Extended supersymmetric sigma-models in 3D AdS

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Building on:

Kuzenko, Lindström & GTM, arXiv:1101.4013

Kuzenko & GTM, arXiv:1109.0496

Based on:

Kuzenko, Lindström & GTM, arXiv:1205.4622

D. Butter, Kuzenko & GTM, arXiv:1209.????

Outline

1. super-AdS in 3D
2. (p,q) AdS superspaces
3. AdS SUSY and target spaces
4. constrained hyperKähler
5. (4,0) AdS superspace
6. Some open problems

Specific features of (super) AdS in three dimensions

- (super)-AdS_3 is the simplest case of supergravity backgrounds.
- useful to understand off-shell SUSY theories on curved backgrounds: Festuccia & Seiberg (2011) (see Klare talk as well); see also localization and AdS/CFT: Pestun (2007), Jafferis (2010).
- In 3D, the anti-de Sitter (AdS) isometry group is reducible,

$$\mathrm{SO}_0(2,2) \cong \Big(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\Big)/\mathbb{Z}_2$$

and so are its supersymmetric extensions,

$$\mathrm{OSp}(p|2;\mathbb{R}) \times \mathrm{OSp}(q|2;\mathbb{R})$$

- $\mathcal{N}$-extended AdS supergravity exists in several incarnations Achúcarro &Townsend (1986)
- same for its maximally symmetric solutions – (p,q) AdS superspaces Kuzenko, Lindström & GTM (2012)
  Different choices of p and q, $p \geq q$, for fixed $\mathcal{N} = p + q$, lead to SUSY field theories with different properties; richer than $D \geq 4$

Superspace techniques versus Chern-Simons construction

- For any values of p and q allowed, the pure (p,q) AdS supergravity was constructed as a Chern-Simons theory with the gauge group

$$\mathrm{OSp}(p|2;\mathbb{R}) \times \mathrm{OSp}(q|2;\mathbb{R})$$

Achúcarro & Townsend (1986)

- Similar ideas used for 3D higher-spin (p,q) AdS supergravity
  Henneaux, Lucena Gomez, Park & Rey (2012)
- Chern-Simons construction becomes less powerful in coupling AdS supergravity to supersymmetric matter. To describe general off-shell supergravity-matter systems in these cases, superspace approaches prove to be useful especially in the cases $\mathcal{N} = 1, 2, 3, 4$.
  Kuzenko, Lindström & GTM (2011)
  Strategy: Supergravity-matter couplings are realised as conformal supergravity coupled to matter supermultiplets.

3D $\mathcal{N}$-extended conformal supergravity in superspace I

Howe, Izquierdo, Papadopoulos &Townsend (1996)
Kuzenko, Lindström & GTM (2011)

$\mathcal{N}$-extended curved superspace parametrized by real bosonic (x^m) and real fermionic $(\theta^\mu_{\rm I})$ coordinates,

$$z^M = (x^m, \theta^\mu_{\rm I}) \ , \qquad m = 0, 1, 2 \ , \quad \mu = 1, 2 \ , \quad {\rm I} = \mathbf{1}, \cdots, \boldsymbol{\mathcal{N}}$$

Structure group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(\mathcal{N})$.
The superspace covariant derivatives (A tangent space index)

$$\mathcal{D}_A \equiv (\mathcal{D}_a, \mathcal{D}^I_\alpha) = E_A{}^M \partial_M + \frac{1}{2}\Omega_A{}^{cd}\mathcal{M}_{cd} + \Phi_A{}^{KL}\mathcal{N}_{KL}$$

- $E_A{}^M(z)$ supervielbein, $\partial_M = \partial/\partial z^M$
- $\Omega_A{}^{cd}(z)$ the Lorentz connection,
- $\Phi_A(z)$ is the $\mathrm{SO}(\mathcal{N})$-connection,

- The covariant derivatives algebra

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C\mathcal{D}_C + \frac{1}{2}R_{AB}{}^{bc}\mathcal{M}_{bc} + R_{AB}{}^{KL}\mathcal{N}_{KL}$$

is constrained by Bianchi Identities $\sum_{[ABC)}[[\mathcal{D}_A, \mathcal{D}_B\}\mathcal{D}_C\} = 0$

3D $\mathcal{N}$-extended conformal supergravity in superspace II

Solve Bianchi identities:

$$\{\mathcal{D}_\alpha^I, \mathcal{D}_\beta^J\} = 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} - 2\mathrm{i}\varepsilon_{\alpha\beta}C^{\gamma\delta IJ}\mathcal{M}_{\gamma\delta} - 4\mathrm{i}S^{IJ}\mathcal{M}_{\alpha\beta} + \Big(\mathrm{i}\varepsilon_{\alpha\beta}X^{IJKL} - 4\mathrm{i}\varepsilon_{\alpha\beta}S^{K[I}\delta^{J]L} + \mathrm{i}C_{\alpha\beta}{}^{KL}\delta^{IJ} - 4\mathrm{i}C_{\alpha\beta}{}^{K(I}\delta^{J)L}\Big)\mathcal{N}_{KL} ,$$

$$[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma^K] = -\Big(\varepsilon_{\gamma(\alpha}C_{\beta)\delta}{}^{KL} + \varepsilon_{\delta(\alpha}C_{\beta)\gamma}{}^{KL} + 2\varepsilon_{\gamma(\alpha}\varepsilon_{\beta)\delta}S^{KL}\Big)\mathcal{D}_L^\delta + \frac{1}{2}R_{\alpha\beta}{}_\gamma^{K\,de}\mathcal{M}_{de} + \frac{1}{2}R_{\alpha\beta}{}_\gamma^{K\,PQ}\mathcal{N}_{PQ} .$$

All the components of the torsion and curvature are expressed in terms of three real mass dimension-one superfields:

$$X^{IJKL} = X^{[IJKL]} , \quad S^{IJ} = S^{(IJ)} , \quad C_a{}^{IJ} = C_a{}^{[IJ]}$$

and their covariant derivatives.
(they are constrained by various differential constraints)

3D $\mathcal{N}$-extended conformal supergravity in superspace III

Geometry invariant under local super-Weyl transformations

$$\begin{aligned}
\mathcal{D}'^I_\alpha &= \mathrm{e}^{\frac{1}{2}\sigma}\Big(\mathcal{D}^I_\alpha + (\mathcal{D}^{\beta I}\sigma)\mathcal{M}_{\alpha\beta} + (\mathcal{D}_{\alpha J}\sigma)\mathcal{N}^{IJ}\Big)\\
\mathcal{D}'_a &= \mathrm{e}^{\sigma}\Big(\mathcal{D}_a + \frac{\mathrm{i}}{2}(\gamma_a)^{\alpha\beta}(\mathcal{D}^K_{(\alpha}\sigma)\mathcal{D}_{\beta)K}\\
&\quad +\varepsilon_{abc}(\mathcal{D}^b\sigma)\mathcal{M}^c - \frac{\mathrm{i}}{8}(\gamma_a)^{\alpha\beta}(\mathcal{D}^\rho_K\sigma)(\mathcal{D}^K_\rho\sigma)\mathcal{M}_{\alpha\beta}\\
&\quad +\frac{\mathrm{i}}{16}(\gamma_a)^{\alpha\beta}([\mathcal{D}^{[K}_{(\alpha},\mathcal{D}^{L]}_{\beta)}]\sigma)\mathcal{N}_{KL} + \frac{3\mathrm{i}}{8}(\gamma_a)^{\alpha\beta}(\mathcal{D}^{[K}_{(\alpha}\sigma)(\mathcal{D}^{L]}_{\beta)}\sigma)\mathcal{N}_{KL}\Big)
\end{aligned}$$

Transformation laws of the dimension-1 torsion and curvature tensors:

$$\begin{aligned}
S'^{IJ} &= \mathrm{e}^{\sigma}\Big(S^{IJ} - \frac{\mathrm{i}}{8}([\mathcal{D}^{\rho(I},\mathcal{D}^{J)}_\rho]\sigma) + \frac{\mathrm{i}}{4}(\mathcal{D}^{\rho(I}\sigma)(\mathcal{D}^{J)}_\rho\sigma) - \frac{\mathrm{i}}{8}\delta^{IJ}(\mathcal{D}^\rho_K\sigma)(\mathcal{D}^K_\rho\sigma)\Big)\\
C'_a{}^{IJ} &= \mathrm{e}^{\sigma}\Big(C_a{}^{IJ} - \frac{\mathrm{i}}{8}(\gamma_a)^{\alpha\beta}([\mathcal{D}^{[I}_{(\alpha},\mathcal{D}^{J]}_{\beta)}]\sigma) - \frac{\mathrm{i}}{4}(\gamma_a)^{\alpha\beta}(\mathcal{D}^{[I}_{(\alpha}\sigma)(\mathcal{D}^{J]}_{\beta)}\sigma)\Big)\\
X'^{IJKL} &= \mathrm{e}^{\sigma}X^{IJKL}
\end{aligned}$$

invariance essential for multiplet of conformal supergravity

components algebraically gauged away using components of σ leaving:

- [0]: vielbein $e_a{}^m$; [1/2]: gravitini $\Psi_a{}^\mu$; [1]: connection $A_a{}^{KL}$;

plus auxiliaries as $X^{IJKL}|_{\theta=0}$ to close SUSY of conformal sugra multiplet

Definition of (p,q) AdS superspaces

$\mathcal{N}$-extended AdS superspaces correspond to those conformal supergravity backgrounds which satisfy the following requirements:

(i) the torsion and curvature tensors are Lorentz invariant;
(ii) the torsion and curvature tensors are covariantly constant.

$$(i) \quad \Longrightarrow \quad C_a{}^{IJ} = 0 \ ;$$
$$(ii) \quad \Longrightarrow \quad \mathcal{D}^I_\alpha S^{JK} = \mathcal{D}_a S^{JK} = 0 \ , \quad \mathcal{D}^I_\alpha X^{JKLM} = \mathcal{D}_a X^{JKLM} = 0 \ .$$

The complete algebra of covariant derivatives takes the form:

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}S^{IJ}\mathcal{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\Big(X^{IJKL} - 4S^{K[I}\delta^{J]L}\Big)\mathcal{N}_{KL} \ ,$$
$$[\mathcal{D}_a, \mathcal{D}^J_\beta] = S^J{}_K(\gamma_a)_\beta{}^\gamma\mathcal{D}^K_\gamma \ ,$$
$$[\mathcal{D}_a, \mathcal{D}_b] = -4\,S^2\,\mathcal{M}_{ab} \ , \qquad S^2 := \frac{1}{\mathcal{N}}S^{IJ}S_{IJ} \geq 0$$

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AdS

Consistency conditions

Together with the Bianchi identities, impose the Integrability conditions

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} S^{KL} = 0 \ , \qquad \{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} X^{KLMN} = 0$$

you get an algebraic constraints on S^{KL}:

$$S^{IK} S_K{}^J = S^2 \delta^{IJ}$$

in the case $S^2 > 0$, S^{IJ} is a nonsingular symmetric $\mathcal{N} \times \mathcal{N}$ matrix, S^{IJ}/S is an orthogonal matrix. Local SO($\mathcal{N}$) transformation to diagonalise

$$S^{IJ} = S\,\mathrm{diag}(\overbrace{+1, \cdots, +1}^{p}, \overbrace{-1, \cdots, -1}^{q=\mathcal{N}-p}) \ , \qquad S > 0$$

Local $\mathrm{SO}(p) \times \mathrm{SO}(q)$ remains unbroken and (p, q) classification arises.

For X^{IJKL}, BI and integrability give two different cases:

$$q > 0 : \qquad \Longrightarrow \qquad X^{IJKL} = 0 \ ,$$
$$(n, 0) : \qquad \Longrightarrow \qquad X_N{}^{IJ[K} X^{LPQ]N} = 0$$

(Deformed) Minkowski superspaces

$$S^2 = 0 \quad \Longleftrightarrow \quad S^{IJ} = 0$$

$$\begin{aligned} \{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} &= 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}X^{IJKL}\mathcal{N}_{KL} \ , \\ [\mathcal{D}_a, \mathcal{D}^J_\beta] &= 0 \ , \qquad [\mathcal{D}_a, \mathcal{D}_b] = 0 \ . \end{aligned}$$

This superspace is of Minkowski type for $\mathcal{N} = 1, 2, 3$.
In the case $\mathcal{N} \geq 4$, there may exist a non-zero constant X^{IJKL}

$$X_N{}^{IJ[K}X^{LPQ]N} = 0 \ ,$$

resulting in a deformation of $\mathcal{N}$–extended Minkowski superspace.
first case $\mathcal{N} = 4$: $X^{IJKL} = \varepsilon^{IJKL}X$.

Conformal flatness of (p, q) AdS and maximally SUSY

- All 3D (p, q) AdS superspaces with $X^{IJKL} = 0$ are conformally flat.
  This is similar to the well-known situation in four dimensions:
  All 4D $\mathcal{N}$-extended AdS superspaces are conformally flat.
  Bandos, Ivanov, Lukierski & Sorokin (2002)
- All $(\mathcal{N}, 0)$ AdS superspaces with $X^{IJKL} \neq 0$ are not conformally flat.
- One can study the maximally symmetric isometry transformations which are generated by (p, q) AdS Killing vector fields

$$\xi = \xi^a \mathcal{D}_a + \xi^\alpha_I \mathcal{D}^I_\alpha$$

$$\left[\xi + \frac{1}{2}\Lambda^{IJ}\mathcal{N}_{IJ} + \frac{1}{2}\Lambda^{ab}\mathcal{M}_{ab}, \mathcal{D}_C\right] = 0$$

AdS supersymmetry and target space geometry

Back to field theory. What is most general SUSY sigma model in AdS_3?
Strategy: use AdS superspaces to approach the problem.

3D AdS SUSY imposes extra restrictions on the target space geometry of sigma models, as compared with the super-Poincare case.

AdS supersymmetry and target space geometry: $\mathcal{N} = 2$

The $\mathcal{N} = 2$ story is pretty simple but still nontrivial.
Study sigma models in term of covariantly chiral superfields $\bar{\mathcal{D}}_\alpha \phi^a = 0$

$$S = \int \mathrm{d}^3x \, \mathrm{d}^4\theta \, E \, K(\phi^a, \bar{\phi}^{\bar{a}}) + \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathcal{E} \, W(\phi^a) + c.c.$$

K is Kähler potential, W superpotential

(1,1) AdS SUSY: Any σ-model target space must be a Kähler manifolds with exact Kähler form. Such manifolds are necessarily non-compact.

(2,0) AdS SUSY: Without superpotential, arbitrary Kähler manifolds as σ-model target spaces, with ϕ^a being neutral under the $\mathrm{U}(1)_R$.
If a superpotential $W(\phi)$ is present, any σ-model target space must possess a U(1) isometry group.

$$\delta\phi^a = \xi^a(\phi) \, , \qquad \xi^a W_a = -2W$$

Izquierdo & Townsend (1995)
Deger, Kaya, Sezgin & Sundell (2000)
Kuzenko & GTM (2011)

AdS SUSY and target space geometry for $\mathcal{N} = 3, 4$?

$\mathcal{N} = 3, 4$ Poincaré SUSY: arbitrary hyperkähler manifolds.
$\mathcal{N} = 3, 4$ AdS SUSY: hyperkähler manifolds of restricted type.

We classified all possible types of hyperkähler target space geometries for $\mathcal{N} = 3, 4$ in AdS by developing two different realizations for the most general (p, q) supersymmetric sigma models:

(i) off-shell formulation in terms of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ projective supermultiplets (see Lindström's talk and arXiv:1101.4013): start with

$$S = \oint_\gamma \frac{\mathrm{d}\zeta}{2\pi \mathrm{i}\zeta} \int \mathrm{d}^3x \, \mathrm{d}^4\theta \, E \, K(\Upsilon^I(\zeta), \breve{\Upsilon}^{\bar{J}}(\zeta))$$

reduce to (2,0) chiral superfields and read target space properties

(ii) on-shell formulation using (2,0) AdS covariantly chiral superfields: impose invariance under extra SUSY ($\bar{\rho}$ encodes extra AdS isometries)

$$\delta\phi^a = \frac{\mathrm{i}}{2} \bar{\mathcal{D}}^2 (\bar{\rho} \, \Omega^a(\phi, \bar{\phi}))$$

and read constraints on $K(\phi, \bar{\phi})$ and $\Omega^a(\phi, \bar{\phi})$

AdS supersymmetry and target space geometry: $\mathcal{N} = 3$

- (3,0) AdS SUSY: For any supersymmetric sigma model, its target space must be a hyperkähler cone.
  Hyperkähler cones are the target spaces of $\mathcal{N} = 3$ superconformal sigma models. All hyperkähler cones are non-compact.
- (2,1) AdS SUSY: Target space must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations of the two-sphere of complex structures.

Kuzenko, Lindström & GTM, arXiv:1205.4622

Target spaces of (2,1) supersymmetric sigma models in AdS_3
is the same as those of $\mathcal{N} = 2$ supersymmetric sigma models in AdS_4

Butter & Kuzenko arXiv:1105.3111

and $\mathcal{N} = 1$ supersymmetric sigma models in AdS_5

Bagger & Xiong, arXiv:1105.4852

Kähler cones

A Kähler manifold $(\mathcal{M}, g_{a\bar{b}})$ parametrized by local complex coordinates ϕ^a is called a Kähler cone if it possesses a homothetic conformal Killing vector or infinitesimal dilatation

$$\chi = \chi^a \frac{\partial}{\partial \phi^a} + \bar{\chi}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \equiv \chi^\mu \frac{\partial}{\partial \varphi^\mu}$$

with the property

$$\nabla_\nu \chi^\mu = \delta_\nu{}^\mu \quad \Longleftrightarrow \quad \nabla_b \chi^a = \delta_b{}^a \ , \qquad \nabla_{\bar{b}} \chi^a = \partial_{\bar{b}} \chi^a = 0 \ .$$

In particular, χ is holomorphic. The properties of χ include the following:

$$\chi_a := g_{a\bar{b}}\, \bar{\chi}^{\bar{b}} = \partial_a K \quad \Longrightarrow \quad \chi^a K_a = K \ ,$$

where $K := g_{a\bar{b}}\, \chi^a \bar{\chi}^{\bar{b}}$ can be used as global Kähler potential, $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$. Complex coordinates for $\mathcal{M}$ can be chosen such that

$$\chi = \phi^a \frac{\partial}{\partial \phi^a} + \bar{\phi}^{\bar{a}} \frac{\partial}{\partial \bar{\phi}^{\bar{a}}} \quad \longrightarrow \quad \phi^a K_a(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) \ .$$

(3,0): Hyperkähler cones

A hyperkähler cone is simply a hyperkähler manifold $(\mathcal{M}, g_{\mu\nu}, J_A{}^\mu{}_\nu)$ admitting an infinitesimal dilatation χ.
$J_A{}^\mu{}_\nu$ are the three integrable quaternionic complex structures

$$J_A J_B = -\delta_{AB}\mathbb{I} + \varepsilon_{ABC} J_C \ .$$

Associated with the conformal Killing vector field χ are three Killing vectors $X_A{}^\mu := J_A{}^\mu{}_\nu \chi^\nu$, which leave the Kähler potential invariant, $X_A{}^\mu \partial_\mu \mathcal{K} = 0$. These obey the $\mathrm{SU}(2)$ algebra

$$[X_A, X_B] = -2\varepsilon_{ABC} X_C \ .$$

(2,1): Hyperkähler with U(1) isometry group rotating the complex structures

Let V^μ be the Killing vector generating the group U(1). Without loss of generality, V^μ is holomorphic w.r.t. J_3

$$\mathcal{L}_V J_1 = -J_2 \ , \quad \mathcal{L}_V J_2 = +J_1 \ , \quad \mathcal{L}_V J_3 = 0 \ .$$

The three closed Kähler two-forms are

$$\Omega_A = \frac{1}{2}(\Omega_A)_{\mu\nu}\, \mathrm{d}\phi^\mu \wedge \mathrm{d}\phi^\nu \ , \qquad (\Omega_A)_{\mu\nu} = g_{\mu\rho}(J_A)^\rho{}_\nu \ .$$

From Ω_1 and Ω_2 construct $(2,0)$ and $(0,2)$ forms with respect to $\mathcal{J}_3$

$$\Omega_\pm = \frac{1}{2}\Omega_1 \pm \frac{\mathrm{i}}{2}\Omega_2 \ , \qquad \mathcal{L}_V \Omega_\pm = \pm \mathrm{i}\, \Omega_\pm$$

Ω_+ is holomorphic with respect to J_3. Ω_+ and Ω_- prove to be exact.
$\rho_+ := -\mathrm{i}\, \imath_V \Omega_+$, holomorphic (1,0) form with respect to $\mathcal{J}_3$. $\mathrm{d}\rho_+ = \Omega_+$.
Because some of the Kähler two-forms are exact, $\mathcal{M}$ is non-compact

AdS supersymmetry and target space geometry: $\mathcal{N} = 4$

Target spaces of 3D $\mathcal{N} = 4$ sigma models in AdS are decomposable

$$\mathcal{M}_L \times \mathcal{M}_R$$

where $\mathcal{M}_L$ and $\mathcal{M}_R$ are certain hyperkähler manifolds.

- (3,1) AdS SUSY: For any supersymmetric sigma model, its left and right target spaces must be hyperkähler cones.
- (2,2) AdS SUSY: Left and right target spaces must be non-compact hyperkähler possessing a Killing vector field which generates an SO(2) group of rotations of the two-sphere of complex structures.

The story is much more interesting in the (4,0) case.

(4,0) AdS superspace

Geometry

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\mathcal{D}_{\alpha\beta} - 4\mathrm{i}\, S\, \delta^{IJ}\mathcal{M}_{\alpha\beta} + \mathrm{i}\varepsilon_{\alpha\beta}\Big(X\varepsilon^{IJKL}\mathcal{N}_{KL} - 4\, S\, \mathcal{N}^{IJ}\Big) ~,$$
$$[\mathcal{D}_a, \mathcal{D}^J_\beta] = S\,(\gamma_a)_\beta{}^\gamma\mathcal{D}^J_\gamma ~, \qquad [\mathcal{D}_a, \mathcal{D}_b] = -4\, S^2\, \mathcal{M}_{ab} ~.$$

X is a free parameter that does not affect the bosonic AdS.
The algebra simplifies if we switch from SO(4) isovector indices to pairs of $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ isospinor indices making use of the isomorphism $\mathrm{SO}(4) \cong \big(\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R\big)/\mathbb{Z}_2$.

$$\begin{aligned}\{\mathcal{D}^{i\bar i}_\alpha, \mathcal{D}^{j\bar j}_\beta\} &= 2\mathrm{i}\varepsilon^{ij}\varepsilon^{\bar i\bar j}\mathcal{D}_{\alpha\beta} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{\bar i\bar j}(2S + X)\mathbf{L}^{ij} + 2\mathrm{i}\varepsilon_{\alpha\beta}\varepsilon^{ij}(2S - X)\mathbf{R}^{\bar i\bar j} \\ &\quad -4\mathrm{i}\, S\, \varepsilon^{ij}\varepsilon^{\bar i\bar j}\mathcal{M}_{\alpha\beta} ~, \\ [\mathcal{D}_a, \mathcal{D}^{j\bar j}_\beta] &= S\,(\gamma_a)_\beta{}^\gamma\mathcal{D}^{j\bar j}_\gamma ~, \qquad [\mathcal{D}_a, \mathcal{D}_b] = -\,4\, S^2\, \mathcal{M}_{ab} ~.\end{aligned}$$

Critical case: $X = \pm 2S$ and either $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$ is flat
Different isometry groups depending on the choice of X.

AdS supersymmetry and target space geometry: $\mathcal{N} = 4$

- (4,0) AdS SUSY with $X = 0$: left and right target spaces must be hyperkähler cones. The sigma model is superconformal.
- (4,0) AdS SUSY with $X \neq \pm 2S$: its left and right target spaces must be hyperkähler cones. The sigma model is not superconformal. X leads to non-trivial scalar potentials in both sectors.
- (4,0) AdS SUSY with $X = \pm 2S$: One of the two target spaces, left or right, must be a hyperkähler cone (nontrivial scalar potential). The other target space is an arbitrary hyperkähler manifold; in particular, it may be compact.
- note that if $S = 0$, the presence of X leads to the appearance of nontrivial potentials in both left and right sectors.
  New mechanism to generate massive sigma models in Minkowski.

Some open problems

- Classification of 3D Lorentzian and Euclidian superspaces admitting various off-shell SUSY
- by using general superspace sugra-matter couplings we then have formalism to define SUSY models in 3D curved manifolds
- QFT in (p, q) AdS superspaces; localization
- Higher-spin theories in (p, q) AdS superspaces;

Conformal flatness of (p,q) AdS superspaces II

Useful local parametrisation of the (p,q) AdS superspace with $X^{IJKL}=0$:

$$\mathcal{D}^I_\alpha = \mathrm{e}^{\frac{1}{2}\sigma}\Big(D^I_\alpha + (D^{\beta I}\sigma)\mathcal{M}_{\alpha\beta} + (D_{\alpha J}\sigma)\mathcal{N}^{IJ}\Big)$$

$$\mathcal{D}_a = \mathrm{e}^{\sigma}\Big(\partial_a + \frac{\mathrm{i}}{2}(\gamma_a)^{\alpha\beta}(D^K_{(\alpha}\sigma)D_{\beta)K} + \varepsilon_{abc}(\partial^b\sigma)\mathcal{M}^c - \frac{\mathrm{i}}{8}(\gamma_a)^{\alpha\beta}(D^\rho_K\sigma)(D^K_\rho\sigma)\mathcal{M}_{\alpha\beta} + \frac{\mathrm{i}}{16}(\gamma_a)^{\alpha\beta}([D^{[K}_{(\alpha}, D^{L]}_{\beta)}]\sigma)\mathcal{N}_{KL} + \frac{3\mathrm{i}}{8}(\gamma_a)^{\alpha\beta}(D^{[K}_{(\alpha}\sigma)(D^{L]}_{\beta)}\sigma)\mathcal{N}_{KL}\Big)$$

where $D_A=(\partial_a, D^I_\alpha)$ are the covariant derivatives of $\mathcal{N}$-extended 3D Minkowski superspace, and

$$\mathrm{e}^{\sigma} = 1 - s^2x^2 - \mathrm{i}\Theta_s - \frac{1}{8}s^2(\Theta)^2\ , \quad \theta^{IJ} := \theta^{\gamma I}\theta^J_\gamma = \theta^{JI}\ ,$$

$$s := \sqrt{s^{KL}s_{KL}/\mathcal{N}} = S\ , \qquad s^{IJ} = \mathrm{const}\ , \qquad \Theta_s := s^{IJ}\theta_{IJ}\ , \quad \Theta := \delta^{IJ}\theta_{IJ}$$

$$S^{IJ} = s^{IJ} + 2\mathrm{i}\, s^2\,\frac{\theta^{IJ} - s^{K(I}s^{J)L}\theta_{KL} + 2s^{K(I}\theta^{J)}_\gamma\theta_{\delta K}x^{\gamma\delta} - \theta^{IJ}\Theta_s + s^{K(I}\theta^{J)}{}_K\Theta}{1 - s^2x^2 - \Theta_s + \frac{1}{4}s^2\Theta^2}$$