Exact solution of the quartic matrix model and application to 4D noncommutative QFT

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joint work with Raimar Wulkenhaar (Münster)

(based on arXiv: 1205.0465v4, arXiv: 1306.2816)

Matrix models

1. 2D quantum gravity is the enumeration of random triangulations of surfaces.
   - Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int dM \;\exp\Big(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n)\Big)\;, \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

   - For $\mathcal{N} \to \infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
2. 2D topological quantum gravity has correlation functions which are intersection numbers of complex curves.
   - They can be arranged into a generating functional with series parameters (t_n).

[Witten, 1990] conjectured that both (t_n)-series are the same.

The Kontsevich model

- [Kontsevich, 1992] computed the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\int dM \ \exp\big(-\frac{1}{2}\mathrm{tr}(EM^2) + \frac{\mathrm{i}}{6}\mathrm{tr}(M^3)\big)}{\int dM \ \exp\big(-\frac{1}{2}\mathrm{tr}(EM^2)\big)}, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

A matrix model inspired by noncommutative QFT

- The simplest QFT on a 4D noncommutative manifold can be written as a matrix model

$$\mathcal{Z}[E, J, \lambda] = \frac{\int dM \ \exp\big(-\mathrm{tr}(EM^2) + \mathrm{tr}(JM) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)}{\int dM \ \exp\big(-\mathrm{tr}(EM^2) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)},$$

  where $E = E^* \in M_{\mathcal{N}}(\mathbb{C})$ is the 4D Laplacian, $\lambda \geq 0$ and $J \in M_{\mathcal{N}}(\mathbb{C})$ generates correlation functions.
- In joint work with Raimar Wulkenhaar [arXiv:1205.0465v4] we achieved the exact solution of $\mathcal{Z}[E, J, \lambda]$ for $\mathcal{N} \to \infty$ and after renormalisation of E, λ.
- Schwinger functions describe a commutative 4D QFT [arXiv:1306.2816]. "Particles" interact without momentum transfer. There are non-trivial topological sectors.

Field-theoretical matrix models

- classical scalar field $\phi \in \mathcal{C}_0(\mathbb{R}^d) \subset \mathcal{B}(H)$, with $\frac{m}{2}\int_{\mathbb{R}^d} dx\, \phi^2(x)$
- translates to $\mathrm{tr}(\phi^2) < \infty$, i.e. nc scalar field is Hilbert-Schmidt compact operator on Hilbert space $H = L^2(I, \mu)$
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I\times I, \mu\times\mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\, M_{ac}M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\, M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\,\mathrm{tr}(E\phi^2 + P[\phi])$$

  E – unbounded positive selfadjoint op. with compact resolvent,
  $P[\phi]$ – polynomial in ϕ with scalar coefficients

- partition function $\mathcal{Z}[J] = \int \mathcal{D}\phi\ \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1 \dots p_{N_1}| \dots |q_1 \dots q_{N_B}|} J^{N_1}_{p_1 \dots p_{N_1}} \cdots J^{N_B}_{q_1 \dots q_{N_B}}$ according to the cycle structure.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

…choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \left(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \right)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

We turn the Ward identity for E injective into formula for $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K} \Big(\sum_{n\in I} G_{|an|P_1|\dots|P_K|} + G_{|a|a|P_1|\dots|P_K|} + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r} \Big)$$
$$+ V^2 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} G_{|a|P_1|\dots|P_K|} G_{|a|Q_1|\dots|Q_{K'}|} \Big\} \mathcal{Z}[J]$$
$$+ \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big)$$

How to use the Ward identity

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\,, \qquad \langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}$$

Example: $G_{|ab|}$ (for $a \neq b$)

$$\begin{aligned} G_{|ab|} &= \frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \\ &= \frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]}\frac{\partial}{\partial J_{ab}}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0} \\ &= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\Big(\phi_{ab}\frac{\partial(-VS_{int})}{\partial\phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0} \end{aligned}$$

$\frac{\partial(-VS_{int})}{\partial\phi_{ab}}$ contains, for any $P[\phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

The previous formula lets the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular):

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a + E_b)} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For every other $G^{(g)}_{a_1 \dots a_N}$ an equation which only depends on

- $G^{(g)}_{a_1 \dots a_k}$ for $k \leq N$,
- $G^{(h)}_{a_1 \dots a_k}$ with $h < g$ and $k \leq N + 2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, M, and λ!

Exact solution for $\phi = \phi^*$

Reality implies invariance under orientation reversal

$$G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1|\dots|p_0^B p_1^B \dots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \dots p_1^1|\dots|p_0^B p_{N_B-1}^B \dots p_1^B|}$$

- empty for $G_{|ab|}$
- cancellations in $(E_a+E_{b_1})G_{ab_1b_2\dots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\dots b_2b_1}$

Theorem (universal algebraic recursion formula)

$$G_{|b_0b_1\dots b_{N-1}|} = (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\dots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\dots b_{N-1}|} - G_{|b_{2l}b_1\dots b_{2l-1}|}G_{|b_0b_{2l+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})} + \frac{(-\lambda)}{V}\sum_{k=1}^{N-1} \frac{G_{|b_0b_1\dots b_{k-1}|b_kb_{k+1}\dots b_{N-1}|} - G_{|b_kb_1\dots b_{k-1}|b_0b_{k+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_k})(E_{b_1}-E_{b_{N-1}})}$$

Last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{N-1}|}$

Further observations

- Non-planar contributions with genus $g \geq 1$ are suppressed by V^{-2g}. In limit $V \to \infty$, full function and its restriction to planar sector satisfy the same equations.
- The non-linear equation

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{V(E_a+E_b)} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

  is not algebraic and to be solved case by case for given E
- Divergent index sums can possibly be renormalised by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2\lambda$.
- Pattern extends to $B \geq 2$ boundary components: Equation for $(N_1 + \ldots + \ldots N_B)$-point functions $G_{|p^1_1 \ldots p^1_{N_1}|\ldots|p^B_1 \ldots p^B_{N_B}|}$ is
  1. universally algebraic if one $N_i \geq 3$
  2. an affine equation to be solved case by case if all $N_i \leq 2$. The coefficients are known by induction.

Renormalisation theorem

The renormalisation leaves algebraic equations invariant:

Theorem

Given a real scalar matrix model with $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p^1_1 \ldots p^1_{N_1}|\ldots|p^B_1\ldots p^B_{N_B}|}$ of $(N_1 + \ldots + \ldots N_B)$-point functions.

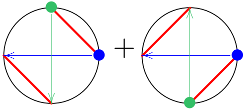
Assume the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2\lambda$.

Then all functions with one $N_i \geq 3$

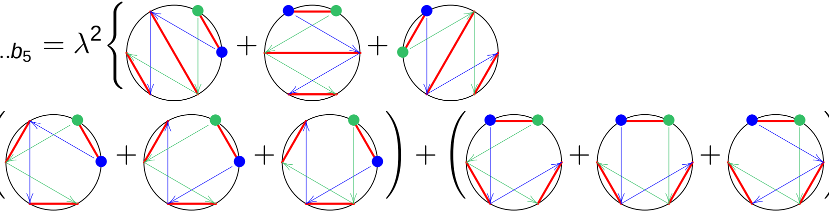
1. are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function.
2. are given by algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.

Graphical realisation ($B = 1, g = 0$)

$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(E_{b_0}-E_{b_2})(E_{b_1}-E_{b_3})} = -\lambda\Big\{ \cdots + \cdots \Big\}$$



$$G_{b_0\dots b_5} = \lambda^2\Big\{ \cdots + \cdots + \cdots + \big(\cdots + \cdots + \cdots \big) + \big(\cdots + \cdots + \cdots \big) \Big\}$$



b_i ——— b_j $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\,\frac{N}{2}!}$

b_i ——→ b_j $= \frac{1}{E_{b_i}-E_{b_j}}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

ϕ^4_4 on Moyal space with harmonic propagation

Moyal product $(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dx\, dk}{(2\pi)^d} f(x+\frac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k,y\rangle}$

$$S[\phi]=64\pi^2\int d^4x \Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu_{bare}^2)\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- renormalisable as formal power series in λ [HG+R.Wulkenhaar, 2004]
  (renormalisation of $\mu_{bare}^2, \lambda, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$)
  means: well-defined perturbative quantum field theory
- Langmann-Szabo duality (2002): theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006]
  means: almost scale-invariant

Is the self-dual (critical) model integrable?

Matrix basis and thermodynamic limit

Moyal algebra has matrix basis [Gracia-Bondía+Várilly, 1988]:

$$\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \phi_{\underline{mn}} f_{\underline{mn}}(x), \qquad f_{\underline{mn}}(x) = f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)$$

$$f_{mn}(y^0,y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big) e^{-\frac{|y|^2}{\theta}}, \qquad y = y^0 + \mathrm{i}y^1$$

- satisfies $(f_{\underline{kl}} \star f_{\underline{mn}})(x) = \delta_{\underline{ml}} f_{\underline{kn}}(x)$, $\int_{\mathbb{R}^4} dx\, f_{\underline{mn}}(x) = (2\pi\theta)^2 \delta_{\underline{mn}}$
- previous action becomes for $\Omega = 1$

$$S[\phi] = V\Bigg(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\, \phi_{\underline{mn}}\phi_{\underline{nm}} + \frac{Z^2\lambda}{4} \sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}}\Bigg)$$

$$E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big), \qquad |\underline{m}| := \underline{m}_1 + \underline{m}_2 \leq \mathcal{N}$$

- $V = \big(\frac{\theta}{4}\big)^2$ is for $\Omega = 1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.
- We do this in a scaling limit $\frac{\mathcal{N}}{\sqrt{V}} = \Lambda^2\mu^2 = \mathrm{const}$

Integral equations

- Matrix indices become continuous $\frac{|p|}{\sqrt{V}} \mapsto \mu^2 p$ with $p \in [0, \Lambda^2]$
- Normalised planar 2-point function $G_{ab} = \mu^2 G^{(0)}_{|ab|}$, $a, b \in [0, \Lambda^2]$
- Difference of eqns for G_{ab} and G_{a0} cancels worst divergence
- Renormalisation $\mu_{bare} \mapsto \mu$ and $Z^{-1} \mapsto (1+\mathcal{Y})$ by normalisation conditions $G_{00} = 1$ and $\frac{dG_{ab}}{db}\Big|_{a=b=0} = -(1+\mathcal{Y})$

Integral equation for Hölder-continuous G_{ab} and $\Lambda \to \infty$

$$\left(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a\big[G_{\bullet 0}\big]}{a G_{a0}}\right) D_{ab} - \lambda\pi \mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

where

- $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$, $\quad \mathcal{Y} = -\lambda \int_0^\infty \frac{dp}{p}\, D_{p0}$
- Hilbert transform $\quad \mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon\to 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\infty \right) \frac{f(q)\, dq}{q-a}$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$h(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)\,, \qquad x \in [-1,1]$$

is for $h(x)$ continuous + Hölder near ± 1 and $f \in L^p$ solved by

$$y(x) = \frac{\sin(\vartheta(x))}{\lambda\pi}\Big(f(x)\cos(\vartheta(x)) + e^{\mathcal{H}_x[\vartheta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\vartheta]}}{1-x}\Big)$$

$$\vartheta(x) = \arctan_{[0,\pi]}\Big(\frac{\lambda\pi}{h(x)}\Big)\,, \quad \sin(\vartheta(x)) = \frac{|\lambda\pi|}{\sqrt{(h(x))^2 + (\lambda\pi)^2}}$$

where C is an arbitrary constant.

Assumption: $C = 0$

Solution

- angle $\vartheta_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{\lambda\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$
- G_{a0} is solved for $\vartheta_0(a)$: $\quad G_{a0} = \frac{\sin(\vartheta_0(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\vartheta_0(\bullet)] - \mathcal{H}_0[\vartheta_0(\bullet)]}$
- Addition theorems and Tricomi's identity $e^{-\mathcal{H}_a[\vartheta_b]}\cos(\vartheta_b(a)) + \mathcal{H}_a\Big[e^{-\mathcal{H}_\bullet[\vartheta_b]}\sin(\vartheta_b(\bullet))\Big] = 1$ give:

Theorem

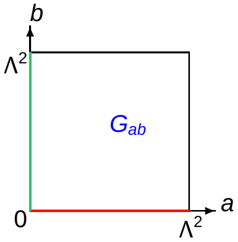
$$G_{ab} = \frac{\sin(\vartheta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\vartheta_b] - \mathcal{H}_0[\vartheta_0]} = \frac{e^{\mathcal{H}_a[\vartheta_b(\bullet)] - \mathcal{H}_0[\vartheta_0(\bullet)]}}{\sqrt{(\lambda\pi a)^2 + \left(b + \frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}\right)^2}}$$

- Consequence: $G_{ab} \geq 0$!
- $\mathcal{Y} = \lambda \displaystyle\int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(\frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}$

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab},
in particular G_{0b}

symmetry forces $G_{b0} = G_{0b}$



Master equation

The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2} \right)$$

Existence proof

The operator T satisfies assumptions of Schauder fixed point theorem. Define

$$\mathcal{K}_\lambda := \Big\{ f \in \mathcal{C}^1_0(\mathbb{R}_+) \;:\quad f(0)=1\,,\quad 0 < f(b) \leq \tfrac{1}{1+b}\,, \quad 0 \leq -f'(b) \leq \big(\tfrac{1}{1+b} + C_\lambda\big) f(b) \Big\}$$

with C_λ from $2\lambda P_\lambda^2(1+C_\lambda)e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. Then:

1. $\mathcal{K}_\lambda$ convex
2. $\overline{T\mathcal{K}_\lambda} \subset \mathcal{K}_\lambda$
3. $(Tf)''(b) \leq \big(\frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2}\frac{1}{(\lambda\pi^2 P_\lambda)^2}\big)(Tf)(b)$ for any $f \in \mathcal{K}_\lambda$.

   $\Rightarrow$ $T\mathcal{K}_\lambda$ is relatively compact in $\mathcal{K}_\lambda$ by variant of Arzelá-Ascoli
4. $T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ is continuous

This provides exact solution of ϕ^4-QFT on 4D Moyal space at $\theta \to \infty$

Translation to 4D Euclidean QFT model

infinite volume limit $V \to \infty$ requires densities

Schwinger functions

$$\mu^N S_c(\mu x_1, \dots, \mu x_N) := \lim_{V\mu^4 \to \infty} \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N \in \mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N) \frac{\mu^{4N}\partial^N \mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2 \mu^8} \log\left(\frac{\int \mathcal{D}[\phi]\, e^{-S[\phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\phi_{\underline{ab}}J_{\underline{ba}}}}{\int \mathcal{D}[\phi]\, e^{-S[\phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2 \\ Z\mapsto(1+\mathcal{Y})}}$$

- J-cycle structure in $\mathcal{F}$ produces $f_{\underline{mn}}$-cycles for every face: $\sum_{\underline{m}_1,\dots,\underline{m}_j} f_{\underline{m}_1\underline{m}_2}\cdots f_{\underline{m}_{j-1}\underline{m}_j} f_{\underline{m}_j\underline{m}_1} G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$
- Write $G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$ for every face as Laplace transform in $\frac{|\underline{m}_1|+\dots+|\underline{m}_j|}{\sqrt{V}}$ and Fourier transform in $\frac{|\underline{m}_{i+1}|-|\underline{m}_i|}{\sqrt{V}}$

Lemma

(with $J+i\equiv i$, $|z_i|<1$)

$$\sum_{m_1,\dots,m_J=0}^{\infty}\prod_{i=1}^{J} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(r_i) = \frac{\exp\left(-\frac{\sum_{i,k=1}^{J} r_i (z_{k+i}\cdots z_{J+i})}{1-(z_1\cdots z_J)}\right)}{1-(z_1\cdots z_J)}$$

- $1-(z_1\cdots z_J) \xrightarrow{V\to\infty} \begin{cases} 2 & (J \text{ odd}) \\ \frac{t}{\sqrt{V}} & (J \text{ even}) \end{cases}$ (t – Laplace par., $r \propto \frac{x^2}{\sqrt{V}}$)
- gives factor $V^{\#(\text{even faces})}$, and G gives factor $V^{-\#(\text{all faces})}$

Proposition

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2}\sum_{\substack{j_1+\dots+j_B=N \\ j_\beta \text{ even}}}\sum_{\sigma\in\mathcal{S}_N}\left(\prod_{\beta=1}^{B}\frac{4^{j_\beta}}{j_\beta}\int_{\mathbb{R}^4}\frac{d^4p_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{j_\beta}(-1)^{i-1}\mu x_{\sigma(j_1+\dots+j_{\beta-1}+i)}\right\rangle}\right)$$

$$\times\, G\Big(\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{j_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{j_B}\Big)$$

Results

- Only a restricted sector of the matrix model contributes to position space: All faces have common matrix indices.
- Schwinger functions are symmetric and invariant under the full Euclidean group (this is limit $\theta \to \infty$!)
- Most interesting sector: every face has $j_i = 2$ indices. This describes propagation and interaction of B particles, without any momentum exchange
- Similar to free particles, but $(N_1 + \ldots + N_B)$-point functions violate clustering. There are non-trivial topological sectors.
- Analytic continuation to Minkowski space and Osterwalder-Schrader reflection positivity would follow (at least for 2-point function) if $a \mapsto G_{aa}$ is a Stieltjes function.
  f Stieltjes $\Leftrightarrow$ f-smooth, $f(x) \geq 0$, $(-1)^n \frac{d^{2n+1}}{dx^{2n+1}}(x^{n+1}f(x)) \geq 0$
- This can at best be the case for wrong sign $\lambda < 0$.

Next steps

(Analysis): The homogeneous Carleman equation has non-trivial solutions not taken into account. They arise from a winding number and seem to be relevant for $\lambda > \frac{1}{\pi}$.

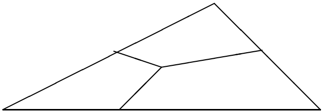
We are currently performing computer simulations.

The (important!) uniqueness proof needs prior clarification of this freedom.

(2D model): Carrying these methods and results over to 2D Moyal space is easy. But the master equation has a singularity at $a = 0$ (infrared) so that the Schauder existence proof does not work in the same way.

Future steps

(2D quantum gravity) should have equivalent descriptions as cubic and quartic matrix model.



Quartic models show positivity and boundedness from below. They admit techniques from constructive QFT (loop vertex expansion) not possible in cubic model.

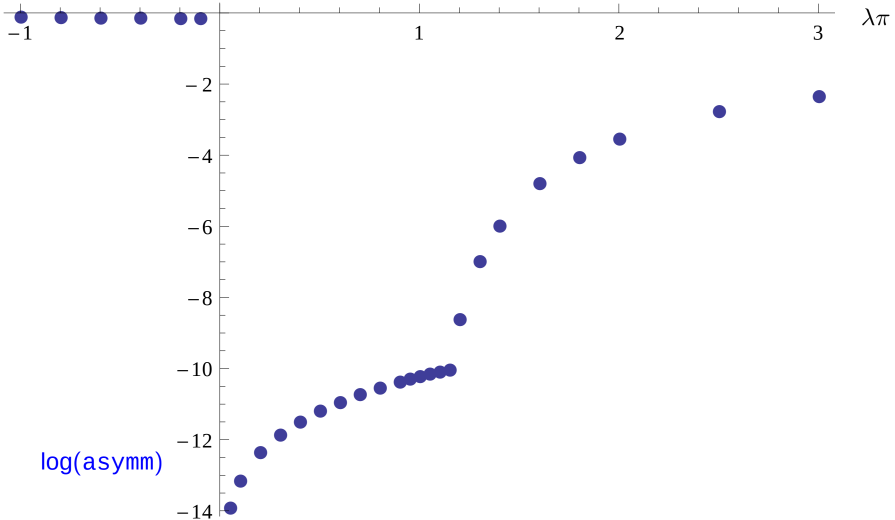
Our solution of the quartic matrix model might be useful in 2D quantum gravity and algebraic geometry.

(Coloured tensor models) extend these methods to quantum gravity in $D \geq 3$. They have Schwinger-Dyson equations and action of $U(\infty)$ group. Our method might generalise to this class.

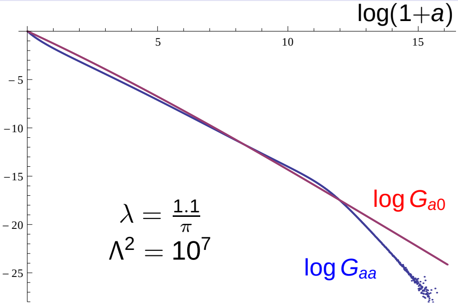
Computer simulations

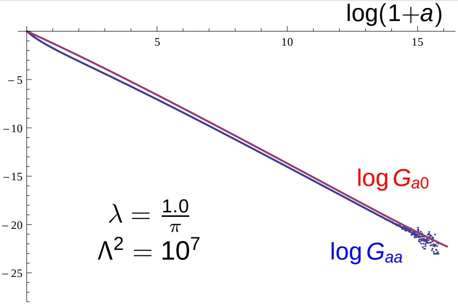
- We implement G_{a0} for $a \in [0, \Lambda^2]$ as piecewise-linear function with edges arranged as geometric progression.
- We find numerically that the operator T in the fixed point equation $G = TG$ satisfies the assumptions of the Banach fixed point theorem in Lipschitz space.
- Convergence of the sequence $G^{n+1}_{a0} = (TG^n)_{a0}$ is established for $\lambda \geq -1$.
- There is no discontinuity of $G_{a0}(\lambda)$ at $\lambda = 0$.
- The required symmetry $G_{ab} = G_{ba}$ is numerically
  – verified for $0 \leq \lambda \leq \frac{1}{\pi}$
  – increasingly violated for $\lambda > \frac{1}{\pi}$
  Solution of homogenous equation to be added for $\lambda < 0$

asymmetry $\sup_{a,b} |G_{ab} - G_{ba}|$



$\log G_{a0}$ and $\log G_{aa}$ as function of $\log(1+a)$





- For $0 \leq \lambda \leq \frac{1}{\pi}$ we have $G_{aa} \approx \frac{C}{(1+a)^{1+\eta}}$ with $\eta > \lambda$.
  Such functions are not Stieltjes.
- For $\lambda \geq \frac{1.1}{\pi}$ the function G_{aa} suddenly bends (here at $a \approx 10^5$) and increases the (negative) slope by 1.

  This signals necessity of the non-trivial solution of the homogeneous Carleman equation

Taking the non-trivial solution into account

G_{ab} is parametrised by a constant C and possibly an arbitrary function $f(b)$. These may depend on (λ, Λ^2).

$$G_{ab} = \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_b] - \mathcal{H}^\Lambda_0[\vartheta_0]} \sin(\vartheta_b(a))}{|\lambda|\pi a} \left(1 + \frac{\Lambda^2 (aC + bf(b))}{\Lambda^2 - a}\right)$$

Assuming $f(b) = 0$, then the fixed-point equation is unchanged, and C can be computed from $\frac{G_{a0}}{G_{0a}}$: