Black holes, the Van der Waals gas, compressibility and the speed of sound

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Outline

- Review of black hole thermodynamics
  - Basics
  - 1st and 2nd laws
  - Smarr relation
- Pressure and Enthalpy
  - Enthalpy and the 1st law
  - Volume
- Equation of state
  - Critical behaviour
  - Compressibility
  - Speed of sound
- Conclusions

Basics

- Entropy: $S \propto \frac{A}{\ell_{Pl}^2}$ ($\ell_{Pl}^2 = \hbar G/c^3$, $G = c = 1$).
  Bekenstein (1972)
- Temperature, $T = \frac{\kappa\hbar}{2\pi}$: $\kappa =$surface gravity. Hawking (1974)

Schwarzschild black-hole, $\kappa = \frac{1}{4M}$

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First and Second laws

- Internal energy $U(S)$, $T = \frac{\partial U}{\partial S}$:
  identify $M = U(S) \Rightarrow dM = TdS$.
  More generally $M = U(S, J, Q)$, (angular momentum, electric charge),

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Smarr (1973)

- Ordinary thermodynamics: $U(S,V,n)$ ($n =$ number of moles) is a function of extensive variables. U is also extensive $\Rightarrow$

$$\lambda^d U(S,V,n) = U(\lambda^d S, \lambda^d V, \lambda^d n)$$

$$\Rightarrow \quad U = S\frac{\partial U}{\partial S} + V\frac{\partial U}{\partial V} + n\frac{\partial U}{\partial n}$$

Euler equation

$$\Rightarrow \quad U = ST - VP + n\mu$$

($\mu =$ chemical potential)

$$\Rightarrow \quad G = U + VP - ST = n\mu\,.$$

Gibbs-Duhem relation

- Black hole in D dimensions, angular momenta J_i: ($Q = 0$) $S \to \lambda^{D-2}S$, $J_i \to \lambda^{D-2}J_i$, $M \to \lambda^{D-3}M$ $\Rightarrow$

$$\lambda^{D-3}M(S,J_i) = M(\lambda^{D-3}S, \lambda^{D-2}J_i)$$

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- Pressure $P = -\frac{\Lambda}{8\pi} \Rightarrow$ mass is the enthalpy:

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$$M = H(S, P, J)$$

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$$dU = T\,dS \; - P\,dV + \Omega dJ$$

- For $\Lambda < 0$, Penrose process can be up to 50% efficient (0.29% for $\Lambda = 0$). (BPD [1008.5023])

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Thermodynamic Volume

- In $4-D$ the thermodynamic volume evaluates to

$$V = \frac{1}{3}\left(r_h \mathcal{A}_h + \frac{J^2}{M}\right),$$

where $\mathcal{A}_h$ is area of the event horizon and r_h the radius.
(Cvetič *et al.* [1012.2888].)

Equation of state $(D = 4, \Lambda < 0)$

- Critical point $(J \neq 0)$:

$$(PJ)_{crit} \approx 0.002857\,, \ \left(T\sqrt{J}\right)_{crit} \approx 0.04175\,, \ \left(\frac{V}{J^{3/2}}\right)_{crit} \approx 115.8$$

- Define

$$t := \frac{T - T_c}{T_c}, \qquad v := \frac{V - V_c}{V_c}, \qquad p := \frac{P - P_c}{P_c}.$$

Expand the equation of state about the critical point:

$$p = 2.42t - 0.81tv - 0.21v^3 + o(t^2, tv^2, v^4).$$

cf. Van der Waals gas: $p = 4t - 6tv - \frac{3}{2}v^3 + o(t^2, tv^2, v^4)$.
(Gunasekaran *et al.* [1208.6251], BPD [1209.1272]).

Compressibility

- Adiabatic compressibility: $\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}\Big|_{S,J}$.
- Rotating black-hole in D-dimensions (Myers-Perry). Dimensionless angular momenta, $\mathcal{J}_i := \frac{2\pi J_i}{S}$,

  Constraint: $T \geq 0 \Rightarrow \sum_i \frac{1}{1+\mathcal{J}_i^2} \geq \begin{cases} \frac{1}{2} & \text{even } D \\ 1 & \text{odd } D. \end{cases}$

Compressibility, $\Lambda \to 0$

$$\kappa = \frac{16\pi r_h^2}{(D-1)(D-2)^2}\left\{\frac{(D-2)\sum_i \mathcal{J}_i^4 - (\sum_i \mathcal{J}_i^2)^2}{D-2+\sum_i \mathcal{J}_i^2}\right\},$$

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- $0 \leq \kappa < \infty$.

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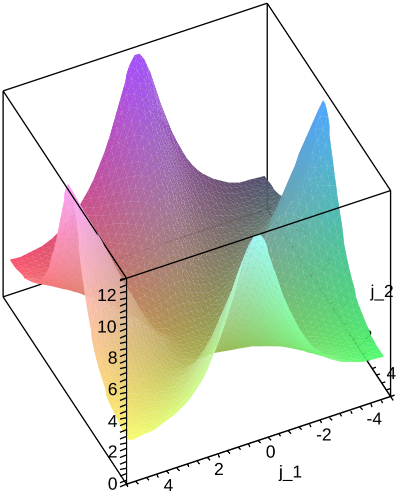
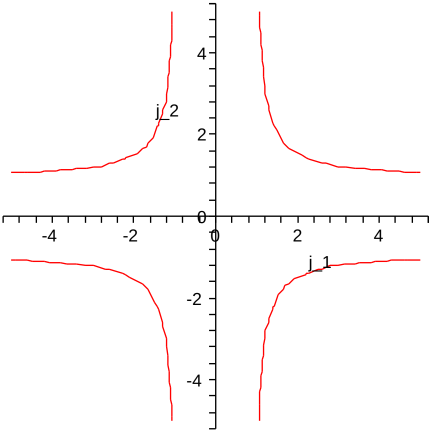
Example of compressibility, $D = 6$, $SO(5)$: J_1, J_2,





Speed of sound

- Define $\rho := \frac{M}{V}$, then the thermodynamic speed of sound is

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S,J}, \qquad \Rightarrow \qquad c_s^{-2} = 1 + \kappa\rho.$$

Speed of sound

$$c_s^2 = \frac{1}{(D-2)} \frac{\left(D - 2 + \sum_i \mathcal{J}_i^2\right)^2}{\left(D - 2 + 2\sum_i \mathcal{J}_i^2 + \sum_i \mathcal{J}_i^4\right)}.$$

- $\frac{1}{D-2} \le c_s^2 \le 1.$

Speed of sound

- Define $\rho := \frac{M}{V}$, then the thermodynamic speed of sound is

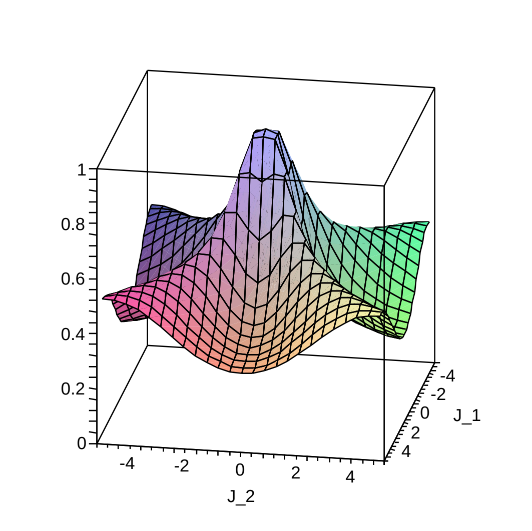
$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{S,J}, \qquad \Rightarrow \qquad c_s^{-2} = 1 + \kappa\rho.$$

Speed of sound

$$c_s^2 = \frac{1}{(D-2)} \frac{\left(D - 2 + \sum_i \mathcal{J}_i^2\right)^2}{\left(D - 2 + 2\sum_i \mathcal{J}_i^2 + \sum_i \mathcal{J}_i^4\right)}.$$

- $\frac{1}{D-2} \leq c_s^2 \leq 1.$

c_s^2 in 6-dimensions:



Conclusions

- $\Lambda \neq 0 \quad \Rightarrow \quad P\,dV$ term in black hole 1st law.
- Black hole mass is identified with **enthalpy**, $H(S,P,J_i)$:

$$dM = dH = T\,dS + V\,dP + \Omega_i J_i\,,$$
$$dU = T\,dS - P\,dV + \Omega_i J_i\,.$$

- Hawking temperature: $T = \left(\frac{\partial H}{\partial S}\right)_P$.
- "Thermodynamic" volume: $V = \left(\frac{\partial H}{\partial P}\right)_T$.
- PdV term affects Penrose processes — more efficient in asymptotically AdS space-times.
- $D = 4$: Van der Waals type equation of state.
- Compressibility, $0 \leq \kappa < \infty$, with $\kappa \to \infty$ for some $J_i \to \infty$.
- Instability of ultra-spinning black-holes.
- Speed of sound: $\frac{1}{D-2} \leq c_s^2 \leq 1$.

Non-zero charge, zero J

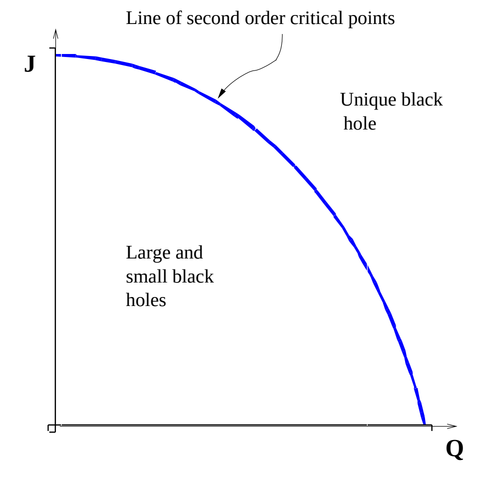
Reissner-Nordström — Anti-de Sitter ($J = 0$, $Q \neq 0$):

- Gibbs free energy is a 'swallowtail', same as Van der Waals
  Champlin, Emparan, Johnson+Myers:
  [hep-th/9902170]; [hep-th9904197].
- Equation of state:

$$p = \frac{8}{3}t - \frac{8}{9}tv - \frac{4}{81}v^3 + o(t^2, tv^2, v^4),$$

Critical exponents are mean field
(Kubizňák+Mann [arXiv:1205.0559]).

Kerr-Reissner-Nordström-AdS



Reissner-Nordström anti-de Sitter ($J \neq 0$, $Q \neq 0$).
No Hawking-Page phase transition in canonical ensemble, if $J \neq 0$ or $Q \neq 0$
Caldarelli, Gognola+Klemm [hep-th/9908022].