Large-N reduction and IIB matrix model

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IIB matrix model is one of the promising candidates of the non-perturbative definition of superstring, and many interesting aspects will be discussed by several speakers in this school.

In order to examine the dynamics of this model, it is useful to understand the large-N reduction, which is one of the universal features of large-N gauge theory.

In my talk, first I would like to explain how the large-N reduction appears in general large-N gauge theory, and then discuss why we expect that the IIB matrix model describes superstring correctly.

PART 1 How does the Large-N reduction appear?

1. What is Large-N reduction

Roughly speaking,

Physics of the large-N gauge theory with periodic boundary condition *does not depend* on the volume of the space-time.

More precisely,

Consider U(N) or SU(N) lattice gauge theoy

$$S_{\mathrm{Wilson}} = -\frac{N}{\lambda} \sum_{n,\mu\neq\nu} Tr\left(U_{n,\mu} U_{n+\hat{\mu},\nu} U^{\dagger}_{n+\hat{\nu},\mu} U^{\dagger}_{n,\nu}\right)$$

in a periodic box of size $L_1 \times L_2 \times \cdots L_d$.

Take the large-N limit $N \to \infty,\ \lambda$: fixed.

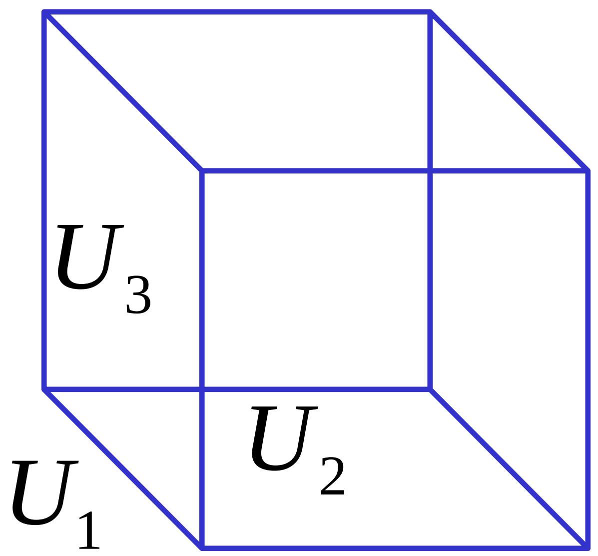
The physics of the system does not depend on the size, at least in the strong coupling region:

$$\lambda > \lambda_c\,,\ \lambda_c \sim 4.$$

In particular,

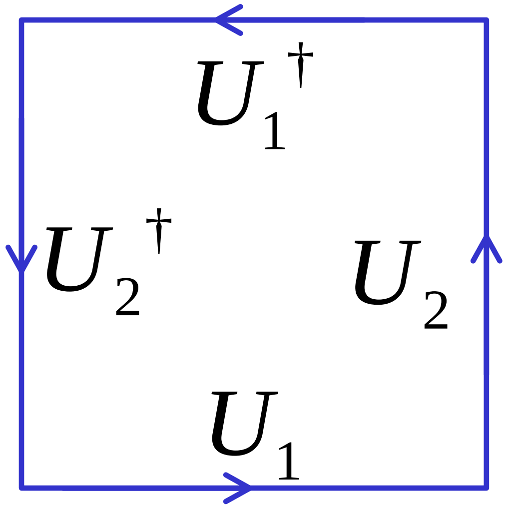
we can consider the minimum size of the box

$$1\times 1\times\cdots 1.$$



Then the system is reduced to a d-matrix model

$$S_{\text{reduced}} = -\frac{N}{\lambda}\sum_{\mu\neq\nu=1}^{d} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right).$$



the large-N reduced model

The meaning of “physics”

(1) Free energy per unit volume $f = \frac{F}{V}$

$$F = -\log Z, \;\; Z = \int [dU] \exp\left(-S_{\text{Wilson}}\right),$$

$$V = L_1 \times L_2 \times \cdots L_d \, ,$$

does not depend on L’s.

In particular,

$$f = -\log Z_{\text{reduced}},$$

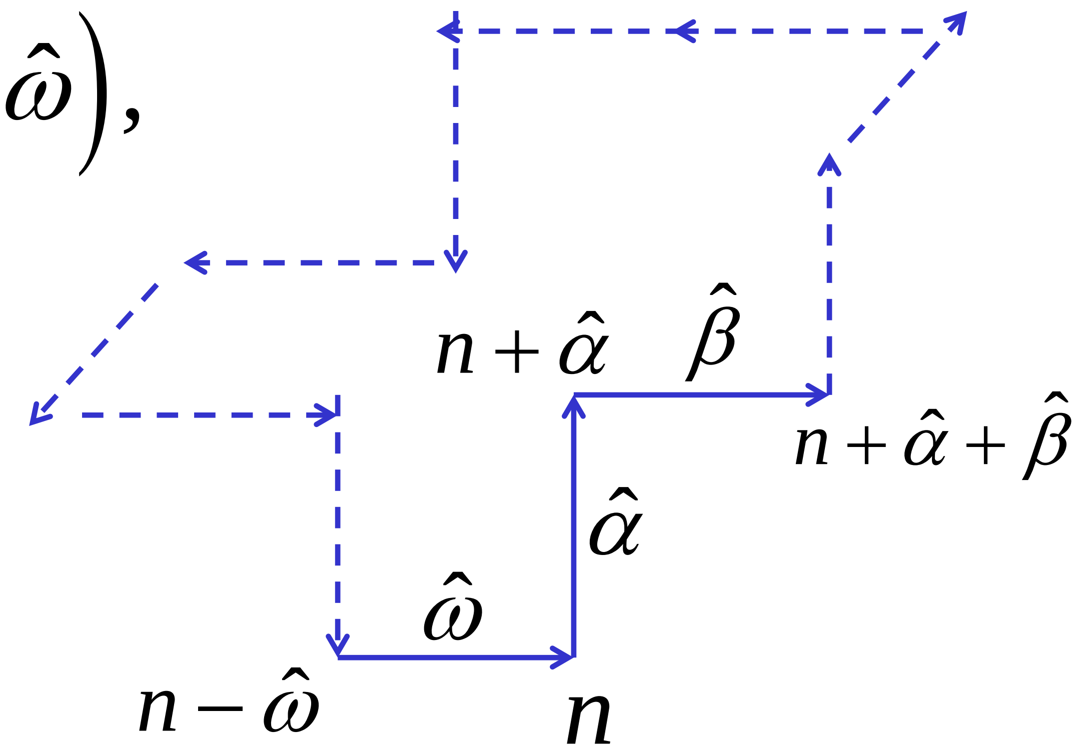
$$Z_{\text{reduced}} = \int \prod_{\mu=1}^{d} dU_\mu \exp\left(-S_{\text{reduced}}\right).$$

(2) Wilson loop

Consider a closed loop *C* in the infinitely extended lattice space

$$C=\left(n,n+\hat{\alpha},n+\hat{\alpha}+\hat{\beta},\cdots,n-\hat{\omega}\right),$$

$$\alpha,\beta,\cdots=\pm 1,\pm 2,\cdots,\pm d\,.$$



***C* is specified by**

the starting point n

the sequence of the directions $\alpha,\beta,\cdots$

Wilson loop in a periodic box:

Define the corresponding loop in the periodic box by the same expression

$$C = \left(n, n+\hat{\alpha}, n+\hat{\alpha}+\hat{\beta}, \cdots, n-\hat{\omega}\right),$$

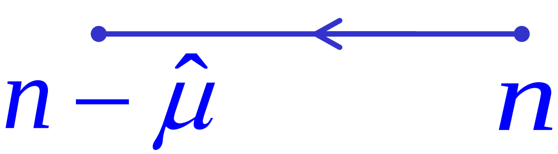
$$\alpha, \beta, \cdots = \pm 1, \pm 2, \cdots, \pm d\,.$$

The Wilson loop is given by

$$W(C) = \langle w(C) \rangle = \frac{\int \prod_{n,\mu} dU_{n,\mu} \exp\left(-S_{\text{Wilson}}\right) w(C)}{\int \prod_{n,\mu} dU_{n,\mu} \exp\left(-S_{\text{Wilson}}\right)}$$

$$w(C) = \frac{1}{N} Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right).$$

Here we assume $U_{n,-\mu} = U^{\dagger}_{n-\hat{\mu},\mu}\,.$



Because of the translational invariance $W(C)$ does not depend on the position n.

The Wilson loop defined in this manner does not depend on the size of the periodic box L's.

In particular the Wilson loop in the infinitely extended space is given by the reduced model:

$$W(C) = \langle w_{\text{reduced}}(C) \rangle = \frac{\int \prod_{\mu=1}^{d} dU_\mu \exp(-S_{\text{reduced}}) \, w_{\text{reduced}}(C)}{\int \prod_{\mu=1}^{d} dU_\mu \exp(-S_{\text{reduced}})}$$

$$w_{\text{reduced}}(C) = \frac{1}{N} Tr\left(U_\alpha U_\beta \cdots U_\omega\right). \qquad U_{-\mu} = U_\mu^\dagger$$

The Large-N reduction

The large-N gauge theory with periodic boundary condition *does not depend* on the volume of the space-time.

In particular, the theory in the infinite space-time is equivalent to that on one point.

The space-time emerges from the internal degrees of freedom of the reduced model.

2. Strong coupling analysis

We show that the coefficients of the strong coupling expansion of the Wilson action and the reduced model agree in the large-N limit.

It indicates that two theories are equivalent at least in the convergence radius of the strong coupling expansion.

Contents of this section

(a) Weingarten model

(b) Wilson action

a) Weingarten model

The Weingarten model is a modification of the Wilson action in which the strong coupling expansion is easily examined:

$$S_{\text{Weingarten}} = -\frac{N}{\lambda}\sum_{n,\mu\neq\nu} Tr\left(V_{n,\mu}V_{n+\hat{\mu},\nu}V^{\dagger}_{n+\hat{\nu},\mu}V^{\dagger}_{n,\nu}\right) + N\sum_{n,\mu} Tr\left(V_{n,\mu}V^{\dagger}_{n,\mu}\right).$$

$V_{n,\mu} : N \times N$ complex matrix

Obtained from the Wilson action by changing the integration measure

$$\int_{UU^{\dagger}=1} dU \rightarrow \int dV \exp\left(-N\, Tr\left(VV^{\dagger}\right)\right).$$

We will show that this model is equivalent to the reduced Weingarten model:

$$S_{\text{reduced Weingarten}} = -\frac{N}{\lambda}\sum_{\mu\neq\nu=1}^{d} Tr\left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right) + N\sum_{\mu=1}^{d} Tr\left(V_\mu V_\mu^\dagger\right).$$

$V_\mu : N \times N$ complex matrix

The arguments are the same for the free energy and Wilson loop.

Wilson loop in Weingarten model

Wilson loop is defined as usual:

For a loop in the lattice space

$$C = \left(n, n+\hat{\alpha}, n+\hat{\alpha}+\hat{\beta}, \cdots, n-\hat{\omega}\right),$$

Wilson loop is defined by

$$W(C) = \left\langle w(C) \right\rangle = \frac{\int \prod_{n,\mu} dV_{n,\mu} \exp\left(-S_{\text{Weingarten}}\right) w(C)}{\int \prod_{n,\mu} dV_{n,\mu} \exp\left(-S_{\text{Weingarten}}\right)}$$

$$w(C) = \frac{1}{N} Tr\left(V_{n,\alpha} V_{n+\hat{\alpha},\beta} \cdots V_{n-\hat{\omega},\omega}\right),$$

where $\quad V_{n,-\mu} = V^{\dagger}_{n-\hat{\mu},\mu}.$

Hopping parameter expansion of surfaces

We regard the quadratic term as the free Lagrangian and the quartic term as the interaction Lagrangian.

$$S_{\text{Weingarten}} = S_0 + S_{\text{int}} \,,$$

$$S_0 = N \sum_{n,\mu} Tr\left(V_{n,\mu} V_{n,\mu}^{\dagger} \right),$$

$$S_{\text{int}} = -\frac{N}{\lambda} \sum_{n,\mu\neq\nu} Tr\left(V_{n,\mu} V_{n+\hat{\mu},\nu} V_{n+\hat{\nu},\mu}^{\dagger} V_{n,\nu}^{\dagger} \right).$$

In the path integral, we keep S_0 in the exponential function, and expand $\exp\left(-S_{\text{int}}\right)$ with respect to S_{int} .

Then we have the following perturbation series:

$$W(C) = \frac{1}{N} \sum_{S(\partial S = C)} \left(\frac{1}{\lambda} \right)^{\mathrm{A}(S)} N^{\chi(S)}.$$

S **runs over all surfaces with boundary** ***C*** **in the lattice space.**

A(*S*) : the number of plaquettes in *S*.

$\chi(S)$ **: the Euler characteristic of *S*.**

Large-N limit

In the large-N limit, only planar surfaces, $\chi(S)=1$, survives.

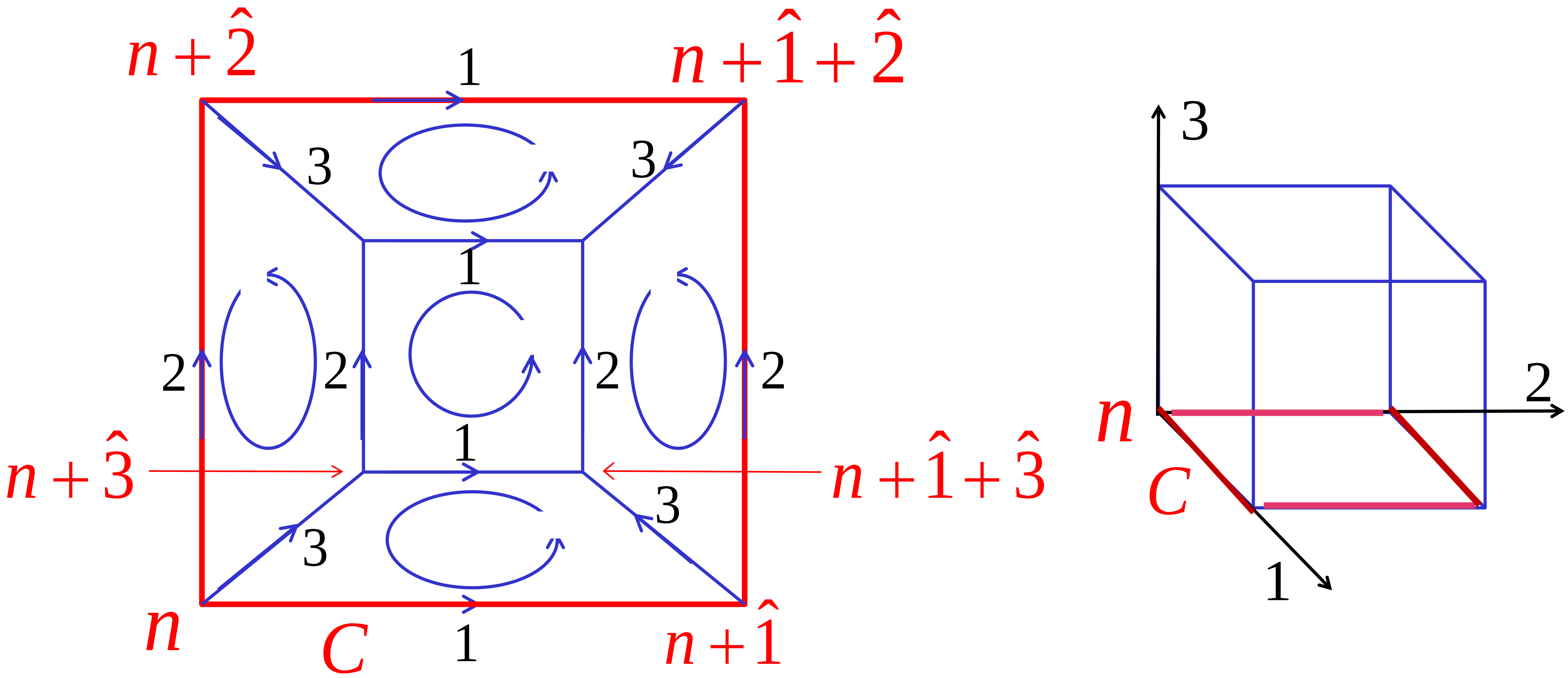
$$\lim_{N\to\infty} W(C) = \sum_{S\,(\partial S=C,\,\chi(S)=1)} \left(\frac{1}{\lambda}\right)^{\mathrm{A}(S)} .$$

Wilson loop of the large-N Weingarten model is given by the sum of planar random surfaces with boundary *C*.

Feynman diagram

We can visualize the surfaces by depicting the Feynman diagrams as

An example



$n+\hat{2}$
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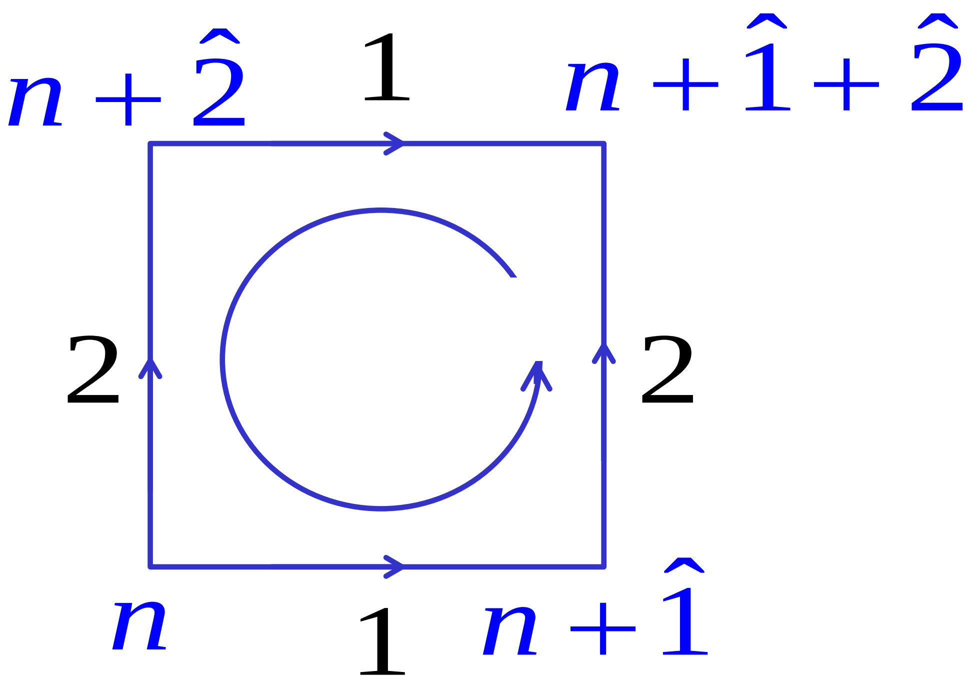
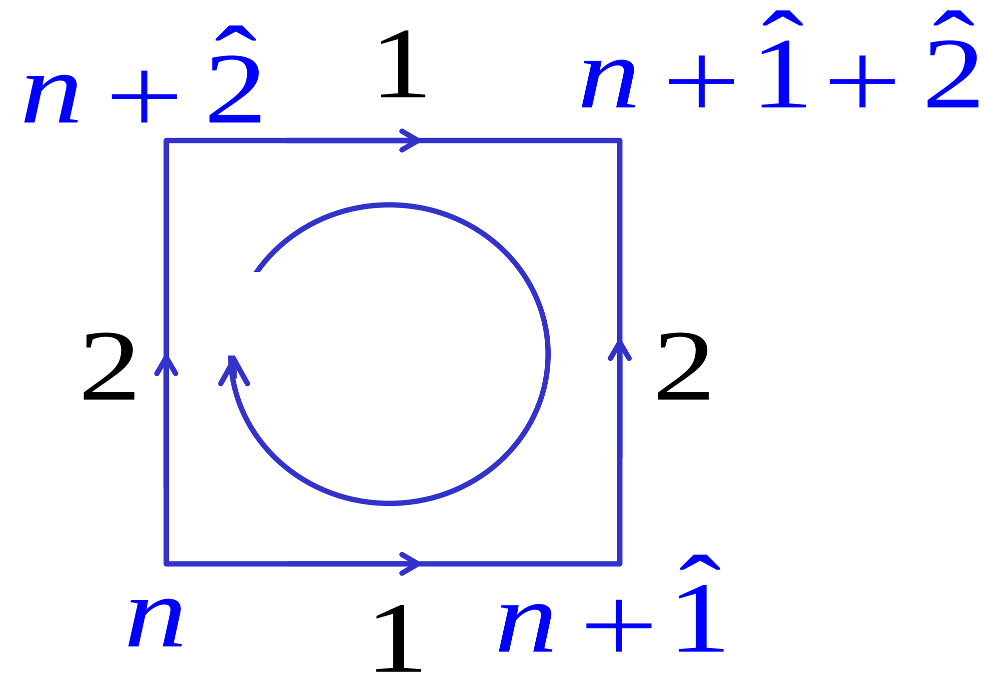
interaction term = square **(next slide)**





$$\frac{N}{\lambda} Tr\left(V_{n,1} V_{n+\hat{1},2} V^{\dagger}_{n+\hat{2},1} V^{\dagger}_{n,2}\right) \qquad \frac{N}{\lambda} Tr\left(V_{n,2} V_{n+\hat{2},1} V^{\dagger}_{n+\hat{1},2} V^{\dagger}_{n,1}\right)$$

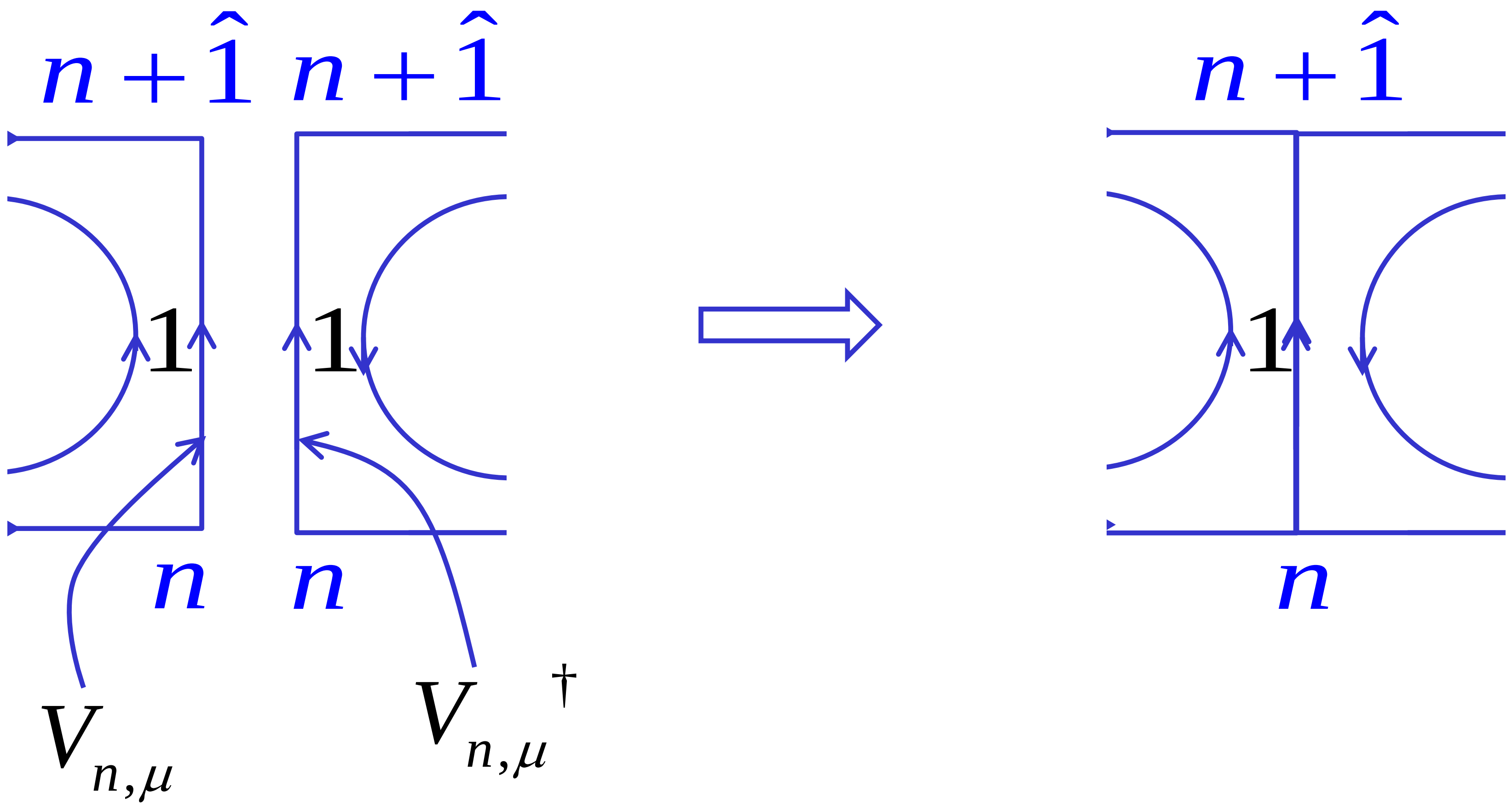
- To each vertex of the square a lattice site is assigned.
- Each edge of the square has a label μ (μ =1,2, .. ,d) and a direction indicated by an arrow.
- Each square has a circular direction along which $V_{n,\mu}$ are multiplied.

  If the circular direction is the same as that of the edge we take $V_{n,\mu}$.

  If not, take $V_{n,\mu}^{\dagger}$.

propagator = gluing two edges

The quadratic part S_0 **glues two edges corresponding to** $V_{n,\mu}$ and $V_{n,\mu}{}^{\dagger}$ **.**



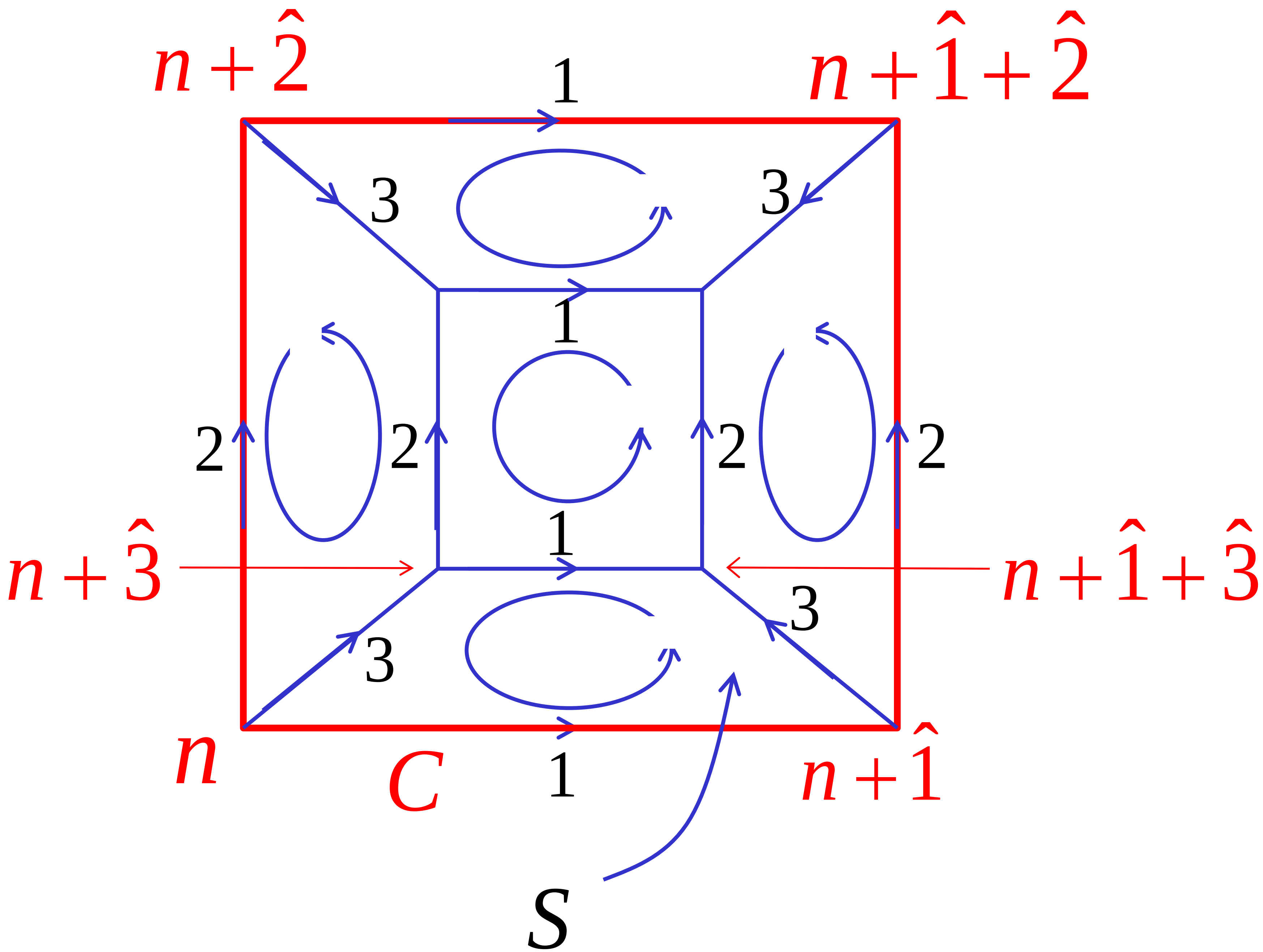
The Feynman diagrams for the Wilson loop are characterized by **(next slide)**

(i) Each diagram is a segmentation of a surface *S* with boundary *C* into a set of squares.

(ii) Each edge of *S* has a label (1,2, .. ,d) and a direction. The opposite edges of each square have the same label and direction.

(iii) If two squares are glued, their circular directions along the common edge are opposite.

(iv) A lattice site is assigned to each vertex. This assignment should be *compatible* with the labels and directions of the edges.



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$n+\hat{1}$
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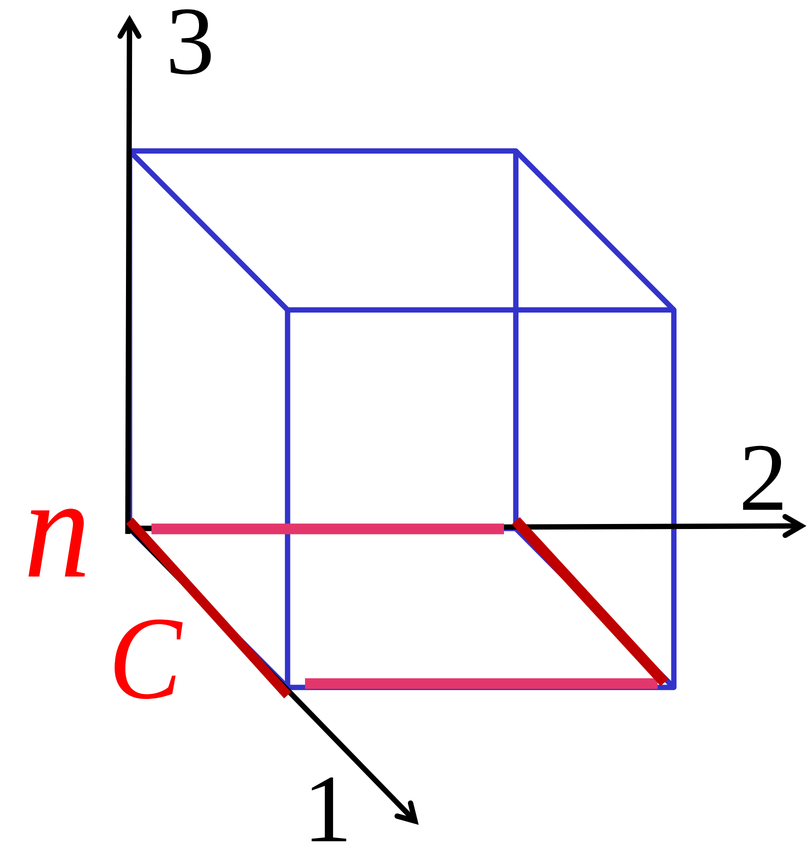

3
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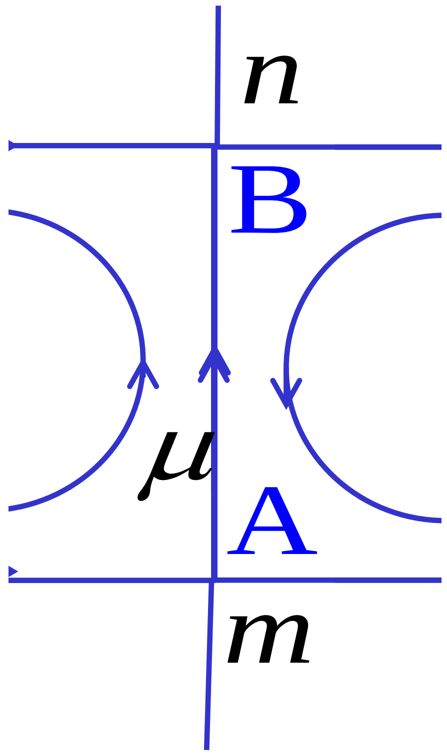
n
C
1

(iv) cont'd

More precisely, suppose that vertices A and B are the endpoints of an edge with label μ , and the arrow points from A to B. Then the sites m and n assigned to A and B, respectively, should satisfy

$$n = m + \hat{\mu}.$$



Value of each Feynman diagram

$$\frac{1}{N}$$

$$\times\left(\frac{N}{\lambda}\right)^{\mathrm{A}(S)}$$

$$\times\left(\frac{1}{N}\right)^{\#\text{ of edges}}$$

$$\times N^{\#\text{ of vertices}}$$

$$=\left(\frac{1}{\lambda}\right)^{\mathrm{A}(S)} N^{\chi(S)-1}.$$

$$w(C)=\frac{1}{N}Tr\left(V_{n,\alpha}V_{n+\hat{\alpha},\beta}\cdots V_{n-\hat{\omega},\omega}\right)$$

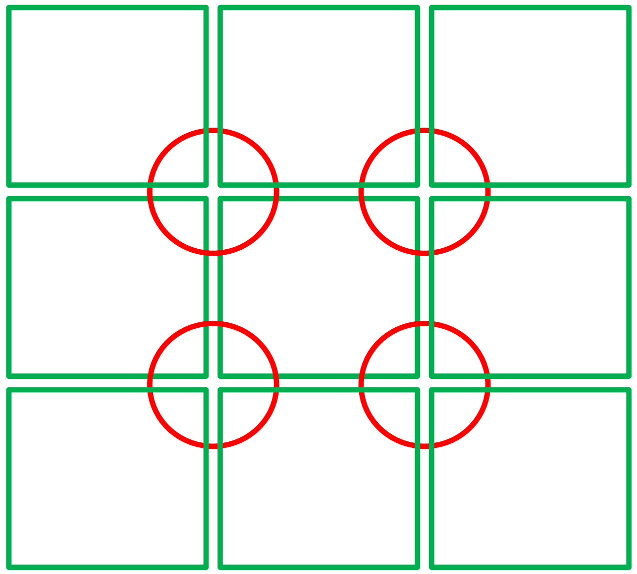
$$S_{\mathrm{int}}=-\frac{N}{\lambda}\sum_{n,\mu\neq\nu}Tr\left(V_{n,\mu}V_{n+\hat{\mu},\nu}V^{\dagger}_{n+\hat{\nu},\mu}V^{\dagger}_{n,\nu}\right)$$

$$S_0=N\sum_{n,\mu}Tr\left(V_{n,\mu}V^{\dagger}_{n,\mu}\right)$$

$\longleftarrow$ **an index loop from each vertex**

$\mathrm{A}(S)$: # of plaqerttes in S

$\chi(S)$: Euler characteristic of S



Large-N limit

In the large-N limit, only the planar surfaces survive, and we have the additional condition:

(0) The surface *S* is planar: $\chi(S) = 1.$

crucial fact:
If the conditions (0) ~ (iii) are satisfied, (iv) is automatically satisfied.

(next slide)

(0) The surface is planar.

(i) Each diagram is a segmentation of a surface *S* with boundary *C* into a set of squares.

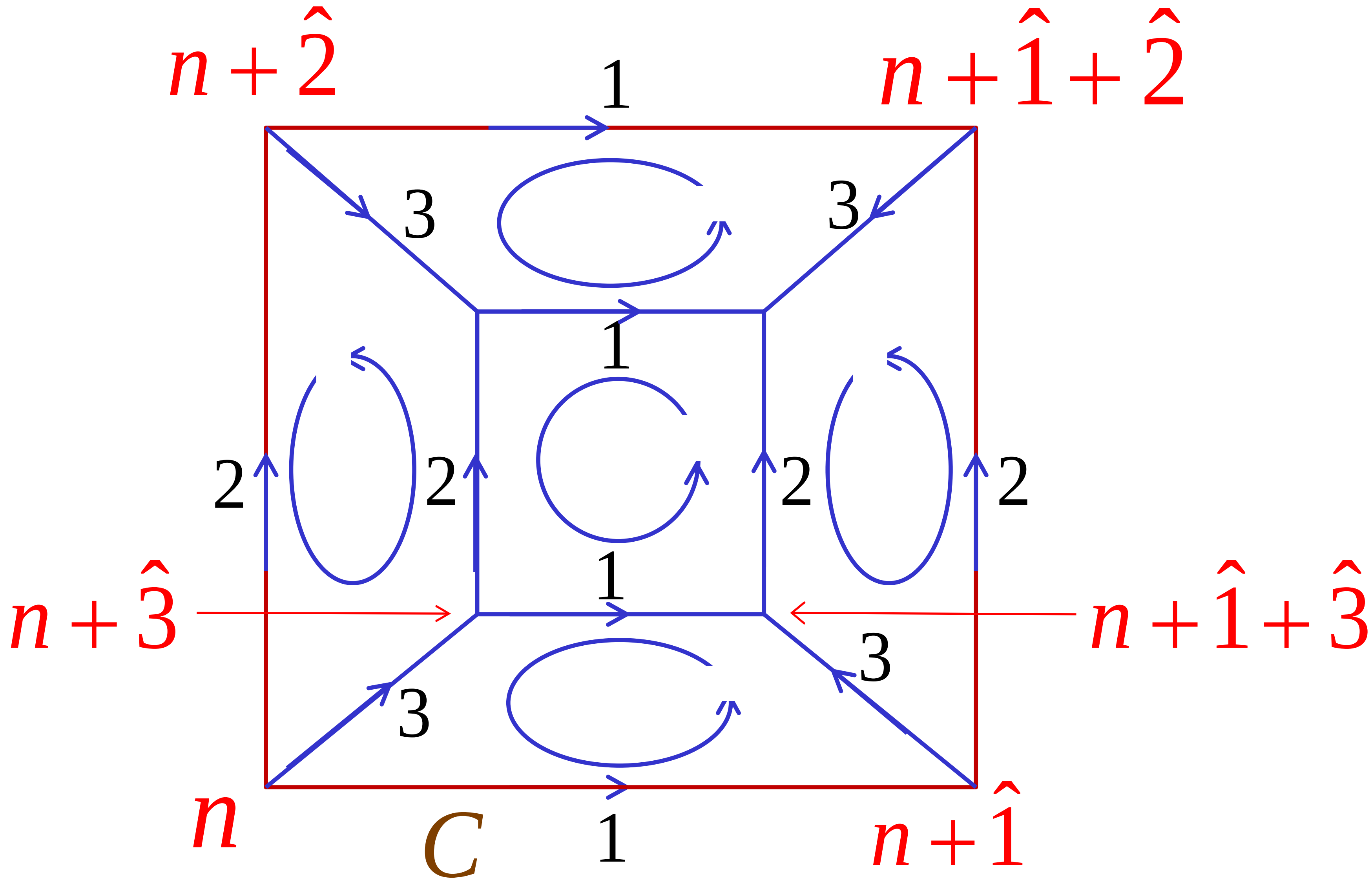
(ii) Each edge of *S* has a label (1,2, .. ,d) and a direction. The opposite edges of each square have the same label and direction.

(iii) If two squares are glued, their circular directions along the common edge are opposite.

(iv) A lattice site is assigned to each vertex. This assignment should be *compatible* with the labels and directions of the edges.

The conditions (0) ~ (iii) tell nothing about the sites.

However, once a site is assigned to one of the vertices, we can uniquely assign sites to the other vertices in a compatible manner with the edges.



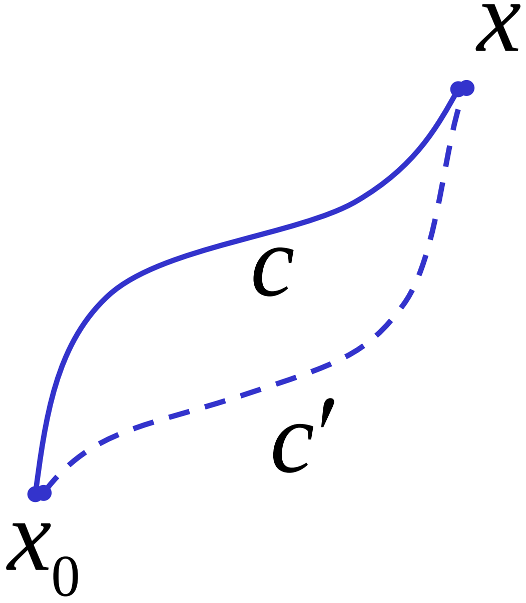
The sum of the displacement vectors around each square is zero. ⇔ rotation free

The sites are consistently determined by the edges, and the condition (iv) is redundant.

This situation is analogous to the well known fact about the existence of the potential.

"A rotation free vector field $v_i(x)$ on a planar surface has a potential $\phi(x)$ that is unique up to an additive constant."

$$\phi(x) = \int_c v_i(y)\,dy^i$$



Wilson loop in the reduced Weingarten model

In the case of the reduced Weingarten model, the Feynman diagrams are characterized by (i) ~ (iii).

The condition (iv) is not there, because we do not have "site".

However, if we take the large-N limit, the planarity condition (0) is added.

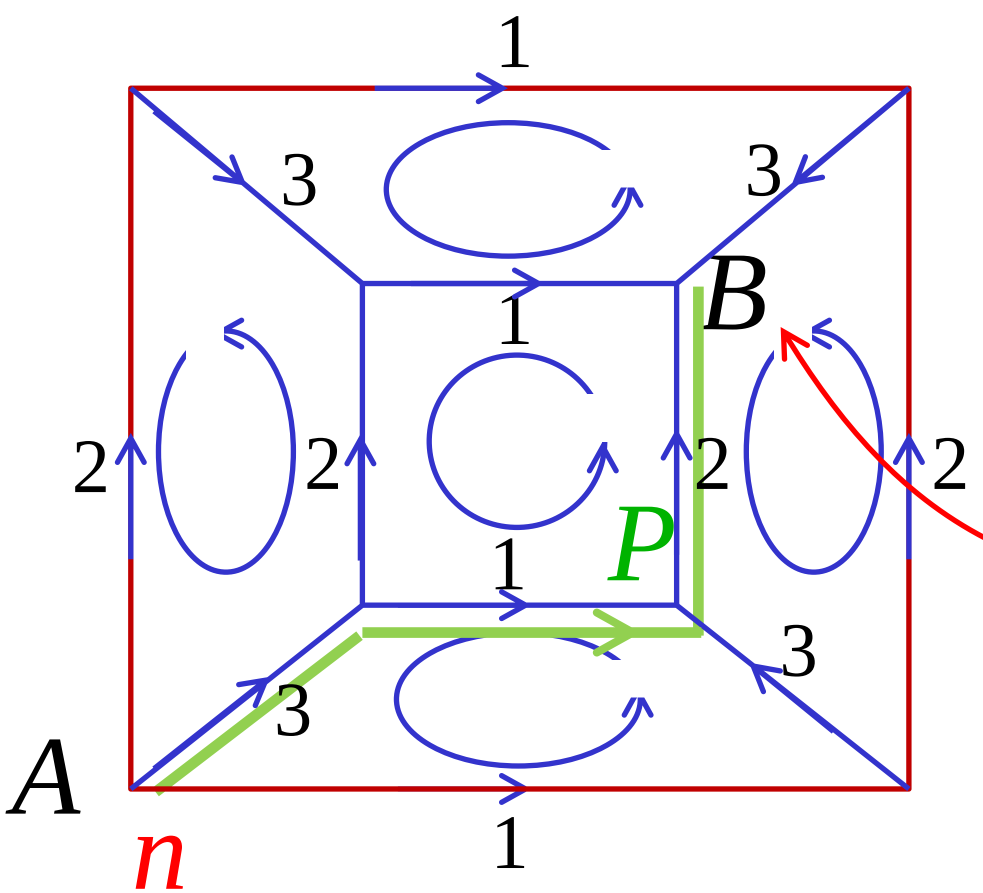
Then the Feynman diagrams of the reduced Weingarten model have one-to-one correspondence with those of the Weingarten model (up to an overall translation of site.)

More precisely,

Suppose a planar Feynman diagram of the reduced Weingarten model is given.

Pick up one vertex *A* and assign a site *n* to it.

For any vertex *B* find a path P from *A* to *B*,



The site that should be assigned to *B* is obtained by summing up the displacement vectors along *P*:

$$n+\hat{3}+\hat{1}+\hat{2}.$$

This does not depend on the choice of P,

because the sum of the displacement vectors around each square is zero and we can deform *P*.

We have seen

“The large-N limits of the Weingarten model and the reduced Weingarten model are the same at laest in the convergence radius of the hopping parameter expansion.”

b) Wilson action

A similar analysis can be applied to the case of Wilson action. The problem is reduced to the case of the Weingarten model by using

$$W(C)=\frac{\left.\exp\left(-S_{\text{Wilson}}\left(\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\right)w\left(C,\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\exp\left(\sum_{n,\mu}f\left(J_{n,\mu}\right)\right)\right|_{\{J=0\}}}{\left.\exp\left(-S_{\text{Wilson}}\left(\left\{\frac{-i}{N}\frac{\partial}{\partial J_{n,\mu}}\right\}\right)\right)\exp\left(\sum_{n,\mu}f\left(J_{n,\mu}\right)\right)\right|_{\{J=0\}}},$$

$$\exp\left(f\left(J\right)\right)=\int dU\exp\left(iN\left(Tr\left(JU\right)+Tr(J^{\dagger}U^{\dagger}\right)\right).$$

This is obtained by plugging the source terms and replacing *U* by *J* derivative.

***f*(*J*) is expanded as the polynomial of *J*:**

$$f\left(J\right)=c_0N^2+c_2N\,Tr\left(JJ^\dagger\right)$$
$$+c_4^1N\,Tr\left(\left(JJ^\dagger\right)^2\right)+c_4^2\left(Tr\left(JJ^\dagger\right)\right)^2+\cdots.$$

We have multiple trace terms, but each term is of order N^2 if we assume each trace is of order N .

We can show that only planar diagrams survive in the large-N limit, and the previous argument for the large-N reduction holds.

We find that the coefficients of the strong coupling expansion of the Wilson action and the reduced model agree in the large-N limit.

Two theories are equivalent at least in the convergence radius of the strong coupling expansion.

3. Schwinger-Dyson equation

So far we have considered strong coupling expansion. But we can make the argument a bit stronger by using the SD equation.

We show

"In the large-N limit, the Schwinger-Dyson equation for the Wilson loops (the loop equations) of the reduced model becomes equivalent to that of the Wilson action as long as the center invariance

$$U_\mu \rightarrow e^{i\theta_\mu} U_\mu$$

is not spontaneously broken."

Loop equations

Consider

$$\int \prod_{n,\mu} dU_{n,\mu} \frac{1}{N^2} Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) \exp\left(-S_{\text{Wilson}}\right),$$

and change the variables as

$$U_{n,\alpha} \to \left(1 + i\varepsilon^b t^b\right) U_{n,\alpha}.$$

Because the integration measure is invariant,

$$\begin{aligned}
0 &= \frac{1}{N^2} \left\langle Tr\left(t^a t^b U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right)\right\rangle \\
&+ \sum_{\mu(\neq\pm\alpha)} \frac{1}{N\lambda} \left\langle Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) Tr\left(t^b U_{n,\alpha} U_{n+\hat{\alpha},\mu} U^\dagger_{n+\hat{\mu},\alpha} U^\dagger_{n,\mu}\right)\right\rangle \\
&- \sum_{\mu(\neq\pm\alpha)} \frac{1}{N\lambda} \left\langle Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) Tr\left(t^b U_{n,\mu} U_{n+\hat{\mu},\alpha} U^\dagger_{n+\hat{\alpha},\mu} U^\dagger_{n,\alpha}\right)\right\rangle \\
&+ \text{ splitting terms}
\end{aligned}$$

The splitting terms appear if one of the link variables in the Wilson loop $,U_{n,\alpha}, U_{n+\hat{\alpha},\beta},\cdots U_{n-\hat{\omega},\omega},$ **coincides with** $U_{n,\alpha}$ or $U^{\dagger}{}_{n,\alpha}$ **.**

For example, if we start with the operator

$$\frac{1}{N^2} Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\alpha},\kappa} U_{n,\alpha} \cdots U_{n-\hat{\omega},\omega}\right),$$

the splitting term

$$\frac{1}{N^2} Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\alpha},\kappa} t^b U_{n,\alpha} \cdots U_{n-\hat{\omega},\omega}\right)$$

appears.

By contracting *a* and *b* , and using

$$\sum_a Tr\left(t^a A\right) Tr\left(t^a B\right) = Tr\left(AB\right),$$

$$\sum_a Tr\left(t^a A\, t^a B\right) = Tr\left(A\right) Tr\left(B\right),$$

we obtain

$$\begin{aligned}
0 = & \left\langle \frac{1}{N} Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) \right\rangle \\
& + \sum_{\mu(\neq\pm\alpha)} \frac{1}{N\lambda} \left\langle Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\mu} U^{\dagger}_{n+\hat{\mu},\alpha} U^{\dagger}_{n,\mu} U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) \right\rangle \\
& - \sum_{\mu(\neq\pm\alpha)} \frac{1}{N\lambda} \left\langle Tr\left(U_{n,\mu} U_{n+\hat{\mu},\alpha} U^{\dagger}_{n+\hat{\alpha},\mu} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\omega},\omega}\right) \right\rangle \\
& + \text{ splitting terms}
\end{aligned}$$

The splitting term for the previous example becomes

$$\left\langle \sum_a \frac{1}{N^2} Tr\left(t^a U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\alpha},\kappa} t^a U_{n,\alpha} \cdots U_{n-\hat{\omega},\omega}\right)\right\rangle$$

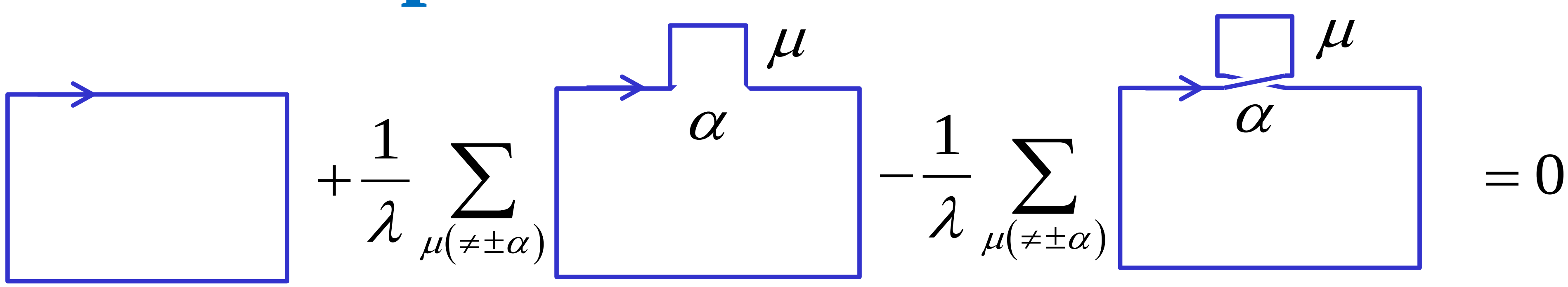
$$= \left\langle \frac{1}{N^2} Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\alpha},\kappa}\right) Tr\left(U_{n,\alpha} \cdots U_{n-\hat{\omega},\omega}\right)\right\rangle$$

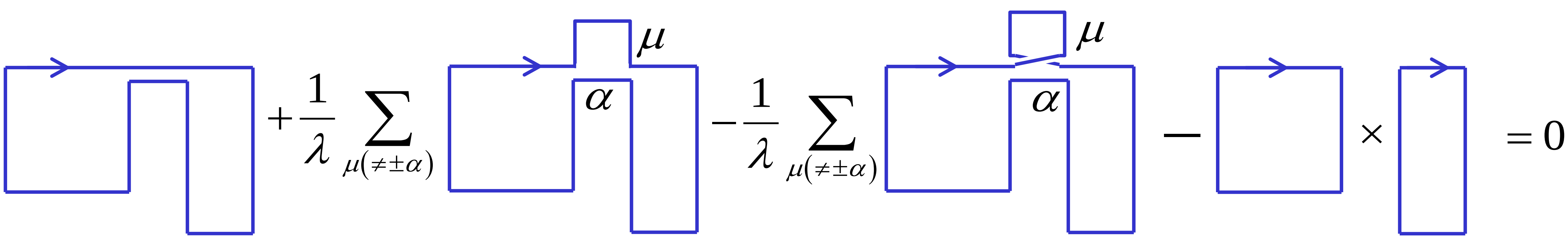
$$= \left\langle \frac{1}{N} Tr\left(U_{n,\alpha} U_{n+\hat{\alpha},\beta} \cdots U_{n-\hat{\alpha},\kappa}\right)\right\rangle \left\langle \frac{1}{N} Tr\left(U_{n,\alpha} \cdots U_{n-\hat{\omega},\omega}\right)\right\rangle.$$

Here we have used the factorization of single trace operators which generally holds in large-N theories:

$$\left\langle Tr(O_1) Tr(O_2)\right\rangle = \left\langle Tr(O_1)\right\rangle \left\langle Tr(O_2)\right\rangle.$$

In this way, splitting terms become a product of Wilson loops in the large-N limit, and the SD equation gives a set of closed relations between Wilson loops.





Splitting terms appear when the loop passes through the link we are considering.

Loop equations of the reduced model

We start with

$$\int \prod_{\mu} dU_{\mu} \frac{1}{N^2} Tr\left(t^a U_{\alpha} U_{\beta} \cdots U_{\omega}\right) \exp\left(-S_{\text{reduced}}\right),$$

and change the variables as

$$U_{\alpha} \to \left(1 + i\varepsilon^b t^b\right) U_{\alpha}.$$

← "Site" does not appear.

Then we have

$$0 = \left\langle \frac{1}{N} Tr\left(U_{\alpha} U_{\beta} \cdots U_{\omega}\right) \right\rangle$$

$$+ \sum_{\mu(\neq \pm \alpha)} \frac{1}{N\lambda} \left\langle Tr\left(U_{\alpha} U_{\mu} U_{\alpha}^{\dagger} U_{\mu}^{\dagger} U_{\alpha} U_{\beta} \cdots U_{\omega}\right) \right\rangle$$

$$- \sum_{\mu(\neq \pm \alpha)} \frac{1}{N\lambda} \left\langle Tr\left(U_{\mu} U_{\alpha} U_{\mu}^{\dagger} U_{\beta} \cdots U_{\omega}\right) \right\rangle$$

$+$ splitting terms

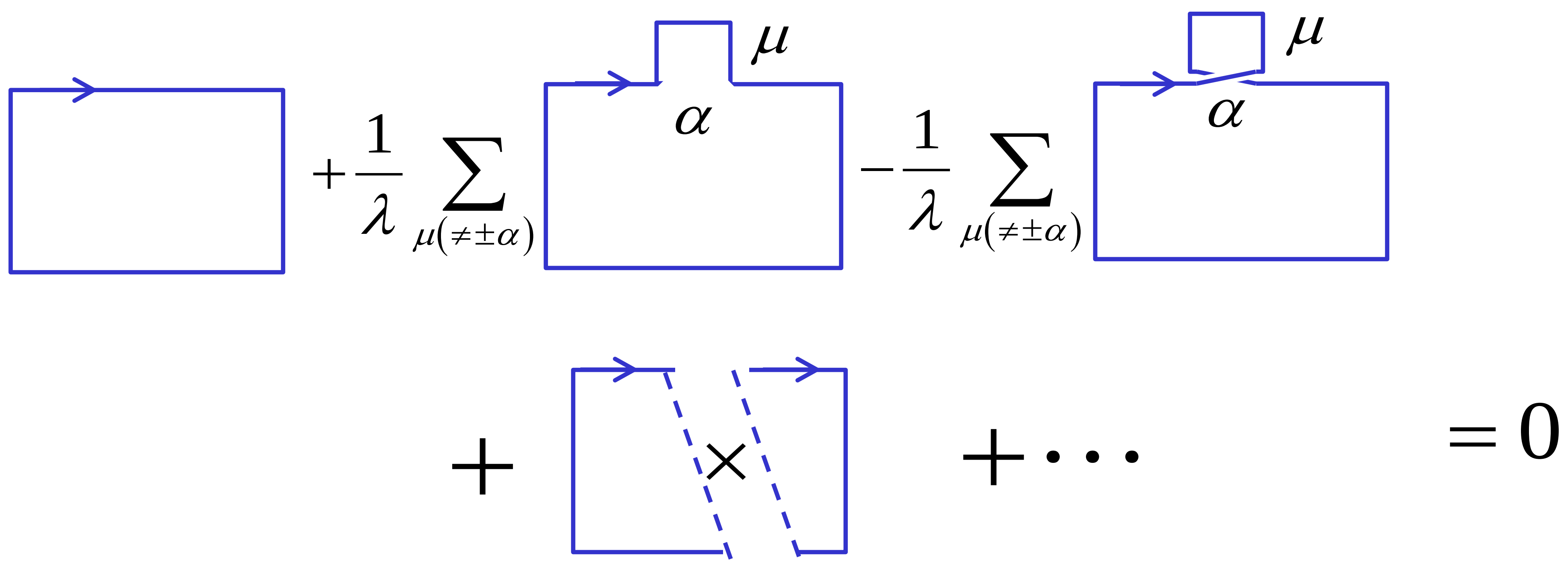
These loop equations are formally obtained from those of the Wilson action by identifying all the $U_{n,\mu}$ that have the same μ , but splitting terms have different structures.

In the reduced model, the splitting terms appear if one of the *U*'s in the Wilson loop, $U_\alpha, U_\beta, \cdots U_\omega,$ coincides with U_α or $U^\dagger{}_\alpha$.

Because this time only the directions are relevant, there appear splitting terms that do not exit in the original Wilson theory.

We can show that those additional terms vanish in the large-N limit as follows:

An additional term is a product of Wilson loops that do not close in the real space, because otherwise such term appears in the case of the Wilson action.



However, for such Wilson loop the number of U_μ and $U_\mu{}^\dagger$ are different at least for one direction μ .

Therefore if the center invariance

$$U_\mu \to e^{i\theta_\mu} U_\mu$$

is not spontaneously broken, such Wilson loop is zero.

The loop equations of the large-N reduced model are equivalent to those of the Wilson action if the center invariance is not broken.

The same argument can be applied to the general periodic boundary condition:
The large-N gauge theory with periodic boundary condition does not depend on the size of the space-time as long as the center invariance is not broken.

4. Perturbative expansion

So far we have considered the strong coupling expansion and the S-D equation.

The large-N reduction can be also understood in the context of the ordinary perturbation series.

Parisi made a general theory that is valid for various matrix field theory, which becomes the reduced model when we apply it to the gauge theory.

(1) Parisi's rule
(2) Application to the gauge theory

Parisi's rule

This rule can be equally applied to both the continuum and lattice theories.

As the simplest example we consider the large-N ϕ^3 theory in the continuum space.

$$S = \int d^d x \, Tr\left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3} \kappa \phi^3 \right),$$

$\phi : N \times N$ hermitian matrix.

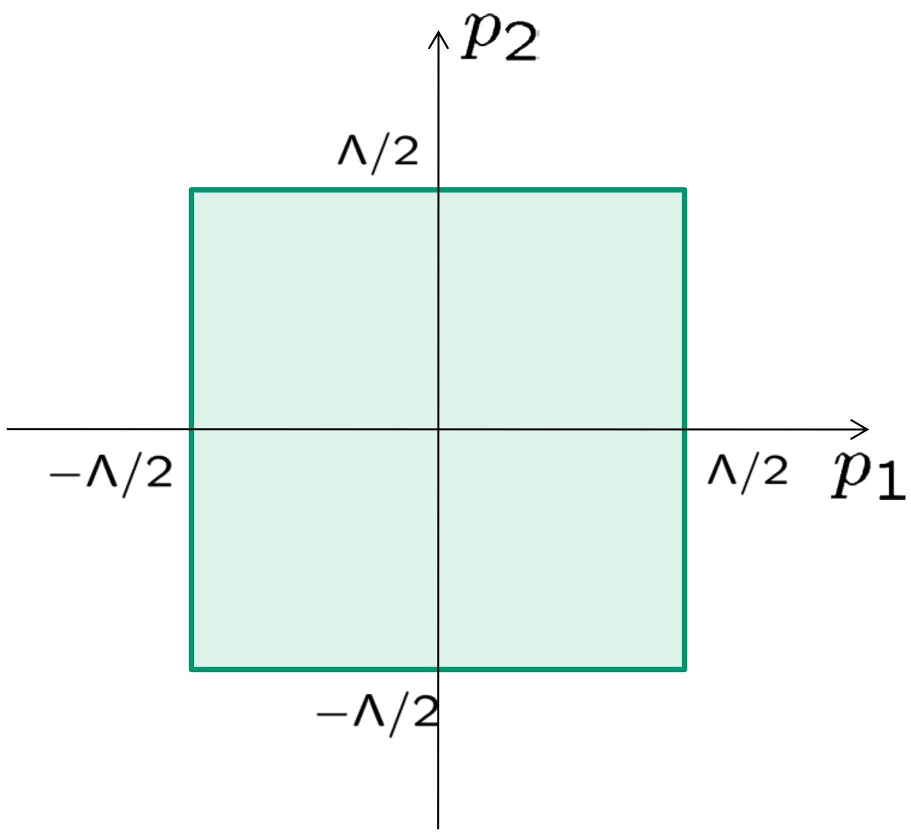
We consider the expectation values of single trace operators such as

$$O_n\left(x_1, x_2, \cdots, x_n\right) = Tr\left(\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right).$$

Corresponding to the action, we construct a matrix model in the following way:

(1) Let $\hat{P}_\mu \ (\mu = 1, \cdots, d)$ be $N \times N$ diagonal matrices whose elements distribute uniformly in the d- dimensional space, which we regard as the momentum space.

$$\hat{P}_\mu = \begin{pmatrix} p_\mu^{(1)} & & & \\ & p_\mu^{(2)} & & \\ & & \ddots & \\ & & & p_\mu^{(N)} \end{pmatrix}$$



(2) Corresponding to each field $\phi(x)$,
introduce a $N \times N$ hermitian matrix $\tilde{\phi}$, and construct the corresponding action and operators by substituting

$$\phi(x) \to \tilde{\phi}(x) = \exp\left(i\,\hat{P}_\mu x^\mu\right)\tilde{\phi}\,\exp\left(-i\,\hat{P}_\mu x^\mu\right),$$

to the original expression.

For the action the space-time integral is replaced by

$$\int d^d x 1 \to \left(\frac{2\pi}{\Lambda}\right)^d .$$

Λ is the cut off that appears in $\hat{P}_\mu$.

Then we have

$$\partial_\mu \phi(x)$$

$$\rightarrow \partial_\mu \tilde{\phi}(x) = \partial_\mu \left(\exp\left(i\,\hat{P}_\mu x^\mu\right) \tilde{\phi} \exp\left(-i\,\hat{P}_\mu x^\mu\right)\right)$$

$$= \exp\left(i\,\hat{P}_\mu x^\mu\right) \left[i\hat{P}_\mu, \tilde{\phi} \right] \exp\left(-i\,\hat{P}_\mu x^\mu\right),$$

$$S = \int d^d x \; Tr\left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3}\kappa\,\phi^3 \right)$$

$$\rightarrow \tilde{S} = \int d^d x \boxed{Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi} \right]^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{3}\kappa \tilde{\phi}^3 \right)} \quad \leftarrow \textbf{no } \boldsymbol{x} \textbf{ dependence}$$

$$\rightarrow \left(\frac{2\pi}{\Lambda} \right)^d Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi} \right]^2 + \frac{1}{2} m^2 \tilde{\phi}^2 + \frac{1}{3}\kappa \tilde{\phi}^3 \right),$$

Then we obtain

action

$$\tilde{S} = \left(\frac{2\pi}{\Lambda}\right)^{d} Tr\left(\frac{1}{2}\left[i\hat{P}_{\mu},\tilde{\phi}\right]^{2} + \frac{1}{2}m^{2}\tilde{\phi}^{2} + \frac{1}{3}\kappa\tilde{\phi}^{3}\right),$$

operators

$$\tilde{O}_{n}\left(x_{1},x_{2},\cdots,x_{n}\right) = Tr\left(\tilde{\phi}\left(x_{1}\right)\tilde{\phi}\left(x_{2}\right)\cdots\tilde{\phi}\left(x_{n}\right)\right).$$

We can show that the expectation value

$$\left\langle \tilde{O} \right\rangle = \frac{\int d\tilde{\phi}\,\tilde{O}\exp\left(-\tilde{S}\right)}{\int d\tilde{\phi}\exp\left(-\tilde{S}\right)}$$

agrees with the original field theory.

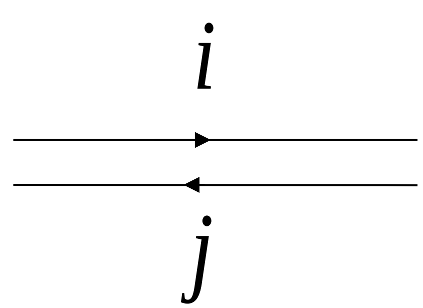
proof

For simplicity, we consider the free energy, The generalization to the expextation values are straightforward.

Because the quadratic part is

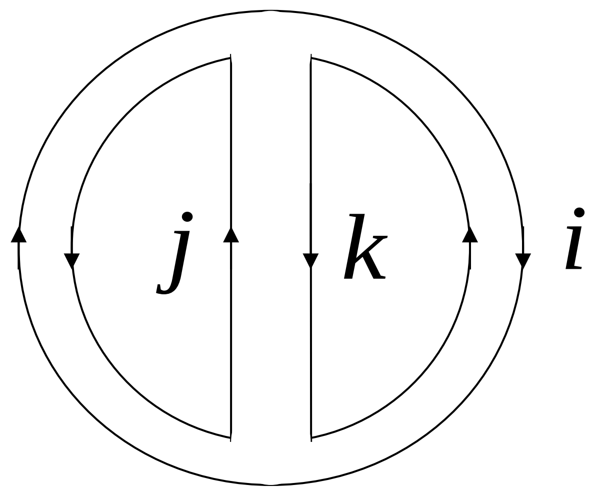
$$Tr\left(\left[\hat{P}_\mu, \tilde{\phi}\right]^2\right) = \sum_{i,j}\left(\left(p_\mu{}^{(i)} - p_\mu{}^{(j)}\right)^2 + m^2\right)\left|\left(\tilde{\phi}\right)_{i,j}\right|^2,$$

the propagator for $\left(\tilde{\phi}\right)_{i,j}$ is given by



$$\frac{1}{\left(p^{(i)} - p^{(j)}\right)^2 + m^2}.$$

Feynman diagrams are something like



$$\sum_i 1 = N = \frac{N}{\Lambda^D}\int d^D p$$

$$\sum_i\sum_j\sum_k \frac{1}{\left(\left(p^{(i)}-p^{(j)}\right)^2+m^2\right)\left(\left(p^{(j)}-p^{(k)}\right)^2+m^2\right)\left(\left(p^{(k)}-p^{(i)}\right)^2+m^2\right)}.$$

In the large-N limit we can replace $\sum_i \to \frac{N}{\Lambda^D}\int d^D p$

$$\to \left(\frac{N}{\Lambda^D}\right)^3 \int\int\int d^D p\, d^D p' d^D p'' \frac{1}{\left(\left(p-p'\right)^2+m^2\right)\left(\left(p'-p''\right)^2+m^2\right)\left(\left(p''-p\right)^2+m^2\right)}$$

$$\to \left(\frac{N}{\Lambda^D}\right)^3 \Lambda^D \int\int d^D k_1\, d^D k_2 \frac{1}{\left(k_1{}^2+m^2\right)\left(k_2{}^2+m^2\right)\left((k_1+k_2)^2+m^2\right)}$$

It might have a wrong overall factor.

$$\left(\frac{N}{\Lambda^d}\right)^3 \Lambda^d \int\int d^d k_1\, d^d k_2 \,\frac{1}{\left(k_1{}^2+m^2\right)\left(k_2{}^2+m^2\right)\left((k_1+k_2)^2+m^2\right)}$$

$$= N^3 \left(\frac{2\pi}{\Lambda}\right)^{2d} \int\int \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d}\frac{1}{\left(k_1{}^2+m^2\right)\left(k_2{}^2+m^2\right)\left((k_1+k_2)^2+m^2\right)}$$

But it is actually *correct*, if we take the prefactor of the action into accout.

$$\tilde{S} = \left(\frac{2\pi}{\Lambda}\right)^d Tr\left(\frac{1}{2}\left[i\hat{P}_\mu, \tilde{\phi}\right]^2 + \frac{1}{2}m^2\tilde{\phi}^2 + \frac{1}{3}\kappa\tilde{\phi}^3\right),$$

$$\Rightarrow N^3 \left(\frac{2\pi}{\Lambda}\right)^{d} \int\int \frac{d^d k_1}{(2\pi)^d}\frac{d^d k_2}{(2\pi)^d}\frac{1}{\left(k_1{}^2+m^2\right)\left(k_2{}^2+m^2\right)\left((k_1+k_2)^2+m^2\right)}$$

$\uparrow$

free energy per unit cell

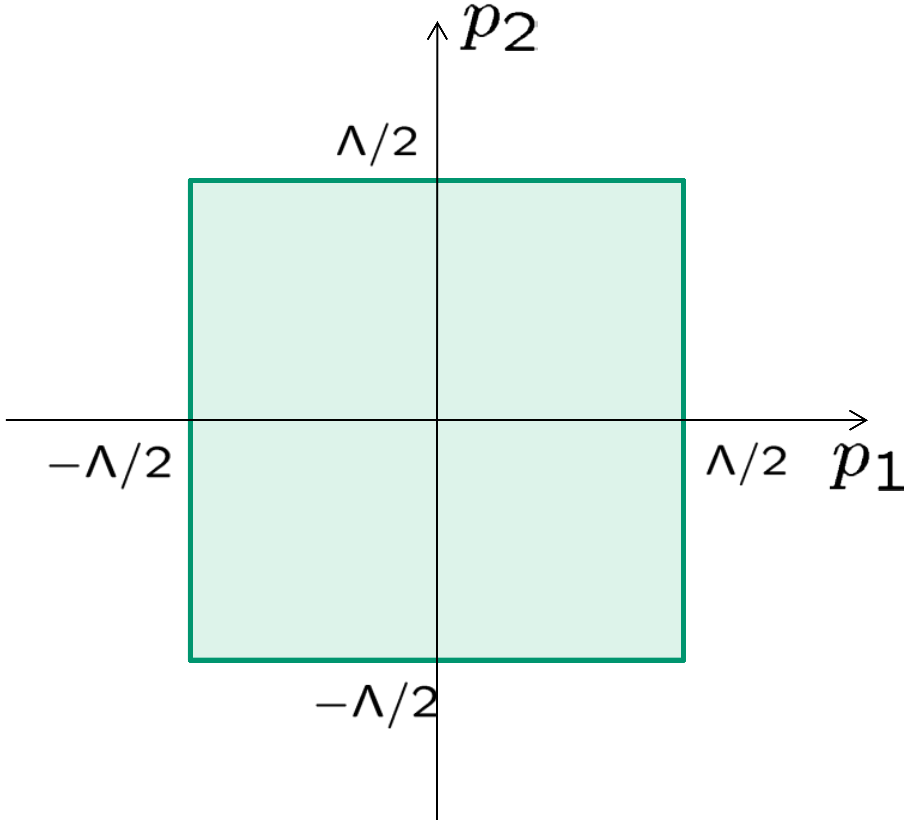
Application to the gauge theory (naive)

We apply Parisi's rule to gauge theory.

Parisi's rule

(1) Let $\hat{P}_\mu \ (\mu = 1, \cdots, d)$ be $N \times N$ diagonal matrices whose elements distribute uniformly in the d- dimensional space.

$$\hat{P}_\mu = \begin{pmatrix} p_\mu^{(1)} & & & \\ & p_\mu^{(2)} & & \\ & & \ddots & \\ & & & p_\mu^{(N)} \end{pmatrix}$$



(2) Corresponding to each field $\phi(x)$,

introduce a $N \times N$ hermitian matrix $\tilde{\phi}$.

Define the action and operators by substituting

$$\phi(x) \to \tilde{\phi}(x) = \exp\left(i\,\hat{P}_\mu x^\mu\right) \tilde{\phi} \exp\left(-i\,\hat{P}_\mu x^\mu\right),$$

in the original field theory.

Space-time integral of 1 should be replaced by the volume of the unit cell:

$$\int d^d x \; 1 \to \left(\frac{2\pi}{\Lambda}\right)^d .$$

The expectation value in the matrix model

$$\left\langle \tilde{O} \right\rangle = \frac{\int d\tilde{\phi}\,\tilde{O}\exp\left(-\tilde{S}\right)}{\int d\tilde{\phi}\exp\left(-\tilde{S}\right)}$$

agrees with that in the original field theory.

We apply Parisi's rule to the continuum gauge theory.

$$A_\mu(x) \to \exp\left(i\hat{P}_\mu x^\mu\right)\tilde{A}_\mu \exp\left(-i\hat{P}_\mu x^\mu\right)$$

$$\partial_\mu A_\nu(x) \to \exp\left(i\hat{P}_\mu x^\mu\right)\left[i\hat{P}_\mu, \tilde{A}_\nu\right]\exp\left(-i\hat{P}_\mu x^\mu\right)$$

$$\begin{aligned}
-iF_{\mu\nu}(x) &= \left[-i\partial_\mu + A_\mu(x), -i\partial_\nu + A_\nu(x)\right] \\
&= -i\partial_\mu A_\nu(x) + i\partial_\nu A_\mu(x) + \left[A_\mu(x), A_\nu(x)\right] \\
&\to \exp\left(i\hat{P}_\mu x^\mu\right)\left(\left[\hat{P}_\mu, \tilde{A}_\nu\right] + \left[\tilde{A}_\mu, \hat{P}_\nu\right] + \left[\tilde{A}_\mu, \tilde{A}_\nu\right]\right)\exp\left(-i\hat{P}_\mu x^\mu\right) \\
&= \exp\left(i\hat{P}_\mu x^\mu\right)\left[\hat{P}_\mu + \tilde{A}_\mu, \hat{P}_\nu + \tilde{A}_\nu\right]\exp\left(-i\hat{P}_\mu x^\mu\right)
\end{aligned}$$

Obtained by replacing $-i\partial_\mu \to \hat{P}_\mu$.

$$S = \int d^d x Tr\left(\frac{N}{4\lambda} F_{\mu\nu}(x)^2\right)$$

$$\rightarrow \tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[\hat{P}_\mu + \tilde{A}_\mu, \hat{P}_\nu + \tilde{A}_\nu\right]^2\right)$$

Remark: Any adjoint matters are allowed.

$$A_\mu(x) \to \exp\left(i\hat{P}_\mu x^\mu\right)\tilde{A}_\mu \exp\left(-i\hat{P}_\mu x^\mu\right)$$

$$\psi(x) \to \exp\left(i\hat{P}_\mu x^\mu\right)\tilde{\psi} \exp\left(-i\hat{P}_\mu x^\mu\right)$$ ← **adjoint fermion**

$$D_\mu\psi(x) = \partial_\mu\psi(x) + i\left[A_\mu(x), \psi(x)\right]$$

$$\to \exp\left(i\hat{P}_\mu x^\mu\right)\left(\left[i\hat{P}_\mu, \tilde{\psi}\right] + i\left[\tilde{A}_\mu, \tilde{\psi}\right]\right)\exp\left(-i\hat{P}_\mu x^\mu\right)$$

$$= \exp\left(i\hat{P}_\mu x^\mu\right) i\left[\hat{P}_\mu + \tilde{A}_\mu, \tilde{\psi}\right]\exp\left(-i\hat{P}_\mu x^\mu\right)$$

$$S' = \int d^d x Tr\left(\frac{1}{2}\bar{\psi}\gamma_\mu D^\mu \psi\right)$$

$$\to \tilde{S}' = \left(\frac{2\pi}{\Lambda}\right)^d Tr\left(\frac{1}{2}\bar{\tilde{\psi}}\gamma_\mu i\left[\hat{P}_\mu + \tilde{A}_\mu, \tilde{\psi}\right]\right).$$

If we define

$$A_\mu = \hat{P}_\mu + \tilde{A}_\mu,$$

the action becomes

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[A_\mu, A_\nu\right]^2\right),$$

and $\hat{P}_\mu$'s disappear from the theory.

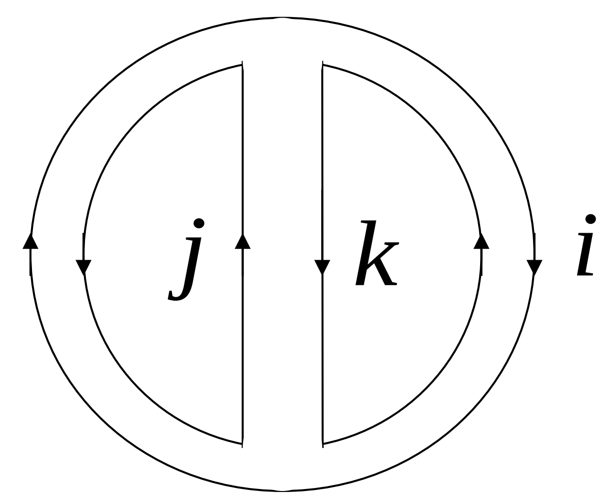
One might conclude that this theory is equivalent to the gauge theory in d-dimensions.

But it is too naïve.

Actually, in Parisi's rule, the diagonal elements are negligible, because we have only N such variables while the action is of order N^2.

But it is not necessarily true in massless theory.

In that case, the propagators for diagonal elements become infinite.



$$\sum_i \sum_j \sum_k \frac{1}{\left(\left(p^{(i)}-p^{(j)}\right)^2+m^2\right)\left(\left(p^{(j)}-p^{(k)}\right)^2+m^2\right)\left(\left(p^{(k)}-p^{(i)}\right)^2+m^2\right)}.$$

We have to be careful, when we apply Parisi's rule to a massless theory such as gauge theory.

Violation of the center invariance

In order to make the problem clearer, we go in the opposite direction.

We start with

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[A_\mu, A_\nu\right]^2\right),$$

and see under what circumstances it becomes gauge theory in d-dimensions.

We first consider the classical solutions.

Because $\hat{P}_\mu$ **commute with each other,** $A^{(0)}{}_\mu = \hat{P}_\mu$ **is a classical minimum of the action.**

Therefore it is natural to consider the expansion around it, $A_\mu = \hat{P}_\mu + \tilde{A}_\mu$ **,** **and we then have**

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[\hat{P}_\mu + \tilde{A}_\mu, \hat{P}_\nu + \tilde{A}_\nu\right]^2\right).$$

If **the fluctuation $\tilde{A}_\mu$ is small and the classical solution $A^{(0)}{}_\mu = \hat{P}_\mu$ is stable, we can safely apply Parisi's rule and conclude that the theory is equivalent to the gauge theory in d-dimensions.**

In the real world, however, the expectation values of the diagonal elements of $\tilde{A}_\mu$ are so large that the classical value $A^{(0)}{}_\mu = \hat{P}_\mu$ is completely cancelled.

In order to see this, we evaluate the one-loop effective Lagrangian for the diagonal elements by integrating out the off-diagonal elements.

$$A_\mu = \begin{pmatrix} p_\mu{}^{(1)} & & * \\ & \ddots & \\ * & & p_\mu{}^{(N)} \end{pmatrix}$$

The quadratic part of the action in the Feynman gauge is

$$S_2 = Tr\left(\left[P_\mu, \tilde{A}_\nu\right]^2 + \left[P_\mu, b\right]\left[P_\mu, c\right]\right)$$

$$= \sum_{i<j}\left(p_\mu{}^{(i)} - p_\mu{}^{(j)}\right)^2 \left|\left(\tilde{A}_\nu\right)_{i,j}\right|^2$$

$$+ \sum_{i<j}\left(p_\mu{}^{(i)} - p_\mu{}^{(j)}\right)^2 \left(\left(b\right)_{i,j}\left(c\right)_{j,i} + \left(b\right)^*{}_{i,j}\left(c\right)^*{}_{j,i}\right),$$

and the effective Lagrangian is given by

$$S_{\text{eff}}{}^{(1\text{–loop})} = \left(d-2\right)\sum_{i<j}\log\left(\left(p_\mu{}^{(i)} - p_\mu{}^{(j)}\right)^2\right).$$

$S_{\text{eff}}^{(1\text{–loop})}$ **is of order N^2 for N variables.**
It is minimized in the large-N limit.

If d > 2, the eigenvalues of $A_\mu = \hat{P}_\mu + \tilde{A}_\mu$
are attractive, and collapse to a point.

This indicates the spontaneous breaking of the translational invariance of the eigenvalues. $A_\mu \to A_\mu + c$

$U_\mu = \exp\left(i a A_\mu\right)$

This is the continuum version of the center invariance. $U_\mu \to e^{i\theta_\mu} U_\mu$

5. Emergence of space-time

The large-N reduced model

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left[A_\mu, A_\nu\right]^2\right)$$

describes d-dimensional space-time
if **the eigenvalues of** A_μ **are uniformly distributed.**
However, it is not automatically realized.
The eigenvalues collapse to one point unless we do something.
Here we consider various ways to make the eigenvalues distributed.

Strong coupling

If the coupling is sufficiently strong, the fluctuation may overwhelm the attractive force.
It actually happens at least for the lattice version of the reduced model.

$$S_{\mathrm{reduced}} = -\frac{N}{\lambda}\sum_{\mu\neq\nu=1}^{d} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right),\ \ \lambda > \lambda_c\,.$$

quenching

Impose the constraint $\left(\tilde{A}_\mu\right)_{i,i}=0$ by hand. Then the perturbation series formally reproduce that of the *d*-dimensional gauge theory. However, this is rather formal, and the gauge invariance is no longer manifest.

A lattice version of the quenching that keeps the manifest gauge invariance was proposed, but it trued out not to work.

Bhanot-Heller-Neuberger,
Gross-Kitazawa

twisting

Gonzalez-Arroyo, Korthals Altes ('83)

If we expand A_μ around the non-commutative back ground

$$A_\mu^{(0)} = \hat{p}_\mu$$

$$\left[\hat{p}_\mu, \hat{p}_\nu\right] = i\, B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

the theory is equivalent to gauge theory in a non-commutative space-time.

Because the equation of motion of the reduced model is given by

$$\left[A_\mu,\left[A_\mu,A_\nu\right]\right]=0,$$

the non-commutative back ground

$$A_\mu^{(0)}=\hat{p}_\mu, \quad \left[\hat{p}_\mu,\hat{p}_\nu\right]=i\,B_{\mu\nu} \quad (B_{\mu\nu}\in\mathbb{R}),$$

is a classical solution.

We expand A_μ **around it:** $A_\mu=A_\mu^{(0)}+\hat{a}_\mu\,.$

Next we introduce the following correspondence between operators and functions:

$$\hat{o} = \int \frac{d^d k}{(2\pi)^d} \tilde{o}(k) \exp(ik_\mu \hat{x}^\mu) \leftrightarrow o(x) = \int \frac{d^d k}{(2\pi)^d} \tilde{o}(k) \exp(ik_\mu x^\mu)$$

where $B_{\mu\nu} C^{\nu\lambda} = \delta_\mu^\lambda, \quad \hat{x}^\mu = C^{\mu\nu} \hat{p}_\nu$ **.**

Then we can show the following rules:

$$\left[\hat{p}_\mu, \hat{o}\right] \leftrightarrow i\partial_\mu o,$$

$$\hat{o}_1 \hat{o}_2 \qquad \leftrightarrow o_1 * o_2,$$

$$Tr(\hat{o}) = \frac{\sqrt{\det B}}{(2\pi)^{d/2}} \int d^d x \, o(x).$$

The matrix model action becomes a field theory on the noncommutative space-time:

$$S = \frac{\sqrt{\det \mathrm{B}}}{(2\pi)^{d/2}} \int d^d x \, \mathrm{tr}\left(-\frac{1}{4} F_{\mu\nu}{}^2 \right)_*$$

One way to make the configuration

$$A_\mu^{(0)} = \hat{p}_\mu\,, \quad \left[\hat{p}_\mu, \hat{p}_\nu\right] = i\,B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

stable is to modify the model to

$$\tilde{S} = -\left(\frac{2\pi}{\Lambda}\right)^d \frac{N}{4\lambda} Tr\left(\left(\left[A_\mu, A_\nu\right] - i\,B_{\mu\nu}\right)^2\right).$$

The lattice version of this is called the twisted reduced model: Gonzalez-Arroyo, Okawa

$$S_{\text{reduced}} = -\frac{N}{\lambda} \sum_{\mu \neq \nu = 1}^{d} e^{i\,\theta_{\mu\nu}} Tr\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right).$$

In the non-commutative gauge theory

$$S = \frac{\sqrt{\det \mathrm{B}}}{(2\pi)^{d/2}} \int d^d x \ \mathrm{tr}\left(-\frac{1}{4} F_{\mu\nu}{}^2 \right)_*$$

only planar diagrams survive in the high energy region, and the theory becomes equivalent to the large-N limit at least formally.

Several MC analyses have been made on the twisted reduced model, and they found some discrepancy from the infinite volume theory.

PART 2 Why do we expect the IIB matrix model describes superstring?

1. Worldsheet as phase space

Worldsheet of string has a structure of phase space.

It becomes manifest when we express the string in terms of the Schild action.

In the Schild action, the worldsheet can be regarded as a symplectic manifold, and the action is given by the integration of a quantity that is expressed in terms of the Poisson bracket.

For simplicity, we start with bosonic string.

Schild action

Nambu-Goto action

$$S_{NG} = -\rho \int d^2\xi \sqrt{-\frac{1}{2}\Sigma^2}, \quad \Sigma^{\mu\nu} = \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu$$

is equivalent to the Schild action

$$S_{\text{Schild}} = \frac{\alpha}{2\pi}\int d^2\xi \sqrt{g}\,\frac{1}{4}\left\{X^\mu, X^\nu\right\}^2 - \frac{\beta}{2\pi}\int d^2\xi \sqrt{g},$$

$\sqrt{g}$: volume density on the world sheet

$\left\{X, Y\right\} = \frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X\, \partial_b Y$: Poisson bracket.

Here we regard the worldsheet as a phase space.

Because $\frac{\delta}{\delta\sqrt{g}} S_{\text{Schild}} = 0 \;\Rightarrow\; \sqrt{g} = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\sqrt{-\left(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu\right)^2}$

$$\Rightarrow S_{\text{Schild}} = -\frac{\sqrt{\alpha\beta}}{2\pi}\int d^2\xi \sqrt{-\frac{1}{2}\left(\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu\right)^2}.$$

symplectic structure of the worldsheet

Schild action has a structure of phase space.

It is given by the integration over the phase space

$$\int d^2\xi\sqrt{g}$$

of a quantity that is expressed by the Poisson bracket

$$\frac{\alpha}{2\pi}\frac{1}{4}\left\{X^\mu, X^\nu\right\}^2 - \frac{\beta}{2\pi}.$$

We have functions X^μ on the phase space, and the action is written in terms of

$\int d^2\xi\sqrt{g}$ **and** $\{\ ,\ \}$ **.**

Worldsheet metric plays no role.

2. Matrix regularization

We want to regularize or discretize the worldsheet in order to perform the path integral.

A natural discretization of phase space is the "quantization".

$$\text{function} \quad \to \quad \text{matrix}$$

$$\{A, B\} \quad \to \quad \frac{1}{i}[A, B]$$

$$\frac{1}{2\pi}\int d^2\xi\sqrt{g}\,A \quad \to \quad TrA$$

$$W_\infty\text{-symmetry} \quad \to \quad U(N)\text{-symmetry}$$

Then the Schild action becomes as

$$S_{\text{Matrix}} = \alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \beta Tr(1),$$

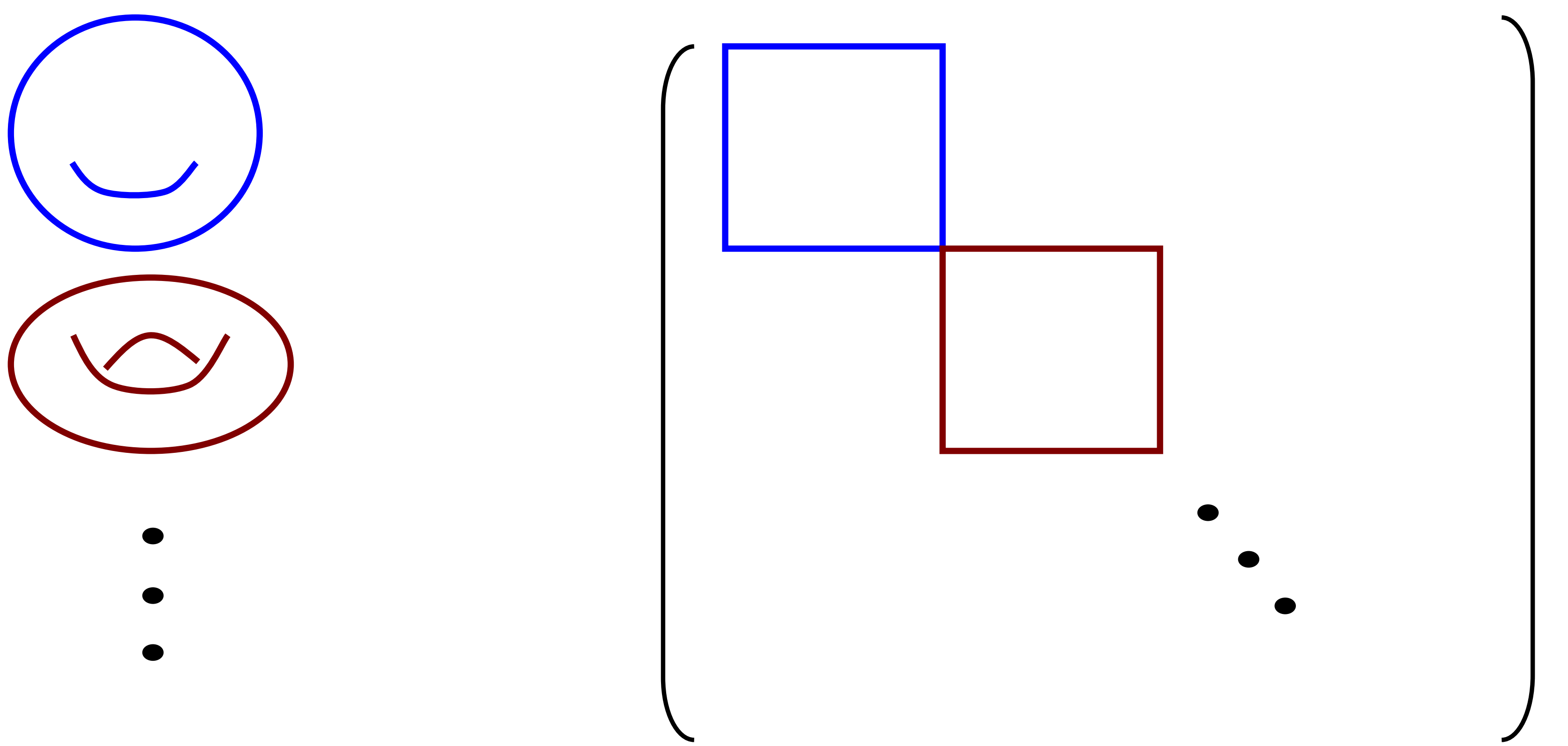
and the path integral is regularized as

$$Z = \int \frac{[dg\, dX]}{\text{vol(Diff)}} \exp\left(iS_{\text{Schild}}\right) \to \sum_{n=1}^{\infty} \int \frac{dA}{SU(n)} \exp\left(iS_{\text{Matrix}}\right).$$

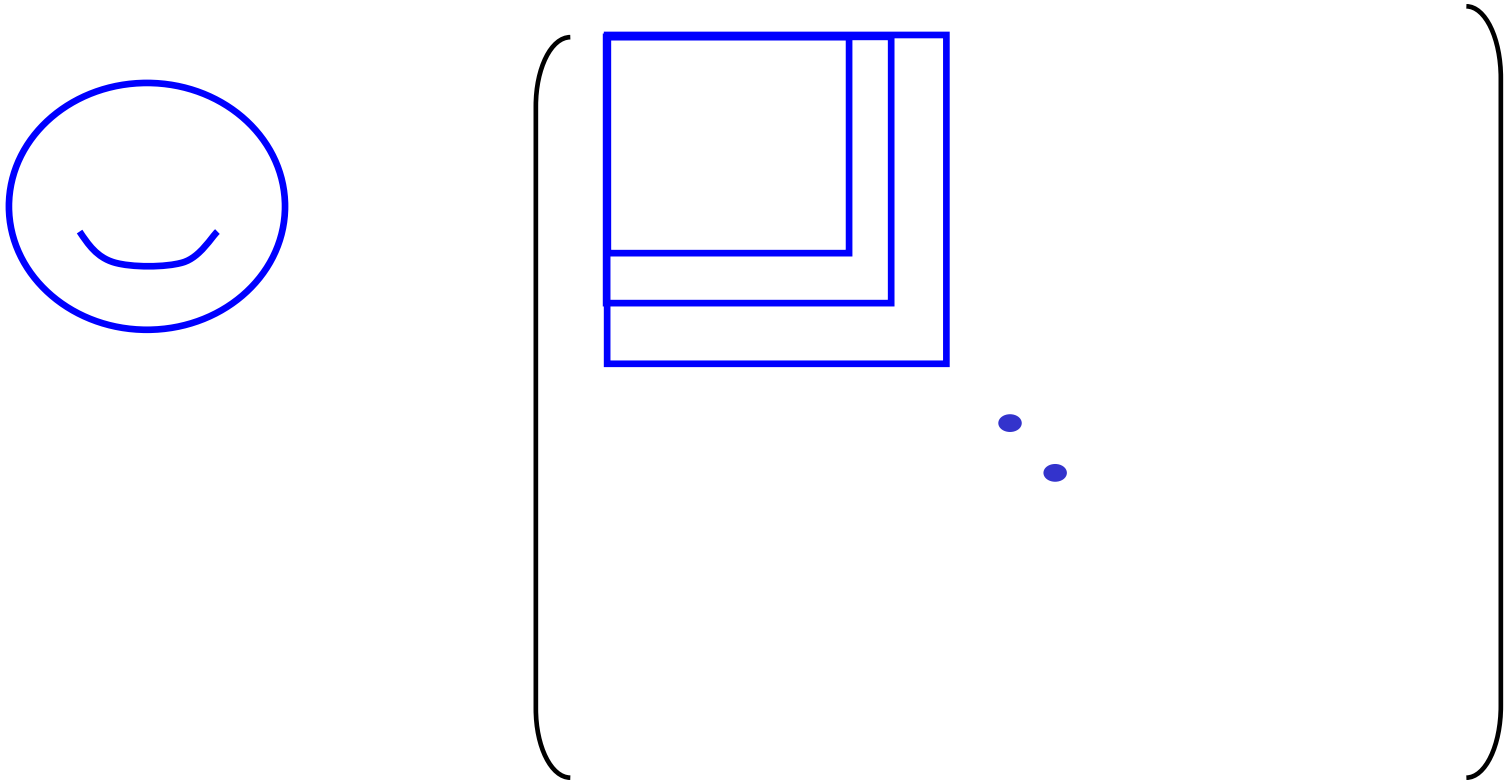
Here we have used the phase space volume $\int d^2\xi \sqrt{g}$ **is diff. invariant and becomes the matrix size** $Tr(1) = n$ **after the regularization.**

Multi-string states

One good point of the matrix regularization is that all topologies of the worldsheet are automatically included in the matrix integral. Disconnected worldsheets are also included as block diagonal configurations.



Furthermore the sum over the size of the matrix that corresponds to a worldsheet is automatically included, if it is imbedded in a larger matrix as a sub matrix.



$$S_{\text{Matrix}} = \alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \beta Tr(1),$$

If we take this picture that all the worldsheets emerge as sub matrices of a large matrix, the second term of

$$S_{\mathrm{Matrix}} = \alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \beta\, Tr(1)$$

can be regarded as describing the chemical potential for the block size.

Thus we expect that the whole universe is described by a large matrix that obeys

$$S = \alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right).$$

This is nothing but the large-N reduced model.

However, as we have seen, in this model the eigenvalues collapse to a point, and it can not describe an extended space-time.
This might be related to the instability of bosonic string by tachyons.

On the other hand, if we start from type IIB superstring, we will get the reduced model for supersymmetric gauge theory. In this case eigenvalues do not collapse, and we can have non-trivial space-time.

3. Schild action of IIB string

We consider the Schild action of the type IIB superstring.

Green-Schwarz action

$X^{\mu}\ (\mu = 0 \sim 9)$

θ^1, θ^2 : 10D Mayorana-Weyl

$$S_{GS} = -\rho \int d^2\xi \Big(\sqrt{-\frac{1}{2}\Sigma^2}$$
$$+ i\varepsilon^{ab}\partial_a X^{\mu}\left(\bar{\theta}^1\Gamma_{\mu}\partial_b\theta^1 + \bar{\theta}^2\Gamma_{\mu}\partial_b\theta^2\right)$$
$$+ \varepsilon^{ab}\bar{\theta}^1\Gamma^{\mu}\partial_a\theta^1\,\bar{\theta}^2\Gamma_{\mu}\partial_b\theta^2 \Big),$$

$$\Sigma^{\mu\nu} = \varepsilon^{ab}\Pi_a^{\mu}\,\Pi_b^{\nu},$$

$$\Pi_a^{\mu} = \partial_a X^{\mu} - i\bar{\theta}^1\Gamma^{\mu}\partial_a\theta^1 + i\bar{\theta}^2\Gamma^{\mu}\partial_a\theta^2$$

κ-symmetry

$$\delta_\kappa \theta^1 = \alpha^1$$

$$\delta_\kappa \theta^2 = \alpha^2$$

$$\delta_\kappa X^\mu = i\bar{\theta}^1 \Gamma^\mu \alpha^1 - i\bar{\theta}^2 \Gamma^\mu \alpha^2$$

$$\alpha^1 = \left(1 + \tilde{\Gamma}\right)\kappa^1$$

$$\alpha^2 = \left(1 - \tilde{\Gamma}\right)\kappa^2,$$

$$\tilde{\Gamma} = \frac{1}{2\sqrt{-\frac{1}{2}\Sigma^2}} \Sigma^{\mu\nu}\Gamma_{\mu\nu}$$

N=2 SUSY

$$\delta_{\mathrm{SUSY}} \theta^1 = \varepsilon^1$$

$$\delta_{\mathrm{SUSY}} \theta^2 = \varepsilon^2$$

$$\delta_{\mathrm{SUSY}} X^\mu = i\bar{\varepsilon}^1 \Gamma^\mu \theta^1 - i\bar{\varepsilon}^2 \Gamma^\mu \theta^2$$

Gauge fixing for the κ-symmetry $\theta^1 = \theta^2 = \psi$

$$S_{GS} = -\rho \int d^2\xi \left(\sqrt{-\frac{1}{2}\sigma^2} + 2i\varepsilon^{ab}\partial_a X^\mu \bar{\psi}\Gamma_\mu \partial_b \psi \right),$$

$$\sigma^{\mu\nu} = \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu .$$

N=2 SUSY

$$\delta\theta^1 = \delta_{\mathrm{SUSY}}\theta^1 + \delta_\kappa \theta^1$$

$$\delta\theta^2 = \delta_{\mathrm{SUSY}}\theta^2 + \delta_\kappa \theta^2$$

$$\delta X^\mu = \delta_{\mathrm{SUSY}} X^\mu + \delta_\kappa X^\mu \qquad \Rightarrow \delta\theta^1 = \delta\theta^2$$

$$\kappa^1 = \frac{-\varepsilon^1 + \varepsilon^2}{2}$$

$$\kappa^2 = \frac{\varepsilon^1 - \varepsilon^2}{2}$$

N=2 SUSY

$$\xi = \frac{\varepsilon^1 + \varepsilon^2}{2}$$

$$\varepsilon = \frac{\varepsilon^1 - \varepsilon^2}{2}$$

$$\delta^{(1)}\psi = -\frac{1}{2\sqrt{-\frac{1}{2}\sigma^2}}\sigma^{\mu\nu}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}X^{\mu} = i\bar{\varepsilon}\,\Gamma^{\mu}\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}X^{\mu} = 0$$

Schild action

$$S_{\mathrm{Schild}}$$

$$= \frac{\alpha}{2\pi}\int d^2\xi\sqrt{g}\left(\frac{1}{4}\left\{X^\mu, X^\nu\right\}^2 - \frac{i}{2}\bar{\psi}\Gamma_\mu\left\{X^\mu, \psi\right\}\right) - \frac{\beta}{2\pi}\int d^2\xi\sqrt{g},$$

N=2 SUSY

$$\delta^{(1)}\psi = -\frac{1}{2}\left\{X^\mu, X^\nu\right\}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}X^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}X^\mu = 0$$

Everything is written in terms of the Poisson bracket.

Matrix regularization

Applying the matrix regularization, we have

$$S_{\text{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu,\ \psi\right]\right) - \beta\, Tr(1).$$

N=2 SUSY

$$\delta^{(1)}\psi = -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\varepsilon \quad \leftarrow F_{\mu\nu} = -i\left[A_\mu, A_\nu\right]$$

$$\delta^{(1)}A^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}A^\mu = 0$$

IIB matrix model

Ishibashi, HK, Kitazawa, Tsuchiya

Drop the second term, and consider large-N

$$S_{\mathrm{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu, \psi\right]\right).$$

IIB matrix model

This is the reduced model of 10D super YM theory.

A good point is that the N=2 SUSY is maintained after the discretization.

N=2 SUSY

$$S_{\text{Matrix}} = \alpha\left(-\frac{1}{4}Tr\left(\left[A_\mu, A_\nu\right]^2\right) - \frac{1}{2}\bar{\psi}\Gamma_\mu\left[A^\mu, \psi\right]\right)$$

One of the N=2 SUSY is nothing but the supersymmety of the 10D super YM theory.

$$\delta^{(1)}\psi = -\frac{1}{2}F^{\mu\nu}\Gamma_{\mu\nu}\varepsilon$$

$$\delta^{(1)}A^\mu = i\bar{\varepsilon}\,\Gamma^\mu\psi$$

The other one is almost trivial.

$$\delta^{(2)}\psi = \xi$$

$$\delta^{(2)}A^\mu = 0$$

Even so, they form non trivial N=2 SUSY:

$$\left\{Q^{(1)}, Q^{(1)}\right\} = 0,\ \left\{Q^{(2)}, Q^{(2)}\right\} = 0,\ \left\{Q^{(1)}, Q^{(2)}\right\} = P.$$

4. Open questions

We expect that the IIB matrix model

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\gamma^\mu [A^\mu, \Psi]\right)$$

gives a constructive definition of superstring.

However there are some fundamental open questions.

(1) Is an infrared cutoff necessary?
(2) How the large-N limit should be taken?
(3) How does the space-time emerge?
(4) Does diff. invariance exist rigorously?

IS IR cutoff necessary?

Because of the supersymmetry the force between eigenvalues cancels between bosons and fermions

$$S^{(1-loop)}{}_{eff} = \left(D-2-d_F\right)\sum_{i,j}\log\left(\left(p^{(i)}-p^{(j)}\right)^2\right)=0$$

It *seems* that we have to impose an infrared cutoff by hand to prevent the eigenvalues from running away to infinity.

$$-l < \mathrm{eigen}\left(A^{\mu}\right) < l$$

But there is a subtlety.
The diagonal elements of fermions are zero modes of the quadratic part of the action.
We should keep them when we consider the effective Lagrangian.
The one-loop effective Lagrangian for the diagonal elements is given by

Aoki, Iso, Kitazawa, Tada, HK

$$A_\mu = \begin{pmatrix} p_\mu{}^{(1)} & & * \\ & \ddots & \\ * & & p_\mu{}^{(N)} \end{pmatrix} \qquad \psi = \begin{pmatrix} \xi^{(1)} & & * \\ & \ddots & \\ * & & \xi^{(N)} \end{pmatrix}$$

$$S_{\text{eff}}{}^{\text{1-loop}}(x,\xi) = \sum_{i<j} tr\left(\frac{S_{(i,j)}{}^4}{4} + \frac{S_{(i,j)}{}^8}{8} \right),$$

$$\left(S_{(i,j)}\right)_{\mu,\nu} = \left(\bar{\xi}^{(i)} - \bar{\xi}^{(j)}\right)\Gamma^{\mu\alpha\nu}\left(\xi^{(i)} - \xi^{(j)}\right)\frac{p_\alpha{}^{(i)} - p_\alpha{}^{(j)}}{\left(\left(p^{(i)} - p^{(j)}\right)^2\right)^2}.$$

Because of the fermionic degrees of freedom, there appears a weak attractive force between the eigenvalues, and the partition function becomes finite.
However it is not clear whether all the correlation functions are finite or not.

⇒Nishimura's talk

Austing and Wheater,
Krauth, Nicolai and Staudacher,
Suyama and Tsuchiya,
Ambjorn, Anagnostopoulos, Bietenholz, Hotta and Nishimura,
Bialas, Burda, Petersson and Tabaczek,
Green and Gutperle,
Moore, Nekrasov and Shatashvili.

To estimate the order of this interaction, we first integrate out the fermionic variables

$$Z^{(1\text{-loop})}(p) = \int \prod_i d^{16}\xi^{(i)} \exp\left(-S_{\text{eff}}^{(1\text{-loop})}(p,\xi)\right)$$

Since $S_{(i,j)}$ **is quadratic in** $\xi^{(i)} - \xi^{(j)}$ **which has only 16 components, we have**

$$S_{(i,j)}{}^{n} = 0, \quad (n > 8)$$

and

$$\exp\left(-S_{\text{eff}}^{(1\text{-loop})}(p,\xi)\right) = \exp\left(-\sum_{i<j} tr\left(\frac{S_{(i,j)}{}^{4}}{4} + \frac{S_{(i,j)}{}^{8}}{8}\right)\right)$$

$$= \prod_{i<j}\left(1 + a\,\mathrm{tr}\left(S_{(i,j)}{}^{4}\right) + b\,\mathrm{tr}\left(S_{(i,j)}{}^{8}\right)\right).$$

Therefore, for each pair of i and j we have 3 choices

$$1, \quad a\,\mathrm{tr}\left(S_{(i,j)}{}^{4}\right), \quad b\,\mathrm{tr}\left(S_{(i,j)}{}^{8}\right)$$

which carry the powers of ξ

0, 8, 16 respectively.

On the other hand, we have $16N$ dimensional fermionic integral $\int \prod_{i=1}^{N} d^{16}\xi^{(i)}$.

Therefore the number of factors other than 1 should be less than or equal to $2N$, and we can estimate as

$$Z^{(1\text{-loop})}(p) = \sum_{\text{various terms}} \underbrace{f\left(p_\alpha{}^{(i)} - p_\alpha{}^{(j)}\right) f\left(p_\alpha{}^{(i')} - p_\alpha{}^{(j')}\right)\cdots}_{\leq 2N}$$

$$\sim \exp\left(O\left(N\right)\right).$$

This should be compared to the bosonic case

$$Z^{(1-\mathrm{loop})}(p) = \exp\left(-(D-2) \sum_{i<j} \log \left(p^{(i)} - p^{(j)} \right)^2 \right)$$

$$\sim \exp\left(O(N^2) \right)$$

SUSY reduces the attractive force by at least a factor 1/N.

In the naïve large-N limit, simultaneously diagonal backgrounds are stable.

It is not clear what happens in the double scaling limit.

How to take large-N limit

In the IIB matrix model , *A* are the space-time coordinates.

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\gamma^\mu[A^\mu, \Psi]\right)$$

g **has dimensions of length squared.**

How is the Planck scale expressed?

If it does not depend on the IR cutoff *l*, as we normally guess, we should have

$$l_{\mathrm{Planck}} = N^\alpha g^{\frac{1}{2}}. \qquad \leftarrow \alpha\,?$$

In other words, we should take the large-N limit keeping this combination finite.

At present we have no definite answer.

How does the space-time emerge?

If we regard the IIB matrix model

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A^\mu, A^\nu]^2 + \frac{1}{2}\overline{\Psi}\gamma^\mu [A^\mu, \Psi]\right)$$

as the matrix regularization of the Schild action, A^μ are space-time coordinates.

On the other hand if we regard it as the large-N reduced model, the diagonal elements of A^μ represent momenta.

It is not a priori clear how the space-time emerges from the matrix degrees of freedom.

One interesting possibility is to consider a non-commutative back ground such as

$$A_\mu^{(0)} = \begin{cases} p_\mu \otimes 1_k \,, & (\mu = 0, \cdots, 3) \\ 0, & (\mu = 4, \ldots, 9) \end{cases},$$

where p_μ **satisfies**

$$\left[\hat{p}_\mu, \hat{p}_\nu \right] = i\, B_{\mu\nu} \quad (B_{\mu\nu} \in \mathbb{R}),$$

we have a flat space with SU(*k*) gauge theory.

There are many possibilities to realize the space-time.

Actually various models that are close to the standard model can be constructed by choosing the background properly.

(ex.) **"Intersecting branes and a standard model realization in matrix models."**
A. Chatzistavrakidis, H. Steinacker, and G. Zoupanos.
JHEP09(2011)115

⇒Aoki, Nishimura, Tsuchiya's talk

Diff. invariance and gravity

Because we have exact N=2 SUSY, it is natural to expect to have graviton in the spectrum of particles.

There are some evidences.

(1) Gravitational interaction appears from one-loop integral.

(2) Emergent gravity by Steinacker. Gravity is induced on the non-commutative back ground.

However, it would be nicer, if we can understand how the diffeomorphism invariance is realized in the matrix model.

5. Prospect

It is important to examine string theory in a non-perturbative manner.

The IIB matrix model is well defined, and in principle it is possible to determine the vacuum structure of string theory.

Although at present we do not know an effective scheme to obtain the ground state, at least numerical analyses are possible and hopefully the problems above will be solved in near future.