# *On the spectral geometry of $\kappa$-Minkowski space*

Marco Matassa

SISSA - Trieste

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- We are interested in studying "non-commutative spaces", where

$$[X^\mu, X^\nu] = \Theta^{\mu\nu}(X) = \underbrace{\Theta^{\mu\nu}_{(0)}}_{\text{Moyal plane}} + \underbrace{\Theta^{\mu\nu}_{(1)\rho} X^\rho}_{\text{Lie algebra-type}} + \cdots \, .$$

- $\kappa$-Minkowski as a "toy model" for spacetime in quantum gravity:
  - space on which deformed symmetries ($\kappa$-Poincaré) act,
  - contains features of 2+1d quantum gravity,
  - related to deformed special relativity (DSR) proposal.
- Can we describe (the Euclidean version) using spectral triples?
- I will try to argue that one needs to use a framework which accomodates the modular properties of this geometry.

- The $\kappa$-Poincaré algebra $\mathcal{P}_\kappa$ is a deformation of the Poincaré algebra [Lukierski, Nowicki, Ruegg, Tolstoy (1991)]. In two dimensions it satisfies the commutation relations (here $\lambda := \kappa^{-1}$)

$$[P_\mu, P_\nu] = 0 \ , \quad [N, P_0] = P_1 \ ,$$
$$[N, P_1] = \frac{1}{2\lambda}(1 - e^{-2\lambda P_0}) - \frac{\lambda}{2} P_1^2 \ .$$

- The coproduct is defined by the relations

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \ , \quad \Delta(P_1) = P_1 \otimes 1 + e^{-\lambda P_0} \otimes P_1 \ .$$

- To avoid formal series we define the element $\mathcal{E} := e^{-\lambda P_0}$, for which $\Delta(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$. The subalgebra generated by $P_\mu$ and $\mathcal{E}$ is the extended momentum algebra $\mathcal{T}_\kappa$.
- $\kappa$-Minkowski space is introduced as a dual Hopf algebra to $\mathcal{T}_\kappa$ [Majid, Ruegg (1991)]. One obtains the relation $[X^0, X^1] = -\lambda X^1$.

- The notion of spectral triple provides the basis for non-commutative geometry in the sense of Alain Connes [Connes (1994)].

## Definition

A compact spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the data of a unital $*$-algebra $\mathcal{A}$, a faithful $*$-representation $\pi$ on a Hilbert space $\mathcal{H}$, and a self-adjoint operator $D$ such that

- $[D, \pi(a)]$ extends to a bounded operator for all $a \in \mathcal{A}$.
- $(D - \mu)^{-1}$ is compact for all $\mu \notin \mathrm{sp}D$.

- With some extra conditions one has the reconstruction theorem for compact spin manifolds, where $D$ is the Dirac operator.
- Modifications needed to treat non-compact cases. We require $\pi(a)(D - \mu)^{-1}$ to be compact.

- The starting point is the introduction of a $*$-algebra associated with the commutation relations $[X^0, X^1] = -\lambda X^1$. We use the algebra and $\star$-product formulation given in [Durhuus, Sitarz (2011)].

**Theorem (Durhuus-Sitarz 2011)**

*The algebra $\mathcal{A}$ is a left $\mathcal{T}_\kappa$-module $*$-algebra with respect to the following representation of the extended momentum algebra*

$$(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x) \ , \qquad (\mathcal{E} \triangleright f)(x) = f(x_0 + i\lambda, x_1) \ .$$

- This means compatibility with the $\kappa$-Poincaré symmetries. That is for all $h \in \mathcal{T}_\kappa$ and $f, g \in \mathcal{A}$, we have

$$h \triangleright (fg) = (h_{(1)} \triangleright f)(h_{(2)} \triangleright g) \ ,$$
$$(h \triangleright f)^* = S(h)^* \triangleright f^* \ .$$

- Now we want to introduce a Hilbert space, on which the algebra $\mathcal{A}$ is represented as bounded operators.
- Use the GNS construction by choosing a weight $\omega$ on $\mathcal{A}$. Choose

$$\omega(f) := \int f(x) d^2 x \ .$$

- Motivated by simplicity, same as in the commutative case, but more importantly by the following invariance property.

**Proposition (Durhuus-Sitarz 2011)**

*We have that $\omega$ is invariant under the action of $\mathcal{P}_\kappa$. This means that for any $h \in \mathcal{P}_\kappa$ and $f \in \mathcal{A}$ we have $\omega(h \triangleright f) = \varepsilon(h)\omega(f)$.*
*It also satisfies the twisted trace property $\omega(fg) = \omega((\mathcal{E} \triangleright g) f)$.*

- Recall the KMS condition, which can be written as

$$\psi(fg) = \psi(\alpha_{-i}(g)f) \ .$$

We can restate the twisted trace property in this language.

**Proposition**

*The weight $\omega$ satisfies the KMS condition at inverse temperature $\beta = 1$ with respect to $\alpha$, defined by $\alpha_t(f)(x) = f(x_0 - \lambda t, x_1)$.*
*The corresponding modular operator is given by $\Delta_\omega = e^{-\lambda \hat{P}_0}$.*

- This means that for $f \in \mathcal{A}$ we have

$$\Delta_\omega^{it} \pi(f) \Delta_\omega^{-it} = \pi(\alpha_t(f))$$

- The modular properties of this geometry appear naturally by choosing the invariant weight $\omega$.

- In this non-commutative geometry framework there is a notion of non-commutative integral, which we define by

$$\varphi(f) = \mathrm{Res}_{s=n} \mathrm{Tr} \left( f (D^2 + \mu^2)^{-s/2} \right) \ .$$

  Here $n$ is the spectral dimension (to be defined shortly).
- It satisfies the trace property $\varphi(fg) = \varphi(gf)$ (hypertrace).
- We know from examples that, if we have a trace $\tau$ on the algebra, then $\varphi(f)$ reduces to $\tau(f)$ up to a constant.
- In the situation $\omega(fg) \neq \omega(gf)$ (that is, the generic one) we can not recover $\omega$ from $\varphi$. But $\omega$ is a natural notion of integration!
- Modifications are needed to handle these cases.

- Twisted spectral triples [Connes, Moscovici (2006)].
Require $[D, f]_\sigma = Df - \sigma(f)D$ to be bounded, where $\sigma$ is an automorphism of $\mathcal{A}$. The non-commutative integral then obeys

$$\varphi(fg) = \varphi\left(\sigma^n(g)f\right) \ .$$

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- Modular spectral triples [Carey, Phillips, Rennie (2010)].
  Use a weight $\Phi$ instead of the operator trace

$$\varphi(f) = \mathrm{Res}_{s=n}\Phi\left(f(D^2+\mu^2)^{-s/2}\right) \ ,$$

where $\Phi(\cdot) = \mathrm{Tr}(\Delta_\Phi \cdot)$. Then we have the modular property

$$\varphi(fg) = \varphi\left(\sigma_i^\Phi(g)f\right)$$

where $\sigma_t^\Phi(f) = \Delta_\Phi^{it} f \Delta_\Phi^{-it}$ is the modular group of $\Phi$.

- We now need a self-adjoint operator $D$ such that, for any $f \in \mathcal{A}$, the commutator $[D, f]$ extends to a bounded operator.
- Consider the classical Dirac operator $D = \Gamma^\mu \hat{P}_\mu$, where $\hat{P}_\mu = -i\partial_\mu$. Using the equivariance of the representation $\pi$ we obtain

$$\hat{P}_1 \pi(f)\psi = \rho(P_1)\pi(f)\psi = \pi(P_1 \triangleright f)\psi + \pi(\mathcal{E} \triangleright f)\rho(P_1)\psi \ .$$

- As a result the commutator is unbounded, since $P_0$ is a derivation while $P_1$ is a twisted derivation.
- Instead we require the boundedness of the twisted commutator

$$[D, f]_\sigma = Df - \sigma(f)D \ .$$

Here $\sigma$ is an automorphism of $\mathcal{A}$.

- $D$ is self-adjoint on $\mathcal{H} \otimes \mathbb{C}^2$ and $\{D, \chi\} = 0$. This implies that $D = \Gamma^\mu \hat{D}_\mu$, where $\hat{D}_\mu$ are self-adjoint operators.
- $D$ should obey the classical limit. We require that for all $\psi \in \mathcal{A}$ we have $\lim \hat{D}_\mu \psi = \hat{P}_\mu \psi$, for $\lambda \to 0$.
- $D$ and $\sigma$ are determined by the symmetries. This means that $\hat{D}_\mu = \rho(D_\mu)$ and $\sigma(f) = \sigma \triangleright f$, for some $D_\mu, \sigma \in \mathcal{T}_\kappa$.

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**Theorem**

*Under the previous assumptions we have that $[D, \pi(f)]_\sigma$ is bounded if and only if we have $D_0 = \frac{1}{\lambda}(1 - \mathcal{E})$, $D_1 = P_1$ and $\sigma = \mathcal{E}$.*

- Recall that $\mathcal{E} = e^{-\lambda \hat{P}_0}$, so formally $\frac{1}{\lambda}(1 - \mathcal{E}) \to P_0$ for $\lambda \to 0$.
- $D$ has nice compatibility properties (and related to the Casimir).

- What about compactness? And summability?
- Since we are in the non-compact case we use the definition given in [Carey, Gayral, Rennie, Sukochev (2012)].

## Definition

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a non-compact spectral triple. We let

$$p := \inf\{s > 0 \ : \ \forall f \in \mathcal{A},\ f \geq 0,\ \mathrm{Tr}\left(f(D^2+1)^{-s/2}\right) < \infty\}$$

and, when it exists, we say that the triple has spectral dimension $p$. In addition we say that it is $\mathcal{Z}_p$-summable if for all $a \in \mathcal{A}$ we have

$$\limsup_{s \downarrow p} \left|(s-p)\mathrm{Tr}\left(f(D^2+1)^{-s/2}\right)\right| < \infty \ .$$

**Proposition**

*The operator $f(D^2+1)^{-s/2}$ is not trace class for any $s>0$.*

- We can give an heuristic explanation. For a twisted spectral triple the non-commutative integral satisfies

$$\varphi(fg) = \varphi(\sigma^n(g)f) \ .$$

- Suppose we had a finite spectral dimension $n=2$. Since the twist is given by $\sigma = \mathcal{E}$ this would imply that

$$\varphi(fg) = \varphi\left((\mathcal{E}^2 \triangleright g)f\right) \ .$$

- But then on the other hand the weight $\omega$ satisfies

$$\omega(fg) = \omega\left((\mathcal{E} \triangleright g)f\right) \ .$$

- This mismatch indicates that we might be able to solve the problem by replacing $\mathrm{Tr}$ with $\Phi$ in the non-commutative integral.
- In this case we expect to have the following modular property

$$\varphi(fg) = \varphi\left(\sigma_i^{\Phi}\left(\sigma^n(g)\right) f\right) \ ,$$

where $n$ is the spectral dimension and $\sigma_t^{\Phi}(f) = \Delta_{\Phi}^{it} f \Delta_{\Phi}^{-it}$.
- If we choose $\Phi(\cdot) = \mathrm{Tr}(\Delta_{\Phi} \cdot)$ with $\Delta_{\Phi} = e^{-\lambda \hat{P}_0}$ then we expect

$$\varphi(fg) = \varphi\left((\mathcal{E}^{n-1} \triangleright g) f\right) \ .$$

- Therefore if $n = 2$ the non-commutative integral has the same modular property as the weight $\omega(fg) = \omega\left((\mathcal{E} \triangleright g) f\right)$.

**Theorem**

*The non-compact modular spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *with weight* $\Phi$ *has spectral dimension* $n = 2$. *Moreover for all* $f \in \mathcal{A}$ *and* $\mu \neq 0$ *we have*

$$\mathrm{Res}_{s=2}\Phi\left(f(D^2+\mu^2)^{-s/2}\right) = \frac{1}{2\pi}\omega(f) \ .$$

- The spectral dimension coincides with the classical dimension.
- We recover the weight $\omega$ from the residue at $s = 2$ of the zeta function. Expected (but not obvious!) since

$$\varphi(f) = \mathrm{Res}_{s=2}\Phi\left(f(D^2+\mu^2)^{-s/2}\right)$$

  was constructed to have the same modular properties of $\omega$.

- In $n$ dimensions we have $\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0$ and $\Delta(P_k) = P_k \otimes 1 + \mathcal{E} \otimes P_k$, with $k = 1, \cdots, n-1$.
- Repeat the construction. The weight is $\omega(f) = \int f(x) d^n x$ and we have $\omega(fg) = \omega((\mathcal{E}^{n-1} \triangleright g) f)$.
- The twisted commutator $[D, f]_\sigma$ is bounded if and only if we have $D_0 = \frac{1}{\lambda}(1 - \mathcal{E})$, $D_k = P_k$ and $\sigma = \mathcal{E}$.
- Repeating the previous considerations we set again $\Delta_\Phi = e^{-\lambda \hat{P}_0}$. Then we find the spectral dimension equal to $n$ and

$$\mathrm{Res}_{s=n} \Phi \left( f (D^2 + \mu^2)^{-s/2} \right) = c_n \omega(f) \ .$$

- The ingredients are the same, little changes!

- We can look more in detail at the associated zeta function.

**Proposition**

*Let $f \in \mathcal{A}$ and $\mathrm{Re}(z) > n$. Then we have*

$$\zeta_f(z) := \Phi\left(f(D^2+\mu^2)^{-z/2}\right) = \frac{2^{[n/2]}}{(2\pi)^n} I(z) \int f(x) d^n x \ ,$$

*where $I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z))$, $I_c(z)$ is the classical result and*

$$I_\lambda(z) = \pi^{(n-1)/2} \mu^{(n-1)-z} \frac{\Gamma\left(\frac{z-(n-1)}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \lambda^{-1} {}_2F_1\left(\frac{1}{2}, \frac{z-(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda\mu)^2}\right) \ .$$

*The function $I(z)$ reduces to the classical one $I_c(z)$ in the limit $\lambda \to 0$.*

- Analytical continuation with only simple poles.

- Another notion of dimension is given by the homological dimension.
- We consider the twisted Hochschild homology of $U(\mathfrak{g}_\kappa)$. This is defined as $H_*(U(\mathfrak{g}_\kappa), {}_\sigma U(\mathfrak{g}_\kappa))$, where ${}_\sigma U(\mathfrak{g}_\kappa)$ the algebra $U(\mathfrak{g}_\kappa)$ with the bimodule structure $a \cdot b \cdot c = \sigma(a)bc$.
- The twisted Hochschild dimension is defined as the maximum of the homological dimension over all the automorphisms of $U(\mathfrak{g})$ [Brown, Zhang (2008)].

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**Theorem**

*The twisted homological dimension of $U(\mathfrak{g}_\kappa)$ is equal to $n$.*

- There are two features worth pointing out:
  1. without the twist we have a dimension drop,
  2. the simplest twist is the inverse modular group of $\omega$.

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- A. Carey, J. Phillips, A. Rennie, *Twisted cyclic theory and an index theory for the gauge invariant KMS state on the Cuntz algebra $O_n$*, Journal of K-theory 6.02 (2010): 339-380.
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Thank you for you attention!