Gauge Fields on Non-Commutative Spaces and Renormalization

Talk presented by Daniel N. Blaschke

Recipient of an APART fellowship of the Austrian Academy of Sciences at

Los Alamos National Laboratory, Theory Division

Collaborators: F. Gieres, F. Heindl, M. Schweda, M. Wohlgenannt;
H. Grosse, J.-C. Wallet

Motivation

- incompatibility between GR and QFT

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \langle T_{\mu\nu} \rangle$$

lhs: classical Einstein tensor, rhs: ev of an operator

- natural limit in experimental length resolution: better length resolution requires higher energy, energy required for resolution of the Planck length has a Schwarzschild radius of the Planck length

$$\Delta x_\mu \simeq \lambda_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33}\mathrm{cm}$$

- Effective models for Landau & Quantum Hall effects

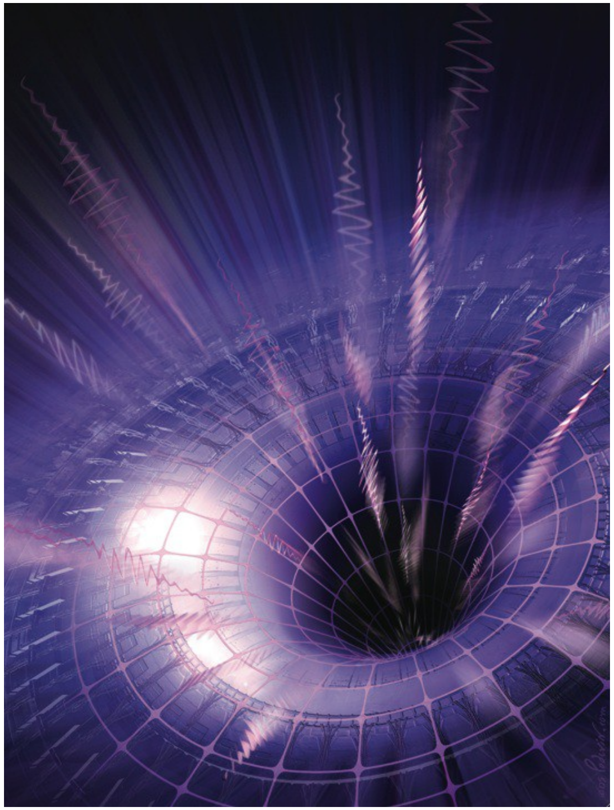


Image source: http://web.physics.ucsb.edu/~giddings/sbgw/physics.html

What is a "non-commutative" space?

Geometric space ⟷ Commutative C*-algebra (according to Gel'fand-Naimark theorem

Non-commutative space ⟷ Non-commutative C*-algebra (A. Connes 1990s)

Commutator of the coordinates has the general form: $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})\,,$

$$\theta^{ij}(x) = \text{const}, \qquad \theta^{ij}(x) = \lambda^{ij}_k \hat{x}^k, \; \ldots$$

e.g. constant case leads to uncertainty relation $\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}| \sim (\lambda_p)^2$

exists isomorphism mapping between NC algebra and commutative one, e.g. Weyl map

$$W : \mathcal{A} \to \widehat{\mathcal{A}}, \qquad x^i \mapsto \hat{x}^i\,,$$

$$\hat{\mathcal{W}}[f]\hat{\mathcal{W}}[g] = \hat{\mathcal{W}}[f \star g]$$

definition of the Groenewold-Moyal *-product:

$$f(x) \star g(x) = \iint \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \tilde{f}(k)\tilde{g}(k') e^{-\frac{i}{2} k_\mu \theta^{\mu\nu} k'_\nu} e^{-i(k_\mu + k'_\mu)x^\mu}$$

QFT on deformed space-time

For a field theory in Euclidean space this means interaction vertices gain phases, whereas propagators remain unchanged, e.g.:

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)$$

and some Feynman integrals (”non-planar diagrams”) have phases which act as UV-regulators

$$\frac{1}{4}\int d^4k \frac{e^{ik\theta p}}{k^2+m^2} = \sqrt{\frac{m^2}{(\theta p)^2}} K_1\left(\sqrt{m^2(\theta p)^2}\right) \approx \frac{1}{(\theta p)^2} + c.m^2 \ln(\theta p)^2$$

➡ origin of the UV/IR mixing problem

<u>Grosse-Wulkenhaar model:</u>

$$S = \int d^4x \left(\tfrac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \tfrac{m^2}{2}\phi^{\star 2} + 2\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \tfrac{\lambda}{4!}\phi^{\star 4} \right), \quad \tilde{x}_\mu := (\theta^{-1})_{\mu\nu} x^\nu$$

propagator is known as the Mehler kernel: $\left(-\Delta + 4\Omega^2 \tilde{x}^2 + m^2\right)^{-1}$

Properties: Langmann-Szabo invariant, renormalizable to all orders in perturbation theory, no Landau ghost and the beta-function vanishes at the self-dual point

Gauge fields on theta-deformed spaces

Non-commutative Yang-Mills action with gauge fixing and ghost terms:

$$S = \frac{1}{4}\int d^4x \left(F_{\mu\nu}\star F^{\mu\nu} + b\star\partial^\mu A_\mu + \tfrac{\xi}{2}b^{\star 2} - \bar{c}\star\partial^\mu D_\mu c\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \underline{- ig[A_\mu \overset{\star}{,} A_\nu]}$$

This action is invariant under the BRST transformations

$$sA_\mu = D_\mu c = \partial_\mu c - ig[A_\mu \overset{\star}{,} c]\,, \qquad sF_{\mu\nu} = -ig[F_{\mu\nu} \overset{\star}{,} c]\,, \qquad sc = igc\star c\,,$$

$$s\bar{c} = b\,, \qquad sb = 0\,, \qquad s^2\varphi = 0\,, \quad \forall\varphi$$

IR divergent terms: $\Pi^{\rm IR}_{\mu\nu}(p) \propto \frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \qquad \tilde{p}^\mu := \theta^{\mu\nu}p_\nu\,,$

$$\Gamma^{3A,\rm IR}_{\mu\nu\rho}(p_1,p_2,p_3) \propto \cos\left(\frac{p_1\theta p_2}{2}\right)\sum_{i=1,2,3}\frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}$$

May try similar techniques as in the scalar case, but gauge symmetry makes matters more complicated...

U(N) gauge fields on Moyal space

Star commutator of two Lie algebra valued functions:

$$[\alpha \stackrel{\star}{,} \beta] = \frac{1}{2}\{\alpha^a \stackrel{\star}{,} \beta^b\}[T^a, T^b] + \frac{1}{2}[\alpha^a \stackrel{\star}{,} \beta^b]\{T^a, T^b\}$$

→ *must always consider enveloping algebras, such as U(N), O(N), etc.*

$$A_\mu = A_\mu^A T^A \,, \qquad \mathrm{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB} \,, \qquad T^0 = \frac{1}{\sqrt{2N}}\mathbb{1}_N$$

couplings between the U(1) and the SU(N) sector

IR divergent terms:

$$\Pi^{\mathrm{IR},AB}_{\mu\nu}(p) \propto N \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}\delta^{A0}\delta^{B0}$$

$$d^{aCD}d^{bCD}\sin^2\left(\frac{k\tilde{p}}{2}\right) + f^{acd}f^{bcd}\cos^2\left(\frac{k\tilde{p}}{2}\right) = N\delta^{ab}, \qquad a, b \neq 0$$

$$\Gamma^{3A,\mathrm{IR}}_{\mu\nu\rho}(p_1, p_2, p_3) \propto \sqrt{N}\cos\left(\frac{p_1\theta p_2}{2}\right) \sum_{i=U(1)\text{-legs}} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{(\tilde{p}_i^2)^2}$$

Slavnov-Taylor identities in NCGFTs

IR divergent terms are consistent with ST-identities (cf. D.B., H. Grosse, J.-C. Wallet, *JHEP* **06** (2013) 038):

- Transversality of 2pt-function: $$\partial^z_\mu \frac{\delta^2\Gamma^{(0)}}{\delta A^A_\nu(y)\delta^B A_\mu(z)}\Big|_{\Phi=0} = 0$$

- Identity connecting 2pt with 3pt function:

$$\partial^z_\mu \frac{\delta^3\Gamma^{(0)}}{\delta A^A_\sigma(x)\delta A^B_\nu(y)\delta A^C_\mu(z)} = \mathrm{i}g d^{DAC}\left[\frac{\delta\Gamma^{(0)}}{\delta A^D_\sigma(x)\delta A^B_\nu(y)} \stackrel{\star}{,} \delta(y-z)\right]$$
$$+\,\mathrm{i}g f^{DAC}\left\{\frac{\delta\Gamma^{(0)}}{\delta A^D_\sigma(x)\delta A^B_\nu(y)} \stackrel{\star}{,} \delta(y-z)\right\} + (\sigma, A, x) \leftrightarrow (\nu, B, y)$$

- And a similar identity connecting 3pt with 4pt function:

➡ *no IR divergence in 4pt function*

Constructing an IR modified gauge field model

E.g.: IR damping similar to scalar model by Gurau et al., *Comm.Math.Phys.* **287** (2009) 275.

$$\int d^4x\, \phi(x) \frac{a^2}{\theta^2 \Box} \phi(x) \quad \Rightarrow \quad \int d^4x\, \frac{1}{4} F^{\mu\nu} \star \frac{a^2}{D^2 \widetilde{D}^2} \star F_{\mu\nu}$$

$$\widetilde{D}_\mu = \theta_{\mu\nu} D^\nu \,, \qquad \delta_\alpha \left(\frac{1}{D^2} F \right) = ig[\alpha \stackrel{\star}{,} \frac{1}{D^2} F]$$

Drawback: infinite number of vertices ...

→ Proposition:

Use techniques known from the Gribov-Zwanziger action in QCD

(cf. D.B., Rofner, Sedmik, Wohlgenannt, *J. Phys.* **A43** (2010) 425401)

In QCD, restriction to 1. Gribov horizon via:

$$\gamma^4 g^2 \int d^4x\, f^{abc} A^b_\mu (\mathcal{M}^{-1})^{ad} f^{dec} A^e_\mu \quad \Rightarrow \quad G^{ab}_{\mu\nu} = \frac{\delta^{ab}}{k^2 + \frac{\gamma^2}{k^2}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)$$

How to prove renormalizability?

- Need a renormalization scheme that preserves gauge symmetry and works also in non-commutative space...

- The scalar model was proven using multiscale analysis which unfortunately breaks gauge symmetry in our case.

- Algebraic renormalization works well with models which have symmetries, but only if they are local in the quantum fields.

- Dimensional regularization does not work well with non-planar graphs.

- BPHZ with dim. Regularization works well with local gauge theories.

  Can we generalize the scheme to a non-commutative setting?

The BPHZ scheme ...

Inventors: Bogoliubov, Parasiuk, Hepp and Zimmermann

- A subtraction scheme
- Proof of locality of these subtractions
- Normalization conditions
- Overlapping (sub-)divergences are treated using Zimmermanns forest formula

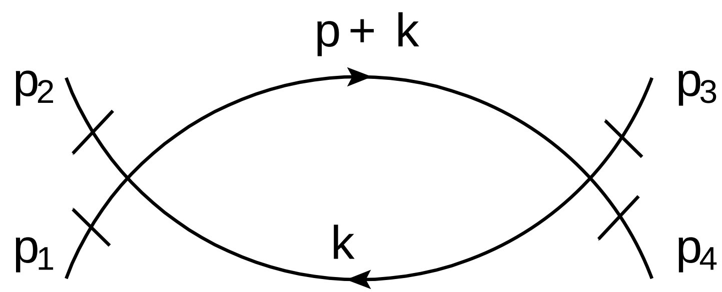
Divergences are subtracted without requiring regularization, e.g.:

$$J_\Gamma^{\text{finite}}(\underline{p}) \equiv \int d^4k \left[1 - t_{\underline{p}}^{\delta(\Gamma)}\right] I_\Gamma(\underline{p}, k) \,,$$

$$\left(t_{\underline{p}}^N I_\Gamma\right)(\underline{p}, k) \equiv \sum_{l=0}^{N} \frac{1}{l!} p_{i_1}^{\mu_1} \cdots p_{i_l}^{\mu_l} \frac{\partial^l I_\Gamma}{\partial p_{i_1}^{\mu_1} \cdots \partial p_{i_l}^{\mu_l}} (\underline{p} = \underline{0}, k) \,.$$

… and its problems in the non-commutative setting

Example: "fish-diagram" of phi^4 theory



$$J_\Gamma^{\text{finite}}(p) \equiv \int d^4k\,[1 - t_p^0]\, I_\Gamma(p,k) = \int d^4k\,[I_\Gamma(p,k) - I_\Gamma(0,k)]$$

$$= \int d^4k \left(\frac{1}{[(p+k)^2 + m^2][k^2+m^2]} - \frac{1}{[k^2+m^2]^2} \right),$$

But: $$J_\Gamma^{\text{NC}}(p) = \int d^4k \left(\frac{A + B\cos(k\tilde{p})}{[(p+k)^2 + m^2][k^2+m^2]} - \frac{A + B}{[k^2+m^2]^2} \right),$$

is not finite!

Suggested remedy: BPHZ for NCQFTs

Treat p and p-tilde independently (cf. D.B., Garschall, Gieres, Heindl, Schweda, Wohlgenannt, *EPJ* **C73** (2013) 2262):

$$J_{\Gamma}^{\mathrm{NC}}(p) = \int d^4k\,(A + B\cos(k\tilde{p}))\left(\frac{1}{[(p+k)^2+m^2][k^2+m^2]} - \frac{1}{[k^2+m^2]^2}\right)$$

General expression with superficial degree of divergence n:

$$R(p_i,\tilde{p}_i,k) = \left(1 - t_p^n\right) I(p_i,\tilde{p}_i,k)\,,$$

$$(t_p^n f)(p_i,\tilde{p}_i) := f(0,\tilde{p}_i) + \sum_j p_j^\mu \left(\frac{\partial}{\partial p_j^\mu} f(p_i,\tilde{p}_i)\right)\Bigg|_{p_i=0} + \ldots$$

$$+ \frac{1}{n!}\sum_{j_1,\ldots,j_n} p_{j_1}^{\mu_1}\ldots p_{i_n}^{\mu_n}\left(\frac{\partial}{\partial p_{j_1}^{\mu_1}}\cdots\frac{\partial}{\partial p_{j_n}^{\mu_n}} f(p_i,\tilde{p}_i)\right)\Bigg|_{p_i=0}$$

Apply new BPHZ scheme to NC scalar theory:

$$S[\phi] = \int d^4x \left[\frac{1}{2}(1+a)\partial^\mu \phi \partial_\mu \phi + \frac{1}{2}(m^2+b)\phi^2 + \frac{1}{4!}(\lambda+c)\phi^{\star 4} \right] .$$

In momentum space, constants b, c are p and θ dependent, but:

$$V(p_1, p_2, p_3, p_4) \equiv -\lambda f(p_i, \theta)$$

$$= \frac{-\lambda}{3} \left[\cos\left(\frac{p_1\tilde{p}_2}{2}\right)\cos\left(\frac{p_3\tilde{p}_4}{2}\right) + \cos\left(\frac{p_1\tilde{p}_3}{2}\right)\cos\left(\frac{p_2\tilde{p}_4}{2}\right) + \cos\left(\frac{p_1\tilde{p}_4}{2}\right)\cos\left(\frac{p_2\tilde{p}_3}{2}\right) \right]$$

Result:

$$c(p_i, \theta) = c' + c''(p_i, \theta) + c_\infty \,,$$

$$c'' \equiv \frac{\lambda^2}{6(4\pi)^2} \left[\sum_{i=1}^{4} \ln(m^2 \tilde{p}_i^2) + \sum_{i=2}^{4} \ln(m^2 (\tilde{p}_1 + \tilde{p}_i)^2) \right] ,$$

$$c_\infty \equiv -\frac{2\lambda^2}{9} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+m^2)^2}$$

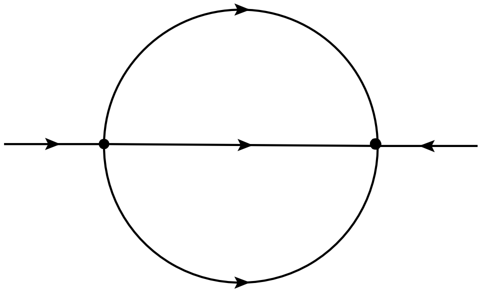
$$\Gamma_r^{(4)}(p_i) = \lambda + c' + \widetilde{\Delta}'(p_i, \theta) \,,$$

c' fixed by normalization condition

New BPHZ scheme seems to work for NC scalar theory

✔ Explicit 1-loop calculations

✔ Also works for sunrise graph which appears at two-loop level and has an overlapping divergence



✔ BPHZ subtractions always involve some ambiguities (finite terms), which in the NC setting lead to additional possible counter terms such as

$$\tilde{\phi}(\tilde{p}^2)^{-1}\tilde{\phi}$$

⇒ this is exactly the additional term in the Gurau et al. model!

In general: allowed terms are polynomials in $1/\tilde{p}^2$ whose degree is determined by the degree of IR divergence.

What about NC gauge theory?

Gauge fields are massless, hence usually need further regularization, ***BUT***

in the model I presented earlier, the propagator's damping term takes care of this!

→ No s-trick of Lowenstein or dim. reg. seem to be required!

New BPHZ scheme works fine for 1-loop vacuum polarization; vertex corrections and higher orders are still work in progress.

$$\Pi^{\text{finite}}_{\mu\nu}(p) = \int d^4k \left(1 - t^2_p\right) I_{\Gamma\mu\nu}(p,\tilde{p},k)$$

$$= 2g^2 \int \frac{d^4k}{(2\pi)^4} (1-\cos(k\tilde{p})) \Bigg\{ \left(p^2 - \frac{a^2p^2}{(k^2)^2} + \frac{4(kp)^2}{N}\left(\frac{3a^2}{(k^2)^2} - 1\right)\right) \frac{(4k_\mu k_\nu - 2k^2\delta_{\mu\nu})}{N^3}$$

$$+ \left(4k_\mu k_\nu - 3p_\mu p_\nu + 2\delta_{\mu\nu}(p^2 - k^2)\right)\frac{1}{N}\left(\frac{1}{\left((k+p)^2 + \frac{a^2}{(k+p)^2}\right)} - \frac{1}{N}\right)\Bigg\},$$

where $\displaystyle \frac{1}{N} := \frac{1}{k^2 + \frac{a^2}{k^2}} = \frac{1}{2}\sum_{\zeta=\pm1} \frac{1}{k^2 + \mathrm{i}a\zeta}\,.$

Conclusion and Outlook

✔ Introduced and motivated the concept of NCQFTs

✔ Have constructed a promising candidate for a renormalizable NC gauge field model (for both U(1) and U(N)), but need to prove renormalizability to all orders

✔ Shown some steps towards this goal by generalizing the BPHZ renormalization scheme to the NC setting

✔ Still some work to do: higher loop orders, systematic proof, ...

References

1. D. N. Blaschke, E. Kronberger, R. I. P. Sedmik and M. Wohlgenannt, “Gauge Theories on Deformed Spaces”, *SIGMA* **6** (2010) 062, [arXiv:1004.2127].
2. D. N. Blaschke, A. Rofner, R. I. P. Sedmik and M. Wohlgenannt, “On Non-Commutative U*(1) Gauge Models and Renormalizability”, *J. Phys.* **A43** (2010) 425401, [arXiv:0912.2634].
3. D. N. Blaschke, “A New Approach to Non-Commutative U(N) Gauge Fields”, *EPL* **91** (2010) 11001, [arXiv:1005.1578].
4. D. N. Blaschke, T. Garschall, F. Gieres, F. Heindl, M. Schweda and M. Wohlgenannt, “On the Renormalization of Non-Commutative Field Theories”, *Eur. Phys. J.* **C73** (2013) 2262, [arXiv:1207.5494].
5. D. N. Blaschke, H. Grosse and J.-C. Wallet, “Slavnov-Taylor identities, non-commutative gauge theories and infrared divergences”, *JHEP* **06** (2013) 038, [arXiv:1302.2903].
6. D. N. Blaschke, F. Gieres, F. Heindl, M. Schweda and M. Wohlgenannt, “BPHZ renormalization and its application to non-commutative field theory”, to appear in EPJC, [arXiv:1307.4650].

Thank you for your attention!