Generalized Complex Geometry

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18th EUROPEAN WORKSHOP ON STRING THEORY
Outline

- Formulations of Generalized Kähler Geometry
- The corresponding Sigma Models

Outline II: Sigma Model geometry

- Sigma models
  - SUSY sigma models and geometry
  - Complex geometry
  - Kähler
  - Bihermitean geometry
  - Generalized complex geometry
  - Generalized Kähler geometry
  - Superspace descriptions
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Outline III: Relation to supergravity

- Pure Spinors
- Generalized Calabi Yau
- Supergravity
- Relation to the Sigma Model formulation
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\[ \phi^i : \Sigma \to \mathcal{T} \]

\[ \xi \mapsto \phi^i(\xi) \]

\[ S = \int_{\Sigma} d\phi^i g_{ij}(\phi) \star d\phi^j \]

\[ \nabla^2 \phi^i := \partial^2 \phi^i + \partial \phi^j \Gamma^i_{jk} \partial \phi^k = 0 \]

\[ S = \int_{\Sigma_B} d\xi \left\{ \eta^{\mu\nu} \partial^i \partial^j g_{ij}(X) \partial \nu X^j \right\} + \ldots \]
Susy Sigma Models

Susy $\sigma$ models $\iff$ Geometry of $\mathcal{T}$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$N=1$</th>
<th>$N=2$</th>
<th>$N=4$</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>Hyperkähler</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>Kähler</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>Riemannian</td>
</tr>
</tbody>
</table>

(Odd dimensions have the same structure as the even dimension lower.)
The (1,1) analysis by Gates Hull and Roček gives:

<table>
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<tr>
<th>Susy</th>
<th>(0,0) (1,1)</th>
<th>(2,2)</th>
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<th>(4,4)</th>
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Manifold $(M^{2d}, J)$

Complex structure: $J \in \text{End}(TM)$ \hspace{1cm} $J^2 = -1$

Projectors: $\pi_{\pm} := \frac{1}{2} (1 \pm iJ)$

These define an involutive distribution if

$$\pi_{\mp}[\pi_{\pm} u, \pi_{\pm} v] = 0 \iff N(J) = 0 \, \text{(Nijenhuis)}$$

This is integrability of $J$.

Local holomorphic coordinates, $M^{2d} \supset O \approx \mathbb{C}^{2k}$, and holomorphic transition functions.
Hermitean Metric: \( J^t g J = g \)

\((g \rightarrow g = g + J^t g J)\)

Symplectic 2-form: \( \omega := gJ \)

Kähler: \( d\omega = 0, \quad \nabla J = 0, \quad g_{z\bar{z}} = \partial_z \partial_{\bar{z}} K(z, \bar{z}) \)

Hyperkähler: \( J^A, A = 1, 2, 3 \quad J^A J^B = -\delta^{AB} + \epsilon^{ABC} J^C \)
Erich Kähler 1906-2000
Gates-Hull-Roček Bihermitean.

\[(M, g, J(\pm), H)\]

\[
J^2(\pm) = -1, \quad J^t(\pm) g J(\pm) = g, \quad \nabla(\pm) J(\pm) = 0
\]

\[
\Gamma(\pm) = \Gamma^0 \pm \frac{1}{2} g^{-1} H, \quad H = dB.
\]

\[
E := g + B
\]
$$J^2_{(\pm)} = -1, \quad J^t(\pm) g J(\pm) = g, \quad \omega(\pm) := g J(\pm)$$

$$d^c(+) \omega(+) + d^c(-) \omega(-) = 0, \quad dd^c(\pm) \omega(\pm) = 0,$$

$$H := d^c(+) \omega(+) = -d^c(-) \omega(-)$$
Generalized Complex Geometry

Complex structure:

\[ \mathcal{J} \in \text{End}(TM \oplus T^*M), \quad \mathcal{J}^2 = -1 \]

\[ \Pi_{\pm} := \frac{1}{2} (1 \pm \mathcal{J}) \]

“Nijenhuis”:

\[ \mathcal{N}_C(\mathcal{J}) = 0 \iff \Pi_{\mp} [\Pi_{\pm} U, \Pi_{\pm} V]_C = 0 \]

where

\[ U = (u, \xi), \quad V = (v, \rho) \]

\[ [U, V]_C = [u, v] + \mathcal{L}_u \rho - \mathcal{L}_v \xi - \frac{1}{2} d(\iota_u \rho - \iota_v \xi) \]
When $\mathcal{J}$ is integrable there are local holomorphic and Darboux coordinates such that $M^{2d}$ looks like $\mathbb{C}^k \times \mathbb{R}^{2d-k}$. 
The automorphisms of this courant bracket are diffeomorphisms and \( b \)-transforms:

\[
e^b(u, \xi) = (u, \xi + \iota_u b), \quad db = 0.
\]

In a coordinate basis \((\partial_x, dx)\) a \( b \)-transform acts on \( \mathcal{J} \) as follows:

\[
\begin{pmatrix}
1 & 0 \\
b & 1
\end{pmatrix}
\mathcal{J}
\begin{pmatrix}
1 & 0 \\
-b & 1
\end{pmatrix},
\]
In such a basis, the natural pairing

\[ \langle (u, \xi), (v, \rho) \rangle = uv \rho + v \xi \]

is represented by the matrix

\[
\mathcal{I} = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

A final requirement of GCG is that

\[ \mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I} \]
Generalized Kähler Geometry

Two commuting generalized complex structures

\[ J_{(1,2)}^2 = -1, \quad [J_{(1)}, J_{(2)}] = 0, \]

\[ J_{(1,2)^t} I J_{(1,2)} = I, \quad G := -J_{(1)} J_{(2)} \]

Ex. Kähler \((\omega = g J)\):

\[ J_1 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \]
The G-map

GKG ↔ Bi-Hermitean:

\[ \mathcal{J}_{(1,2)} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_{(+) \pm J_{(-)} } & \omega_{(-)}^{-1} + \omega_{(+)}^{-1} \\ \omega_{(+)} + \omega_{(-)} & \omega_{(-)}^{-1} + \omega_{(+)}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \]

\[ [U, V]_H = [U, V]_C + \iota_{\mu} \iota_{\nu} H \]
$(M, J_{(\pm)})$

Locally, $\exists$ “symplectic” two-forms $\mathcal{F}_{(\pm)}$ such that

$$\mathcal{F}_{(\pm)}(v, J_{(\pm)}v) > 0, \quad d(\mathcal{F}_{(+)} J_{(+)} - J_{(-)}^t \mathcal{F}_{(-)}) = 0.$$  

$$\mathcal{F}_{(\pm)} = \frac{1}{2} i (B_{(\pm)}^{(2,0)} - B_{(\pm)}^{(0,2)}) \mp \omega_{(\pm)}$$

$$\mathcal{F}_{(+)} = -\frac{1}{2} E_{(+)}^t J_{(+)} , \quad \mathcal{F}_{(-)} = -\frac{1}{2} J_{(-)}^t E_{(-)}^t$$
Geometric data: \((M, g, H, J_{(\pm)})\) or \((M, g, J_{(\pm)})\) or \((M, F_{(\pm)}, J_{(\pm)})\). In each case, there is a complete description in terms of a Generalized Kähler potential \(K\). Unlike the Kähler case, the expressions are non-linear in second derivatives of \(K\). E.g.,

\[
J_{(+)} = \begin{pmatrix} J & 0 \\ (K_{LR})^{-1}[J, K_{LL}] & (K_{LR})^{-1}JK_{LR} \end{pmatrix}
\]

\[
g = \Omega[J_{(+)}, J_{(-)}]
\]

\[
F_{(+)} = d\lambda_{(+)} , \quad \lambda_{(+)}\ell = iK_RJ(K_{LR})^{-1}K_{L\ell} , \ldots
\]
There are two special sets of Darboux coordinates for the symplectic form $\Omega$. One set, $(X^L, Y_L)$, is also canonical coordinates for $J_\(+\)$ and the other set, $(X^R, Y_R)$ is canonical coordinates for $J_\(-\)$. The symplectomorphism that relates the two sets of coordinates has thus a generating function. This generating function is in fact the generalized Kähler-potential $K(X^L, X^R)$.

\[
\begin{array}{c|c|c}
(X^L, Y_L) & \rightarrow K(X^L, X^R) & (X^R, Y_R) \\
J_\(+\) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & J_\(-\) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\end{array}
\]

This fact is a key ingredient in the proof that we have a complete description or GKG.
\( d = 2 \, , \, N = (2, 2) \)

Algebra:

\[ \{ \mathcal{D}_\pm, \bar{\mathcal{D}}_\pm \} = i \partial_\pm \]

Constrained superfields:

\[
\begin{align*}
\bar{\mathcal{D}}_\pm \phi^a &= 0 , \\
\bar{\mathcal{D}}_+ \chi^{a'} &= \mathcal{D}_- \chi^{a'} = 0 , \\
\bar{\mathcal{D}}_+ X^\ell &= 0 , \\
\bar{\mathcal{D}}_- X^r &= 0 .
\end{align*}
\]

Notation: \( c := a, \bar{a} , \, t := a', \bar{a}' , \, L := \ell, \bar{\ell} , \, R := r, \bar{r} . \)
The (2, 2) formulation uses the generalized Kähler Potential.

\[ S = \int \mathcal{D}+\bar{\mathcal{D}}+\mathcal{D}-\bar{\mathcal{D}} K(\phi^c, \chi^t, X^L, X^R) \]

\[ K \rightarrow K(X^L, X^R) \]

Reduction to (2, 1) superspace

\[ \mathcal{D}_- =: D_- - iQ_- , \quad X| =: X , \quad Q_- X^L| =: \psi_- \]

\[ S = \int \mathcal{D}+\bar{\mathcal{D}}+D_- \left( K_L \psi_-^L + K_R JD_- X^R \right) \]

\[ S = i \int \mathcal{D}+\bar{\mathcal{D}}+D_- \left( \lambda(+)\alpha D_- \varphi^\alpha + \text{c.c.} \right) \]

which uses the “Liouville form” \( \mathcal{F}(+) = d\lambda(+) \)
Reduction to (1, 1) finally yields

\[ S = \int D_+ D_- (D_+ X E D_- X) . \]

The reduction goes via \( D_+ =: D_+ - iQ_+ \), \( Q_+ X^R | =: \psi_+^R \) and both the auxiliary spinors \( \psi_-^L \) and \( \psi_+^R \) have been eliminated.

The (1, 1) formulation uses \( E = g + B \) directly.
Superspace encodes and dictates all the geometric formulations of Generalized Kähler Geometry.
Multi forms $\rho$ on $M$ are spinors of $T \oplus T^*$. 

$U = (u, \xi) \in T \oplus T^*$ acts on a form $\rho$ according to

$$U \cdot \rho = u \rho + \xi \wedge \rho$$

This satisfies the Clifford algebra identity for the indefinite metric $\mathcal{I}$:

$$\{U, V\} \cdot \rho = (U \cdot V + V \cdot U) \cdot \rho = 2\mathcal{I}(U, V)\rho$$
The null space of a spinor $\rho$

$$L_\rho = \{ U \in TM \oplus T^* M | U \cdot \rho = 0 \}$$

is isotropic. A spinor $\rho$ is pure if its null space is maximally isotropic, rank $d$.

A GCS $\mathcal{J}$ may alternatively be defined via decomposition

$$(T \oplus T^*) \otimes \mathbb{C} = L + \bar{L}$$

where $L$ is the $+i$ eigenbundle of $\mathcal{J}$.

To every GCS $\mathcal{J}$ with $+i$ eigenbundle $L_\mathcal{J}$ is associated a complex pure spinor $\rho_\mathcal{J}$:

$$L_\mathcal{J} = L_{\rho_\mathcal{J}}$$
A generalized Calabi-Yau structure:

\[ d\rho = 0 , \quad (\rho, \bar{\rho}) \neq 0 \]

A generalized Calabi-Yau metric structure is defined as a pair of closed pure spinors \( \rho_1 \) and \( \rho_2 \) such that the corresponding generalized complex structures \( J_1 \) and \( J_2 \) give rise to a GKS and

\[(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) \neq 0\]

Mukai pairing:

\[
(\rho_1, \rho_2) = \sum_j (-1)^j [\rho_1^{2j} \wedge \rho_2^{d-2j} + \rho_1^{2j+1} \wedge \rho_2^{d-2j-1}] ,
\]
The conditions for Type II supergravity solutions in
\[ ds^2_{(10)} = e^{2A(y)} ds^2_{(4)} + g_{mn} dy^m dy^n \]
is
\[ d_H(e^{4A-\phi} \mathcal{R} \rho_1) = e^{4A} \tilde{F} \]
\[ d_H(e^{3A-\phi} \rho_2) = 0 \]
\[ d_H(e^{2A-\phi} \mathcal{S} \rho_1) = 0 \]
where $\tilde{F}$ is (part of) the polyform of RR fields.
The generalized CY metric structure defines a Type II supersymmetric supergravity solution. (No RR fluxes).

Use the Gualtieri map to find \((g_{\mu\nu}, H_{\mu\nu\rho}, \Phi)\).

\[
(\rho_1, \bar{\rho}_1) = \alpha(\rho_2, \bar{\rho}_2) = e^{-2\Phi} \sqrt{g} \, dx^1 \wedge \ldots \wedge dx^D
\]

\[
R^{(+)}_{\mu\nu} + 2\nabla^{(-)}_{\mu} \partial_{\nu} \Phi = 0,
\]

automatically satisfied.
Construction from the sigma model

Ansatz:

\[ \rho_{1,2} = N_{1,2} \wedge e^{R_{1,2} + iS_{1,2}}, \]  

(0.1)

where

\[ N_1 = e^{f(\phi)} d\phi^1 \wedge \ldots \wedge d\phi^{d_c}, \]
\[ N_2 = e^{g(\chi)} d\chi^1 \wedge \ldots \wedge d\chi_{d_t}, \]
\[ R_1 = -d(K_L dX_L), \]
\[ R_2 = -d(K_R dX_R), \]
\[ S_1 = d(K_T J d\chi + K_L J dX_L - K_R J dX_R), \]
\[ S_2 = -d(K_C J d\phi + K_L J dX_L + K_R J dX_R), \]

These are pure spinors with the correct properties.
\[ (\rho_1, \rho_1) = \alpha(\rho_2, \rho_2) \implies \]
\[ (-1)^{d_s d_c} \ e^f(\phi) \ e^{\bar{f}(\bar{\phi})} \ det \left( \begin{array}{ccc} -K_{\bar{l}l} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}l} & -K_{rr} & -K_{r\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{array} \right) \]
\[ = \alpha \ e^g(\chi) \ e^{\bar{g}(\bar{\chi})} \ det \left( \begin{array}{ccc} K_{l\bar{r}} & K_{\bar{l}l} & K_{l\bar{c}} \\ K_{r\bar{l}} & K_{\bar{r}l} & K_{r\bar{c}} \\ K_{c\bar{r}} & K_{\bar{c}l} & K_{c\bar{c}} \end{array} \right) \] (0.2)

\[ e^{2\phi} = (-1)^{d_s d_c} \frac{e^{-f(\phi)} e^{-\bar{f}(\bar{\phi})}}{det \ K_{LR}} \ det \left( \begin{array}{ccc} -K_{\bar{l}l} & -K_{lr} & -K_{l\bar{t}} \\ -K_{\bar{r}l} & -K_{rr} & -K_{r\bar{t}} \\ -K_{t\bar{l}} & -K_{tr} & -K_{t\bar{t}} \end{array} \right) \] (0.3)
Include RR fields in the geometry. See recent work by Waldram and collaborators.

Understand reduction of GKG. Cavalcanti, Gualtieri,....

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THANK YOU FOR YOUR ATTENTION!