane with 3-form field strength ($\rightsquigarrow$ M2-M5 system)
- p-branes on $2(p+1)$-brane with $p+1$ form field strength

consider $p = 2$, $p' = 5$ and expand further ($k = \mathcal{F}_i^{kl}\mathcal{F}_{jkl}$):

$$det^{\frac{1}{6}}(1+k) = \sqrt{1 + \frac{1}{3}\mathrm{tr}k - \frac{1}{6}\mathrm{tr}k^2 + \frac{1}{18}(\mathrm{tr}\,k)^2 + \dots}.$$

$\Rightarrow$ exact match with κ-symmetry computation of

Cederwall, Nilsson, Sundell, "An Action for the superfive-brane" (1998)

From higher gauge theory to matrix model...

dual NC model in background independent gauge $\Pi C^T = -1$

expanding to lowest order (ignore a non-cosmological constant) $\Rightarrow$

semi-classical/infinite-dimensional version of a matrix model

$$\int d^{p+1}x \, \frac{1}{|\Pi|^{\frac{1}{p+1}}} \, \frac{1}{2(p+1)\widehat{g}_m} \cdot \hat{g}_{i_0 j_0} \cdots \hat{g}_{i_p j_p} \{\widehat{X}^{j_0}, \ldots, \widehat{X}^{j_p}\}\{\widehat{X}^{i_0}, \ldots, \widehat{X}^{i_p}\}$$

quantize:

$$\rightsquigarrow \frac{1}{2(p+1)\widehat{g}_m} \, \mathrm{Tr}\left(\hat{g}_{i_0 j_0} \cdots \hat{g}_{i_p j_p} \left[\widehat{X}^{j_0}, \ldots, \widehat{X}^{j_p}\right] \left[\widehat{X}^{i_0}, \ldots, \widehat{X}^{i_p}\right]\right)$$