Noncommutative Quantum Field Theory Corfu 2013

Harald Grosse

Faculty of Physics, University of Vienna

Introduction

- Motivation: Improve QFT in 4 dimensions, add "gravity" effects
- Minkowski-Euclidean QFT, Axioms
- Renormalizable Quantum Fields (not summmable) Landau ghost
- Space-Time Non-Commutative Geometry
- Noncommutative Quantum Fields
  formulation, renormalization, IR/UV mixing
- Special Models H G + Wulkenhaar, French group, Italian group,...
  - Renormalizable ncQFTs: Scalar Higgs model
  - Taming the Landau Ghost $\beta = 0$
  - Ward identity, Schwinger-Dyson equs., integral equations
- Fermions, Gauge model: Blaschke, H G, Wallet,...
- Minkowski: Wedge locality, analytic continuation?

Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin
- 4 Fermi interaction is not renormalisable, needs cutoff below $300GeV, W^+, Z^0, W^-, \ldots$
- Quantum field theory for standard model (electroweak+strong) is renormalisable t'Hooft, Veltman
- Gravity is not renormalisable

Renormalisation group interpretation

- space-time being smooth manifold $\Rightarrow$ gravity scaled away
- weakness of gravity determines Planck scale where geometry is something different

promising approach: noncommutative geometry
unifies standard model with gravity as classical field theories

Requirements

Quantum mechanical properties

- states are vectors of a separable Hilbert space H
- $\Phi(f)$ on D dense, $\Phi(f) = \int d^4x \Phi(x) f^*(x)$, Ω is cyclic
- Space-time translations are symmetries:
  spectrum $\sigma(P_\mu)$ in closed forward light cone
  Ground state $\Omega \in H$ invariant under $e^{ia_\mu P^\mu}$

Relativistic properties

- $U_{(a,\Lambda)}$ unitary rep. of Poincaré group on H, Covariance
- Locality
  $[\Phi(f), \Phi(g)]\Psi = 0 \qquad \text{for} \qquad suppf \subset (suppg)'$

Define Wighman functions $W_N(f_1 \otimes ... \otimes f_N) :=< \Omega\Phi(f_1) \cdots \Phi(f_N)\Omega >$
Define Schwinger functions $S_N(z_1, ..., z_N) = \int \Phi(z_1)...\Phi(z_N) d\nu(\Phi)$

Perturb

Schwinger distributions: $S_N(x_1,...,x_N) = \int \phi(x_1)...\phi(x_N)d\nu(\phi)$

$$S_N(x_1 \dots x_N) = \int [d\phi]\, e^{-\int dx \mathcal{L}(\phi)} \prod_i^N \phi(x_i)$$

extract free part $\qquad d\mu(\phi) \propto [d\phi]\, e^{-\frac{m^2}{2}\int \phi^2 - \frac{1}{2}\int (\partial_\mu \phi)(\partial^\mu \phi)}$

Two point correlation: $\langle \phi(x_1)\phi(x_2)\rangle = C(x_1, x_2)$

$C(p_1, p_2) = \delta(p_1 - p_2)\frac{1}{p_1^2 + m^2}$

$$\int d\mu(\phi)\phi(x_1)\dots\phi(x_N) = \sum_{\text{pairings}} \prod_{l\in\gamma} C(x_{i_l} - x_{j_l})$$

add $\frac{\lambda}{4!}$ Φ^4 interaction, expand

Φ^4 Interaction

$$S_N(x_1 \ldots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu(\phi) \prod_j^N Z^{1/2} \phi(x_j) \left(\int dx\, Z^2 \frac{\phi^4(x)}{4!} \right)^n$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\kappa(x_l - y_l) \sim \Lambda^{\omega_D(G)}$$

put cutoff, degree of divergence $\omega_4(G) = 4 - N$, N $\#$ of external lines.
BPHZ theorem, require 3 normalisation conditions:

$$C_4^{ren}(p^2 = M^2) = -Z_{ren}^2 \lambda_{ren},$$

$$C_2^{ren}(p^2 = 0) = \frac{1}{Z_{ren} m_{ren}^2}, \qquad \frac{d}{dp^2} C_2^{ren}(p^2 = 0) = -\frac{1}{Z_{ren} m_{ren}^4}.$$

Use formal power expansion, express $\lambda_{bare}, Z_{bare}, m_{bare}^2$ for all Schwartz functions through $\lambda_{ren}, Z_{ren}, m_{ren}^2$

$\Rightarrow$ order by order finite coefficients

Expansions will diverge - no nontrivial model constructed.

Ideas...

- Limited localization of events in space-time

$$D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D$$

gives Planck lenght as a lower limit to localization

- Replace manifold by algebra, Gelfand, Naimark deform it: canonical, Lie, QG
- keep differential calculus derivations, cov. derivative, curvature... $d^2 = 0$, Leibniz,...: universal,..., DIRAC operator!
- Replace fields by projective modules Serre, Swan
- Replace integrals by traces
- Replace de Rham cohomology by cyclic cohomology
- study spectral triples, spectral action principle
- use renormalized perturbation expansion, extend to nc geometry

Approach

renormalisation of QFTs on noncommutative geometries

Can we make sense out of renormalisation in NCG?

First step: construct quantum field theories on simple nc geometries, e.g. the Moyal space related to $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$

Moyal space

algebra of rapidly decaying functions over D-dimensional Euclidean space with $\star$-product

$$(a \star b)(x) = \int d^D y d^D k a(x+\tfrac{1}{2}\Theta\cdot k) b(x+y)\, e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

like Weyl algebra, Rieffel, deform operator product, operators, Buchholz-Summers: warped convolution

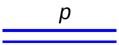
- $\star$-product is associative, noncommutative, and non-local
- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

ϕ^4-action

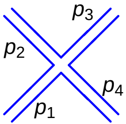
- naïve ϕ^4-action (ϕ-real, Euclidean space) on Moyal plane

$$S = \int d^4x \left(\frac{1}{2} Z \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} Z \phi \star \phi + \frac{\lambda}{4} Z^2 \phi \star \phi \star \phi \star \phi \right)(x)$$

- Feynman rules:



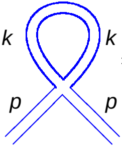
$$= \frac{1/Z}{p^2 + m^2}$$



$$= -\frac{Z^2 \lambda}{4} \exp\left(-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu} \right)$$

- cyclic order of vertex momenta is essential
  $\Rightarrow$ ribbon graphs

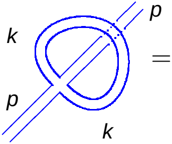
- one-loop two-point function, *planar contribution*:



$$= \frac{-Z^2\lambda}{6}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}$$

to be treated by usual regularisation methods, can be put to 0

- planar nonregular contribution:



$$= \frac{-Z^2\lambda}{12}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot\Theta\cdot p}}{k^2+m^2} \sim (\Theta p)^{-2}$$

- non-planar graphs finite (noncommutativity as a regulator) but $\sim p^{-2}$ for small momenta (renormalisation not possible)

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing

The UV/IR-mixing problem and its solution

- *observation*: ncQFT suffers from UV/IR mixing, which destroys renormalisability if quadratic divergences are present

Theorem

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{Z}{2} \phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{Z^2 \lambda}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Euclidean metric is perturbatively renormalisable to all orders in λ.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

- implements mixing between large and small distance scales
- results from the renormalisation proof
- results as a curvature term Buric, Wohlgenannt

Intuitive remarks

Langmann-Szabo duality

$$\left.\begin{array}{ccc} \tilde{x} & \longmapsto & p \\ \phi(x) & \longmapsto & \hat{\phi}(p) \end{array}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\, (\phi \star \phi \star \phi \star \phi)(x)$ and $\int d^4x\, (\phi \star \phi)(x)$ invariant
- transforms $\int d^4x\, (\phi \star \Delta\phi)(x)$ into $\int d^4x\, (\phi \star \tilde{x}^2 \phi)(x)$

- exact renormalisation group equation in matrix base H G, Wulkenhaar
  simple interaction, complicated propagator, power-counting...
- multi-scale analysis in matrix base Rivasseau, Vignes-Tourneret, Wulkenhaar
  rigorous bounds for the propagator
- multi-scale analysis in position space Gurau, Magnen, Rivasseau, Vignes-Tourneret, simple propagator (Mehler kernel), oscillating vertex

The matrix base of the Moyal plane

- central observation (in 2D):
  $f_{00} := 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad \Rightarrow \quad f_{00} \star f_{00} = f_{00}$

- left and right creation operators:

$$f_{mn}(x_1,x_2) = \frac{(x_1+\mathrm{i}x_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \left(2e^{-\frac{1}{\theta}(x_1^2+x_2^2)}\right) \star \frac{(x_1-\mathrm{i}x_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$

$$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{\mathrm{i}\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}}\rho\right)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m}\left(\frac{2}{\theta}\rho^2\right)$$

- satisfies:

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$$

$$\int d^2x\, f_{mn}(x) = \delta_{mn}$$

- Fourier transformation has the same structure

- extension to 4 dimensions: double indices

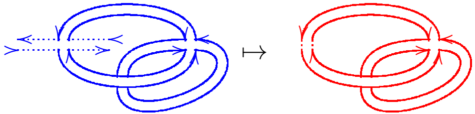
non-local $\star$-product becomes simple *matrix product*

$$S[\phi] = \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right)$$

important: $\Delta_{mn;kl} = 0$ unless $m - l = n - k$

$SO(2) \times SO(2)$ angular momentum conservation

- diagonalisation of Δ: Use Meixner polynomials
- $G_{\substack{m\\0}\substack{m\\0};\substack{m\\0}\substack{m\\0}} \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1) + \Omega^2 (m+1)^2}}$
- $G_{\substack{m_1\\m_2}\substack{m_1\\m_2};\substack{0\\0}\substack{0\\0}} = \frac{\theta}{2(1+\Omega)^2 (m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega} \right)^{m_1+m_2}$
- Feynman graphs are ribbon graphs I edges N external legs
- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes

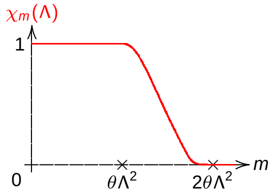


$F = 1$	$g = 1$
$I = 3$	$B = 1$
$V = 2$	$N = 2$

Renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi, \Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

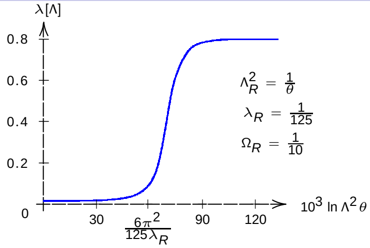
The β-function

one-loop calculation

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\frac{d\lambda}{d\Lambda} = \beta_\lambda = \lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

$\lambda[\Lambda]$ diverges in commutative case



- perturbation theory remains valid at all scales!
- non-perturbative construction: tomorrow,..."solvable"

How does this work?

- four-point function renormalisation with usual sign
- $\exists$ one-loop wavefunction renormalisation which compensates four-point function renormalisation for $\Omega \to 1$

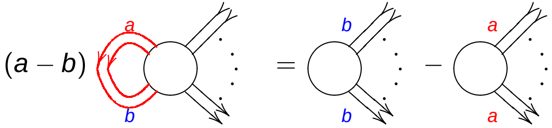
$\Omega = 1$: constant matrix indices for each face

Asymptotic safety to all orders

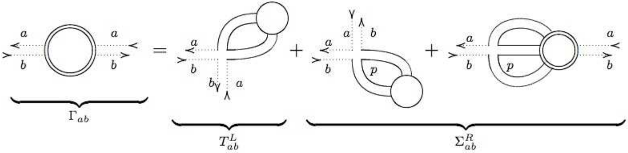
Theorem

$\beta_\lambda = 0$ *at* $\Omega = 1$ *to all orders in* λ, *Disertori, Gurau, Magnen, Rivasseau*

Ward identity:



Schwinger-Dyson equation and resulting integral equations, example



1 particle irreducible 2 point fct — left tadpole — Use WI to reduce to 2 particle amplitudes, get self-consistent equs,...

Theorem

$\beta_\lambda = 0$ *at* $\Omega = 1$, *nonperturbativly (tomorrow)*

Induced Gauge Models

Couple gauge field to scalar field,

H G + Wohlgenannt, de Goursac, Wallet, Wulkenhaar

$$S = \int d^D x \left(\frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{\tilde{X}^\nu, \{\tilde{X}_\nu, \phi\}_\star\}_\star + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x)$$

use covariant coordinates Madore, Wess et al $\tilde{X}_\nu = \tilde{x}_\nu + A_\nu$

$$F_{\mu\nu} = [\tilde{X}_\mu, \tilde{X}_\nu]_\star - i\Theta^{-1}_{\mu\nu}$$

gauge transformation

$$A_\mu \mapsto \mathrm{i} u^* \star \partial_\mu u + u^* \star A_\mu \star u$$

One loop calculation in four dimensions

quadratic divergent; use Duhamel expansion...

$$\Gamma^\epsilon_{1l}[\phi] = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t} \operatorname{Tr} \left(e^{-tH} - e^{-tH^0} \right)$$

Gauge action

$$\Gamma^{\epsilon}_{1l} = \frac{1}{192\pi^2}\int d^4x \Big\{ \frac{24}{\tilde\epsilon\theta}(1-\rho^2)(\tilde X_\nu \star \tilde X^\nu - \tilde x^2) + \ln\epsilon\Big(\frac{12}{\theta}(1-\rho^2)(\tilde\mu^2-\rho^2)(\tilde X_\nu \star \tilde X^\nu - \tilde x^2) + 6(1-\rho^2)^2((\tilde X_\nu \star \tilde X^\nu)^{\star 2} - (\tilde x^2)^2) + \rho^4 F_{\mu\nu}F^{\mu\nu}\Big)\Big\},$$

where $F_{\mu\nu} = [\tilde x_\mu, A_\nu]_\star - [\tilde x_\nu, A_\mu]_\star + [A_\mu, A_\nu]_\star \qquad \rho = \frac{1-\Omega^2}{1+\Omega^2}$

- $\mathcal{O}(1/\epsilon) \ + \ \mathcal{O}(\ln\epsilon)$ gauge invariant renormalizable?
- *quantize* $A = 0$ not stable, nontrivial vacuum solutions
- TrF^2 connects to the IKKT mattrix model: Blaschke, Steinacker, Zahn,...
- Generalize BRST complex to nc gauge models with oscillator H G, Blaschke, Schweda, Wallet, Wohlgenannt,... renormalization ?

A spectral triple

H G, Wulkenhaar Take Dirac operator on Hilbert space $L^2(R^4) \otimes C^{16}$

$$D_8 = (i\Gamma^\mu \partial_\mu + \Omega\Gamma^{\mu+4}\tilde{x}_\mu)$$

$\mu = 1, ...4$, Γ_k generate 8-dim Clifford algebra $\{\Gamma_k \Gamma_l\} = 2\delta_{kl}$

$$D_8^2 = (-\Delta + \Omega^2 ||\tilde{x}||^2)1 - i\Omega\Theta^{-1}_{\mu\nu}[\Gamma^\mu, \Gamma^{\nu+4}]$$

compute action of Dirac operator on sections of spinor bundle

$$[D_8, f] * \psi = i[\Gamma^\mu + \Omega\Gamma^{\mu+4}](\partial\mu f) * \psi$$

only 4 dim. differential appears (leads to spectral triple)
configuration space dimension 4 phase space dim. 8
Clifford alg. dim., KO dim., finite volume: length prop. $\sqrt{\Theta}$...Gayral, Wulkenhaar

QFT on noncommutative Minkowski space

Definition of quantum fields on NC Minkowski space

$$\phi_{\otimes}(x) := \int dp\, e^{ip\cdot\hat{x}} \otimes e^{ip\cdot x}\, \tilde{\phi}(p)$$

- $\phi_{\otimes}(f)$ acts on $\mathcal{V} \otimes \mathcal{H}$, Vacuum $\omega_\theta := \nu \otimes \langle \Omega, .\Omega \rangle$ independent of ν
- We relate antisymmetric matrices to Wedges:

  $W_1 = \left\{ x \in \mathbb{R}^D | x_1 > |x_0| \right\}$ act by Lorentz transformations.
- Stabilizer group is $SO(1,1)xSO(2)$
- Get isomorphism $(\mathcal{W}, i_\Lambda) \cong (\mathcal{A}, \gamma_\Lambda = \Lambda\Theta\Lambda^\dagger)$. $\Phi_W(x) := \Phi_{\Theta(W)}(x)$.

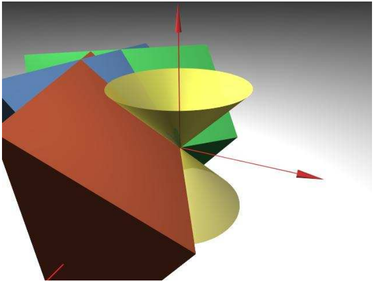
Theorem

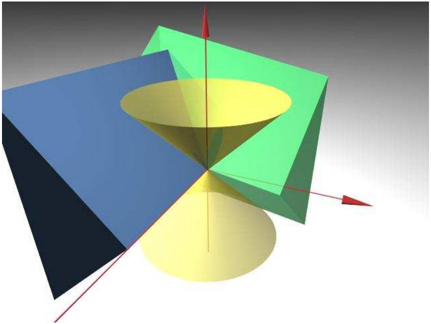
Family $\Phi_W(x)$ is a wedge local quantum field on Fockspace:

$$[\phi_{W_1}(f), \phi_{-W_1}(g)](\psi) = 0,$$

for $supp(f) \subset W_1$, $supp(g) \subset -W_1$.

Use warped convolution,... H G, Lechner; Buchholz, Summers,...





- If rank $\theta = 2$ analytic continuation possible H G, Lechner, Ludwig, Verch
- Mixing occurs rank $\theta = 4$ Bahns; Fischer, Szabo; Zahn
- other models Langmann, Szabo, Zarembo; H G, Steinacker,...Martinetti, Vitale, Wallet

- tomorrow: Nonperturbative construction of Φ^4_θ at $\Omega = 1$ H G, Wulkenhaar
- Model becomes local and Euclidean invariant (in a limit),

it might have analytic continuation for negative coupling,...