Recent developments in the type IIB matrix model

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1. Introduction

Particle physics

- Standard Model
- discovery of SM-like Higgs particle at LHC
- What is the origin of Higgs particle? SM fermions? (3 generations) the SM gauge group?
- hierarchy problem

Cosmology

- Inflation and Big Bang
- CMB (WMAP, PLANCK) structure formation of galaxies nucleosynthesis
- What is the origin of inflaton, its potential, its initial condition? dark matter, dark energy?
- cosmological constant problem

successful phenomenological models

experiments and observations

missing fundamental understanding

serious naturalness problems

All these problems may be solved by superstring theory!

string phenomenology

string cosmology
The most fundamental issue in superstring theory is **space-time dimensionality**

(10-dimensional space-time is required for consistency.)

**conventional approach:**

- compactify extra dimensions.
- infinitely many **perturbative** vacua
  - with various
    - space-time dimensionality
    - gauge symmetry
    - matter (#generations)
- **D-branes** (representing certain nonperturbative effects)
  - enriched both string phenomenology and string cosmology
  - intersecting D-brane models
  - D-brane inflation models
  - flux compactification …

Too many models, and no predictive power !!!
However, we should not forget that all these perspectives are obtained from mostly perturbative studies of superstring theory including at most the nonpertubative effects represented by the existence of D-branes.

A totally new perspective might appear if one studies superstring theory in a complete nonperturbative framework.

c.f.) lattice gauge theory in the case of QCD confinement of quarks, hadron mass spectrum, etc.
(Can never be understood from perturbation theory !!!)
type IIB matrix model  (Ishibashi, Kawai, Kitazawa, Tsuchiya, 1996)

nonperturbative formulation of superstring theory
based on type IIB superstring theory in 10d

• The connection to perturbative formulations can be seen manifestly by considering 10d type IIB superstring theory in 10d.

  worldsheets action,
  light-cone string field Hamiltonian, etc.

• A natural extension of the “one-matrix model”, which is established as a nonperturbative formulation of non-critical strings regarding Feynman diagrams in matrix models as string worldsheets.

• It is expected to be a nonperturbative formulation of the unique theory underlying the web of string dualities.

(Other types of superstring theory can be represented as perturbative vacua of type IIB matrix model)
type IIB matrix model

\[
S_b = -\frac{1}{4g^2} \text{tr}([A_\mu, A_\nu][A^\mu, A^\nu]) \\
S_f = -\frac{1}{2g^2} \text{tr}(\Psi_\alpha(\mathcal{C} \Gamma^\mu)_{\alpha\beta}[A_\mu, \Psi_\beta])
\]

$N \times N$ Hermitian matrices

$A_\mu \ (\mu = 0, \ldots, 9)$ Lorentz vector

$\Psi_\alpha \ (\alpha = 1, \ldots, 16)$ Majorana-Weyl spinor

Lorentzian metric $\eta = \text{diag}(-1, 1, \ldots, 1)$ is used to raise and lower indices

Wick rotation $(A_0 = -iA_{10}, \ \Gamma^0 = i\Gamma_{10})$

Euclidean matrix model

SO(10) symmetry
Lorenzian v.s. Euclidean

There was a reason why no one dared to study the Lorentzian model for 15 years!

\[ S_b \propto \text{tr} \left( F_{\mu\nu} F^{\mu\nu} \right) = -2 \text{tr} \left( F_{0i} \right)^2 + \text{tr} \left( F_{ij} \right)^2 \]

\[ F_{\mu\nu} = -i \left[ A_\mu, A_\nu \right] \]

opposite sign! extremely unstable system.

Once one Euclideanizes it, \[ A_0 = -i A_{10} \]

positive definite!

flat direction ( \[ \left[ A_\mu, A_\nu \right] \sim 0 \] ) is lifted up due to quantum effects.

Aoki-Iso-Kawai-Kitazawa-Tada ('99)
Euclidean model is well defined without the need for cutoffs.

Krauth-Nicolai-Staudacher ('98),
Austing-Wheater ('01)
Recent developments

- Euclidean model
  
  $SSB: SO(10) \rightarrow SO(3)$

  $\frac{R}{r} \sim 5$

  Interpretation of these results is unclear, though.
  Wick rotation is not justifiable unlike in ordinary QFT!

- Lorentzian model
  
  Can be made well-defined by introducing IR cutoffs and removing them in the large-$N$ limit.
  Real-time evolution can be extracted from matrix configurations.
  Expanding 3d out of 9d (SSB), inflation, big bang,…
  A natural solution to the cosmological constant problem.
  Realization of the Standard Model (Tsuchiya’s talk)

Loventzian version of type IIB matrix model is indeed the correct nonperturbative formulation of superstring theory, which describes our Universe.
Plan of the talk

1. Introduction
2. Euclidean type IIB matrix model
3. Lorentzian type IIB matrix model
4. Expanding 3d out of 9d
5. Exponential/power-law expansion
6. Time-evolution at much later times
7. Summary and future prospects
2. Euclidean type IIB matrix model


Most recent results based on the Gaussian expansion method


$d=3$ gives the minimum free energy

This is an interesting dynamical property of the Euclideanized type IIB matrix model.

However, its physical meaning is unclear...

$SO(10) \rightarrow SO(3)$
Results for 6d SUSY model (Gaussian expansion method)


$d=3$ gives the minimum free energy

The mechanism for the SSB is demonstrated by Monte Carlo studies in this case. "phase of the fermion determinant"

$d=3$ gives the minimum free energy

universal shrunken direction

extended direction

$SO(6) \rightarrow SO(3)$

The mechanism for the SSB is demonstrated by Monte Carlo studies in this case. "phase of the fermion determinant"
No SSB if the phase is omitted


\[ T_{\mu\nu} = \frac{1}{N} \Tr(A_\mu A_\nu) \]

6 × 6 real symmetric

eigenvalues: \( \lambda_1 > \lambda_2 > \cdots > \lambda_6 \)

All the eigenvalues converge to the same value at large \( N \)!
Effects of the phase: universal shrunken direction


\[ \text{SO}(2) \text{ vacuum} \]

\[ \text{SO}(3) \text{ vacuum} \]

\[ \text{SO}(4) \text{ vacuum} \]

\[ \text{SO}(5) \text{ vacuum} \]

Effects of the phase of fermion determinant

Analysis based on the factorization method

Anagnostopoulos – J. N.
hep-th/0108041

The extent of the shrunken direction \( x \sim 0.35 \) for all vacua.

Consistent with the GEM calculations!

c.f.) \( x = \frac{r^2}{\ell^2} \sim \frac{0.223}{0.627} \sim 0.355 \)
After all, the problem was in the Euclideanization?

- In QFT, it can be fully justified as an **analytic continuation**. (That’s why we can use **lattice gauge theory**.)

- On the other hand, it is subtle in **gravitating theory**. (Although it might be OK at the classical level...)

  - Quantum gravity based on dynamical triangulation (Ambjorn et al. 2005) (Problems with Euclidean gravity may be overcome in Lorentzian gravity.)

  - Coleman’s **worm hole scenario for the cosmological constant problem** (A physical interpretation is possible only by considering the Lorentzian version instead of the original Euclidean version.) Okada-Kawai (2011)

- Euclidean theory is useless for studying the **real time dynamics** such as the expanding Universe.
3. Lorentzian type IIB matrix model

Kim-J.N.-Tsuchiya  
PRL 108 (2012) 011601  
[arXiv:1108.1540]
Definition of Lorentzian type IIB matrix model

Kim-J.N.-Tsuchiya
PRL 108 (2012) 011601
[arXiv:1108.1540]

partition function

\[ Z = \int dA \, d\psi \, e^{iS} = \int dA \, e^{iS_b} \text{Pf} \mathcal{M}(A) \]

This seems to be natural from the connection to the worldsheet theory.

\[ S = \int d^2 \xi \sqrt{g} \left( \frac{1}{4} \{X^\mu, X^\nu\}^2 + \frac{1}{2} \bar{\psi} \gamma^\mu \{X^\mu, \psi\} \right) \]
\[ \xi_0 \equiv -i \xi_2 \quad (\text{The worldsheet coordinates should also be Wick-rotated.}) \]
Regularization and the large-N limit

Unlike the Euclidean model, the Lorentzian model is NOT well defined as it is.

- The extent in temporal and spatial directions should be made finite. (by introducing cutoffs)

\[
\frac{1}{N} \text{tr} \left( A_0 \right)^2 \leq \kappa L^2
\]
\[
\frac{1}{N} \text{tr} \left( A_i \right)^2 \leq L^2
\]

In what follows, we set \( L = 1 \) without loss of generality.

- It turned out that these two cutoffs can be removed in the large-N limit. (highly nontrivial dynamical property)

- Both SO(9,1) symmetry and supersymmetry are broken explicitly by the cutoffs. The effect of this explicit breaking is expected to disappear in the large-N limit. (needs to be verified.)
Sign problem can be avoided!

\[ Z = \int dA \, d\Psi \, e^{iS} = \int dA \, e^{iS_b} \, \text{Pf} \, \mathcal{M}(A) \]  

The two possible sources of the problem.

1. Pfaffian coming from integrating out fermions  
\[ \text{Pf} \, \mathcal{M}(A) \in \mathbb{R} \]  
The configurations with positive Pfaffian become dominant at large \( N \).

2. Pfaffian coming from integrating out fermions  
\[ \text{Pf} \, \mathcal{M}(A) \in \mathbb{C} \]  
This complex phase induces the SSB of SO(10) symmetry.

J.N.-Vernizzi ('00), Anagnostopoulos-J.N.('02)
Sign problem can be avoided! (Con’d)

(2) What shall we do with $e^{i S_b}$

$$Z = \int dA \, e^{i S_b} \text{Pf} \mathcal{M}(A)$$

$A_\mu = \rho \tilde{A}_\mu$  

First, do $\int d\rho$

$$S_b \propto \text{tr}([A_\mu, A_\nu][A^\mu, A^\nu]) \propto \rho^4$$

$$\text{Pf} \mathcal{M}(A) \propto \rho^{16(N^2-1)}$$

The same problem occurs in QFT in Minkowski space
(Studying real-time dynamics in QFT is a notoriously difficult problem.)

homogenous in $A_\mu$

$$Z \propto \int d\tilde{A} \, \delta(S_b(\tilde{A})) \text{Pf} \mathcal{M}(\tilde{A})$$
4. Expanding 3d out of 9d

Kim-J.N.-Tsuchiya
[arXiv:1108.1540]

PRL 108 (2012) 011601
How to extract time-evolution

$A_0 = \begin{pmatrix}
\alpha_1 \\
. \\
. \\
. \\
\alpha_{\nu} + n \\
. \\
. \\
\alpha_N \\
. \\
. \\
. \\
\end{pmatrix}$

diagonalize $A_0$
$\alpha_1 < \cdots < \alpha_N$

$SU(N)$ transformation

$A_i = \begin{pmatrix}
\text{small} \\
\text{small} \\
\text{small} \\
\end{pmatrix}$

definition of time “$t$”

$$t = \frac{1}{n} \sum_{i=1}^{n} \alpha_{\nu+i}$$

The state of the universe $\bar{A}_i(t)$ at time $t$

$A_i$ has a band diagonal structure

non-trivial dynamical property
Band-diagonal structure

\[ N = 16 \]

We choose \( n = 4 \) as the block size
Spontaneous breaking of SO(9)

\[ T_{ij}(t) = \frac{1}{n} \text{tr} \{ \bar{A}_i(t) \bar{A}_j(t) \} \]

SO(9) → SO(3)

\[ N = 16, \quad \kappa = 4.0 \]


“critical time”
5. Exponential/power-law expansion

Ito-Kim-Koizuka-J.N.-Tsuchiya, in prep.
Ito-Kim-J.N.-Tsuchiya, work in progress
Exponential expansion

Inflation

$I$to-$K$im-$J$.N.-$T$suciya, work in progress

\[ R(t)^2 \equiv \frac{1}{n} \text{tr} \bar{A}_i(t)^2 \]

fitted well to
\[ f(x) = a + (1 - a) e^{bx} \]
Effects of fermionic action

\[ S_f = \text{tr}(\bar{\Psi}_\alpha(\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi_\beta]) \]
\[ = \text{tr}(\bar{\Psi}_\alpha(\Gamma^0)_{\alpha\beta}[A_0, \psi_\beta]) + \text{tr}(\bar{\Psi}_\alpha(\Gamma^i)_{\alpha\beta}[A_i, \psi_\beta]) \]

dominant term at early times

keep only the first term

simplified model at early times

\[ \text{Pf} \mathcal{M}(A) \sim \Delta^{d-1} = \prod_{i<j} (\alpha_i - \alpha_j)^{2(d-1)} \]

repulsive force between eigenvalues of \( A_0 \)

dominant term at late times

simplified model at late time

\[ \text{Pf} \mathcal{M}(A) \sim 1 \]

quench fermions
Exponential expansion at early times

Ito-Kim-Koizuka-J.N.-Tsuchiya, in prep.

- simplified model at early times

\[ \text{Pf} \mathcal{M}(A) \simeq \Delta^{d-1} = \prod_{i<j} (\alpha_i - \alpha_j)^{2(d-1)} \]

The first term is important for exponential expansion.
Power-law expansion at late times

- simplified model at late times

\[ P^f_M(A) \approx 1 \]

\[ g(x) = ax + b \]

\[ R^2 \sim t \Rightarrow R \sim t^{1/2} \]

\[ f(x) = a + (1 - a)e^{bx} \]

\[ t^{1/2} \text{ behavior} \]

Radiation dominated FRW universe
Expected scenario for the full Lorentzian IIB matrix model

\[ R(t)^2 = r^2 + \alpha e^{\Lambda t} \]

\[ R(t) = C t^{1/2} \]

radiation dominated FRW universe

E-folding is determined by the dynamics!
6. Time-evolution at much later times

Time-evolution at much later times

\[ Z = \int dA \, d\Psi \, e^{iS} = \int dA \, e^{iS_b} \text{Pf} \mathcal{M}(A) \]

The cosmic expansion makes each term in the action larger at much later times.

- There are infinitely many classical solutions. (Landscape)
- There is a simple solution representing a (3+1)D expanding universe, which naturally solves the cosmological constant problem.
- Since the weight for each solution is well-defined, one should be able to determine the unique solution that dominates at late times.
- By studying the fluctuation around the solution, one should be able to derive the effective QFT below the Planck scale.

General prescription for solving EOM

- variational function \((i = 1, \ldots, 9)\)

\[
\tilde{S} = \text{tr} \left( -\frac{1}{4} [A_\mu, A_\nu][A^\mu, A^\nu] + \frac{\lambda}{2} (A_0^2 - \kappa L^2) - \frac{\lambda}{2} (A_i^2 - L^2) \right)
\]

- classical equations of motion

\[
- [A_0, [A_0, A_i]] + [A_j, [A_j, A_i]] - \lambda A_i = 0
\]

\[
[A_j, [A_j, A_0]] - \tilde{\lambda} A_0 = 0
\]

- commutation relations

\[
[A_i, A_j] = iC_{ij} \quad \text{Eq. of motion & Jacobi identity}
\]

\[
[A_i, C_{jk}] = iD_{ijk} \quad \text{Lie algebra}
\]

\[
[A_0, A_i] = iE_i \quad \text{Can be made finite-dimensional by imposing simplifying Ansatz.}
\]

\[
[A_0, E_i] = iF_i
\]

\[
[A_i, E_j] = iG_{ij} \quad \text{Unitary representation}
\]

\[
[A_i, E_j] = iG_{ij} \quad \text{classical solution}
\]
An example of SO(4) symmetric solution (RxS\(^3\) space-time)

\[
EOM \quad \frac{\delta}{\delta A_{\mu}} \left( -\frac{1}{4} \text{tr}([A_{\mu}, A_{\nu}]^2) - \frac{\lambda}{2} \text{tr}(A_i)^2 + \frac{\bar{\lambda}}{2} \text{tr}(A_0)^2 \right) = 0
\]

\[
A_0 = bT_0 \otimes 1_k \\
A_i = \alpha bT_1 \otimes M_i \quad (i = 1 \sim 4) \\
A_5 \sim A_9 = 0
\]

*SL(2, \(R\)) algebra*

\[
[T_0, T_1] = iT_2 \quad [T_0, T_2] = -iT_1 \quad [T_1, T_2] = -iT_0
\]

\[
M_i = \text{diag}(n_i^{(1)}, \ldots, n_i^{(k)}) \quad (i = 1, \ldots, 4)
\]

\[
|n^{(I)}| = 1 \quad (I = 1, \ldots, k) \quad \text{uniformly distributed on a unit } S^3
\]

\[
\lambda = -b^2, \quad \bar{\lambda} = -\alpha^2 b^2
\]

Space-space is commutative.
An example of SO(4) symmetric solution (RxS$^3$ space-time) cont’d

primary unitary series

$$(T_0)_{mn} = n\delta_{mn}$$

$$(T_1)_{mn} = -\frac{i}{2} \left(n - i\rho + \frac{1}{2}\right) \delta_{m,n+1} + \frac{i}{2} \left(n + i\rho - \frac{1}{2}\right) \delta_{m,n-1}$$

Block size can be taken to be $n=3$.

the extent of space

$$R(n) = \sqrt{\frac{1}{3} \text{tr}(\bar{A}_i(n)^2)} = \frac{\alpha b}{\sqrt{3}} \sqrt{n^2 + \rho^2 + \frac{1}{4}}$$

$$R(t) = \frac{\alpha}{\sqrt{3}} \sqrt{t^2 + t_0^2}$$

space-time noncommutativity

$$\frac{-\frac{1}{3} \text{tr}([\bar{A}_0(n), \bar{A}_1(n)]^2)}{\frac{1}{3} \text{tr}(\bar{A}_0(n))^2 \frac{1}{3} \text{tr}(\bar{A}_1(n))^2} = \frac{1}{n^2 + \frac{2}{3}} \quad n \to \infty \quad 0$$

cont. lim. $b \to 0$ with $t = nb$, $\rho b = t_0$

$$\lambda = -b^2 \to 0$$

$$\bar{\lambda} = -\alpha^2 b^2 \to 0$$

consistent!
An example of $SO(4)$ symmetric solution ($R \times S^3$ space-time) cont’d

$$R(t) = \frac{\alpha}{\sqrt{3}} \sqrt{t^2 + t_0^2} = a(t)$$

$$H = \frac{\dot{a}}{a} \sim a^{-\frac{3}{2}(1+w)} \quad w = -\frac{1}{3} \left( \frac{2t_0^2}{t^2} + 1 \right)$$

\[\begin{align*}
  t = t_0 & \quad \Rightarrow \quad w = -1 \\
  \text{cosmological const.} & \sim (1/t_0)^4
\end{align*}\]

explains the accelerated expansion at present time

\[\begin{align*}
  t \to \infty & \quad \Rightarrow \quad w = -\frac{1}{3} \\
  \text{Cosmological constant disappears} & \text{ in the far future.}
\end{align*}\]

This part can be identified as a viable late-time behavior.

A natural solution to the cosmological constant problem.
7. Summary and future prospects
Summary

A nonperturbative formulation of superstring theory based on type IIB theory in 10d.

A well-defined theory can be obtained by introducing cutoffs and removing them in the large-$N$ limit.

The notion of “time evolution” emerges dynamically when we diagonalize $A_0$, has band-diagonal structure.

After some “critical time”, the space undergoes the SSB of SO(9), and only 3 directions start to expand.

Exponential expansion observed (Inflation, no initial condition problem.)

Power-law $^{1/2}$ expansion observed in a simplified model for later times.

Classical analysis is valid for much later times.

A natural solution to the cosmological constant problem suggested.

The problems with the Euclidean model have become clear.

Lorentzian model: untouched until recently because of its instability. Monte Carlo simulation has revealed its surprising properties.
Future prospects

- Observe directly the transition from the exponential expansion to the power-law expansion by Monte Carlo simulation.
- Does the transition to commutative space-time (suggested by a classical solution) occur at the same time?
- Can we calculate the **density fluctuation** to be compared with CMB?
- Can we read off the effective QFT below the Planck scale from fluctuations around a classical solution?
- Does **Standard Model** appear at low energy?  

(Tsuchiya’s talk)

Various fundamental questions in particle physics and cosmology:
- the mechanism of inflation, the initial value problem,
- the cosmological constant problem,
- the hierarchy problem, dark matter, dark energy, baryogenesis,
- the origin of the Higgs field, the number of generations etc.

It should be possible to understand all these problems in a unified manner by using the nonperturbative formulation of superstring theory.
Backup slides
Previous works in the Euclidean matrix model

- Perturbative expansion around diagonal configurations, branched-polymer picture
  Aoki-Iso-Kawai-Kitazawa-Tada (1999)

- The effect of complex phase of the fermion determinant (Pfaffian)
  J.N.-Vernizzi (2000)

- Monte Carlo simulation

- Gaussian expansion method
  J.N.-Sugino (2002)
  $S^2 \times S^2$

- Fuzzy

Dynamical generation of 4d space-time?

SSB of SO(10) rotational symmetry

A model with SO(10) rotational symmetry instead of SO(9,1) Lorentz symmetry
Emergence of the notion of “time-evolution”

\[ A_0 = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{pmatrix} \]

\[ A_i = \begin{pmatrix} t
\end{pmatrix} \]

\[ \bar{A}_i(t) \]

\[ \delta t \]

\[ t \]

\[ t_{\nu+1} \]

\[ t_{\nu+n} \]

\[ \nu = 0, 1, \ldots, N - n \]

\[ t = \frac{1}{n} \sum_{a=1}^{n} t_{\nu+a} \]

\[ \bar{A}_i(t) \] represents the state at the time \( t \)
The emergence of “time”

Supersymmetry plays a crucial role!

Calculate the effective action for

\[ A_0 = \text{diag}(\alpha_1, \cdots, \alpha_N) \]

at one loop.

- \[ A_i \ (i = 1, \cdots, d) \] contributes \[ \Delta^{d} \]
- \[ \psi_\alpha \ (\alpha = 1, \cdots, p) \] contributes \[ \Delta^{p/2} \]
- Contribution from van der Monde determinant

Altogether, \[ \Delta^{d+p/2+1} \]

Zero, in a supersymmetric model!
(e.g., \( d = 9, \ p = 16 \))

Attractive force between the eigenvalues in the bosonic model, cancelled in supersymmetric models.
The time-evolution of the extent of space

\[ R(t)^2 \equiv \frac{1}{n} \text{tr} \, \bar{A}_i(t)^2 \]

\[ N = 16 \, , \, n = 4 \]

symmetric under \( t \rightarrow -t \)

We only show the region \( t < 0 \)
SSB of SO(9) rotational symmetry

\[ T_{ij}(t) = \frac{1}{n} \text{tr} \{ \bar{A}_i(t)\bar{A}_j(t) \} \]

SO(9) $\xrightarrow{\text{SSB}}$ SO(3)

\[ N = 16, \quad \kappa = 4.0 \]
What can we expect by studying the time-evolution at later times

- What is seen by Monte Carlo simulation so far is: the birth of our Universe
- What has been thought to be the most difficult from the bottom-up point of view, can be studied first. This is a typical situation in a top-down approach!
- We need to study the time-evolution at later times in order to see the Universe as we know it now!

- Does inflation and the Big Bang occurs? (First-principles description based on superstring theory, instead of just a phenomenological description using “inflaton”; comparison with CMB etc.)
- How does the commutative space-time appear?
- What kind of massless fields appear on it?
- accelerated expansion of the present Universe (dark energy), understanding the cosmological constant problem
- prediction for the end of the Universe (Big Crunch or Big Rip or...)
Ansatz

extra dimension is small \( A_i = 0 \) for \( i > d \)
(extra dimension is small compared with Planck scale)

commutative space \( [A_i, A_j] = 0 \)

\[
\begin{align*}
[A_i, A_j] &= iC_{ij} \\
[A_i, C_{jk}] &= iD_{ijk} \\
[A_0, A_i] &= iE_i \\
[A_0, E_i] &= iF_i \\
[A_i, E_j] &= iG_{ij} \ldots \\
G_{ij} &= M_{ij} + N_{ij} + \frac{1}{d} \delta_{ij} H \\
[A_0, [A_i, A_j]] + [A_i, [A_j, A_0]] + [A_j, [A_0, A_i]] &= 0
\end{align*}
\]
Simplification

\[ M_{ij} = 0 \quad \text{for} \quad i \neq j \]

\[ M_i \equiv M_{ii} \quad \sum_{i=1}^{d} M_i = 0 \]

\[ \downarrow \]

Lie algebra

\[ [A_i, A_j] = 0 , \quad [A_0, A_i] = iE_i , \quad [A_0, E_i] = i\lambda A_i , \]

\[ [E_i, E_j] = 0 , \quad [A_i, E_j] = i\delta_{ij} \left( \frac{\tilde{\lambda}}{d} A_0 + M_i \right) , \quad [A_0, M_i] = 0 , \]

\[ [A_i, M_j] = i\frac{\tilde{\lambda}}{d} (1 - d\delta_{ij}) E_i , \quad [E_i, M_j] = i\frac{\lambda \tilde{\lambda}}{d} (1 - d\delta_{ij}) A_i , \quad [M_i, M_j] = 0 \]

e.g.)

\[ d = 2, \quad \lambda > 0, \quad \tilde{\lambda} > 0 \quad \rightarrow \quad SO(2, 2) \]
d=1 case

\[ [A_0, A_1] = iE, \quad [A_0, E] = i\lambda A_1, \quad [A_1, E] = i\bar{\lambda}A_0 \]

SO(9) rotation \[ r_i A_1 \quad (i = 1, \cdots, 9) \text{ with } r_i^2 = 1 \]

Take a direct sum

\[
\begin{align*}
A'_0 &= A_0 \otimes 1_K \\
A'_i &= A_1 \otimes \text{diag}(r_i^{(1)}, r_i^{(2)}, \cdots, r_i^{(K)}) \\
\text{where} \quad r_i^{(m)} 2 &= 1 \quad (m = 1, \cdots, K)
\end{align*}
\]

\[ r^{(m)} \text{ distributed on a unit } S^3 \]

\((3+1)\text{D space-time} \sim \mathbb{R} \times S^3\)

A complete classification of d=1 solutions has been done. Below we only discuss a physically interesting solution.
SL(2,R) solution

\[ [A_0, A_1] = iE , \quad [A_0, E] = i\lambda A_1 , \quad [A_1, E] = i\tilde{\lambda} A_0 \]

\[
A_0 = aT_2 , \quad A_1 = bT_0 , \quad E = cT_1 \\
\lambda = a^2 , \quad \tilde{\lambda} = b^2 , \quad ab = c
\]

\[ [T_0, T_1] = iT_2 , \quad [T_2, T_0] = iT_1 , \quad [T_1, T_2] = -iT_0 \]

realization of the SL(2,R) algebra on \( \{ e^{in\theta} ; n \in \mathbb{Z} \} \)

\[
\mathcal{T}_0 = i \frac{d}{d\theta} + \epsilon
\]

\[
\mathcal{T}_1 = \frac{i}{2} \left[ (\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2\sin \theta \frac{d}{d\theta} \right]
\]

\[
\mathcal{T}_2 = \frac{1}{2} \left[ -(\tau + \epsilon)e^{i\theta} + (\tau - \epsilon)e^{-i\theta} - 2i \cos \theta \frac{d}{d\theta} \right]
\]
Space-time structure in SL(2,R) solution

- primary unitary series representation

\[
(T_0)_{mn} = n\delta_{mn}
\]
\[
(T_1)_{mn} = -\frac{i}{2}(n - i\rho + \frac{1}{2})\delta_{m,n+1} + \frac{i}{2}(n + i\rho - \frac{1}{2})\delta_{m,n-1}
\]
\[
(T_2)_{mn} = -\frac{1}{2}(n - i\rho + \frac{1}{2})\delta_{m,n+1} - \frac{1}{2}(n + i\rho - \frac{1}{2})\delta_{m,n-1}
\]

tri-diagonal

\[
\bar{A}_0(n) = a \begin{pmatrix} n - 1 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n + 1 \end{pmatrix} \otimes 1_K
\]
\[
\bar{A}_1(n) = \frac{ib}{2} \begin{pmatrix} 0 & n + i\rho - \frac{1}{2} & 0 \\ -n + i\rho + \frac{1}{2} & 0 & n + i\rho + \frac{1}{2} \\ 0 & -n + i\rho - \frac{1}{2} & 0 \end{pmatrix} \otimes \text{diag}(r_1^{(1)}, \cdots, r_1^{(K)})
\]

Space-time noncommutativity disappears in the continuum limit.
Cosmological implication of SL(2,R) solution

- the extent of space

\[ R(n) \equiv \sqrt{\frac{1}{3K}} \text{tr}(\overline{A}_1(n))^2 = \sqrt{\frac{b^2}{3} \left( n^2 + \rho^2 + \frac{1}{4} \right)} \]

- Hubble constant and the \( w \) parameter

\[ R(t) = \sqrt{\frac{\alpha^2}{3}} (t^2 + t_0^2) \]

\[ t = na, \quad t_0 = \rho a \]

cont. lim. \quad a \to 0, \quad n \to \infty, \quad \rho \to \infty

- Hubble constant and the \( w \) parameter

\[ H(t) = \frac{\dot{R}(t)}{R(t)} = c \ R(t)^{-\frac{3}{2}(1+w)} \]

\[
\begin{cases}
  w = \frac{1}{3} & \text{radiation dominant} \\
  w = 0 & \text{matter dominant} \\
  w = -1 & \text{cosmological constant}
\end{cases}
\]
Cosmological implication of SL(2,R) solution (cont’d)

$t_0$ is identified with the present time.

\[ R(t) = \sqrt{\frac{\alpha^2}{3}}(t^2 + t_0^2) \]

This part is considered to give the late-time behavior of the matrix model.

\[ w = -\frac{2t_0^2}{3t^2} - \frac{1}{3} \]

Cosmological constant disappears in the future.

$w \rightarrow -\frac{1}{3}$ for $t \rightarrow \infty$  

\[ w = -1 \text{ present accelerated expansion} \]

$\text{cosmological const.} \sim \left(\frac{1}{t_0}\right)^4 \text{ a solution to the cosmological constant problem} \]
Seiberg’s rapporteur talk (2005) at the 23rd Solvay Conference in Physics

“Emergent Spacetime”

Understanding *how time emerges* will undoubtedly shed new light on some of the most important questions in theoretical physics including *the origin of the Universe*.

Indeed in the Lorentzian matrix model, not only space but also time emerges, and the origin of the Universe seems to be clarified.
The significance of the unique determination of the space-time dimensionality

It strongly suggests that superstring theory has a unique nonperturbative vacuum. By studying the time-evolution further, one should be able to see the emergence of commutative space-time and massless fields propagating on it.

It is conceivable that the SM can be derived uniquely.

This amounts to “proving” the superstring theory.

It is sufficient to identify the classical configuration which dominates at late times by studying the time-evolution at sufficiently late times.

Independently of this, it is important to study classical solutions and to study the fluctuations around them.