# Lifshitz Holographic Renormalization from AdS

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Wissam Chemissany, David Geissbühler, Jelle Hartong, B.R., arXiv:1205.5777

## Motivations

In recent years we observed an intensive search for more realistic realizations of the Gauge/Gravity duality.

- The holographic approach to condensed matter :
  - Vast class of strongly coupled systems
  - Systems can be fine tuned

- Motivation from a theoretical point of view :

  Construct holographic techniques for non-AdS space-times
  e.g. Flat, Kerr, deSitter, ...

  - Lifshitz and Schrödinger spaces might be easier to deal with.

Review papers : [Hartnoll, 2009] [McGreevy, 2009] [Sachdev, 2011]

- Typically : Condensed matter systems near a quantum critical point.
  - Effective description in terms of a scale invariant theory
  - Strong coupling
  - Physics extracted via the concept of universality

- Many such systems actually exhibit anisotropic scaling ($z \neq 1$) at the QCP

$$D_z : \quad t \to \lambda^z t, \quad x^i \to \lambda x^i$$

- These have either Lifshitz or Schrödinger symmetries.

  The Lifshitz algebra :

$$\begin{aligned} [D_z, H] &= zH \,, & [M_{ij}, P_k] &= \delta_{ik} P_j - \delta_{jk} P_i \,, \\ [D_z, P_i] &= P_i \,, & [M_{ij}, M_{kl}] &= \delta_{ik} M_{jl} - \delta_{il} M_{jk} - \delta_{jk} M_{il} + \delta_{jl} M_{ik} \,. \end{aligned}$$

- The Lifshitz symmetry group forms a subgroup of the full conformal group.

# The Lifshitz space-time

A geometric realization of the Lifshitz algebra is achieved by

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right) \, .$$

- Boundary at $r = 0$, bulk at $r = \infty$.
- For $z \geq 1$ timelike/null geodesics reach infinity in finite proper time/affine parameter. Tidal forces (in a parallel propagated frame) go like $(z-1)r^{2z}$, hence the space is singular in the bulk for $z > 1$.
- Focus on the dynamical critical exponent $z = 2$ in the near boundary region.

Lifshitz holography initiated by [Kachru, Liu, Mulligan, 2008].

# Summary

We can obtain a 4D $z = 2$ Lifshitz space-time by reduction of a 5D $z = 0$ Schrödinger space-time [Balasubramanian, Narayan, 2010], [Donos, Gauntlett, 2010] :

$$\begin{aligned} ds^2 &= \frac{1}{r^2}\left(dr^2 + 2du dt + d\vec{x}^2\right) + du^2 \\ &= -\frac{dt^2}{r^4} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right) + \left(du + \frac{dt}{r^2}\right)^2 \end{aligned}$$

- In 5D the Sch$_{z=0}$ space supported by $\phi =$ cte and $\chi \propto u$ is a solution of [Cassani, Faedo, 2011], [Chemissany, Hartong, 2011]

$$S = \int d^5x\sqrt{-g}\left(R + 12 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{2\phi}\partial_\mu\chi\partial^\mu\chi\right)$$

- String theory origin :
  - Freund-Rubin compactification of IIB supergravity over $S^5$.
  - $Sch_{z=0} \times S^5$ corresponds to the near horizon geometry of a stack of D3-branes deformed by an axion wave.

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The $z = 0$ Schrödinger space is AAdS

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The general strategy :

1. Perform holographic renormalization for AdS gravity with Axion-Dilaton
2. Restrict the full set of 5D asymptotically locally AdS solutions to those satisfying the reduction ansatz and the $z = 0$ Schrödinger asymptotics (we keep $\text{AlSch}_{z=0} \subset$ AlAdS)
3. Perform a Scherk-Schwarz reduction
4. In 4D we read off
   - The Fefferman-Graham expansions for asymptotically locally $z = 2$ Lifshitz space-times and the associated matter fields
   - The counterterms & anomalies

# HR for AdS gravity coupled to an Axion-Dilaton

We consider $S_{\text{ren}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{ct}}$ with

$$S_{\text{bulk}} = \int d^5x\sqrt{-\hat{g}}\left(\hat{R} + 12 - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2}e^{2\hat{\phi}}\,(\partial\hat{\chi})^2\right)$$

An AlAdS solution in Fefferman-Graham coordinate is given by $\hat{g}_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} = \frac{dr^2}{r^2} + \hat{h}_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}}$ with an expansion [Papadimitriou, 2011]

$$\begin{aligned}
\hat{h}_{\hat{a}\hat{b}} &= r^{-2}\hat{h}_{(0)\hat{a}\hat{b}} + \hat{h}_{(2)\hat{a}\hat{b}} + r^2\log r\hat{h}_{(4,1)\hat{a}\hat{b}} + r^2\hat{h}_{(4)\hat{a}\hat{b}} + \mathcal{O}(r^4\log r)\,,\\
\hat{\phi} &= \hat{\phi}_{(0)} + r^2\hat{\phi}_{(2)} + r^4\log r\hat{\phi}_{(4,1)} + r^4\hat{\phi}_{(4)} + \mathcal{O}(r^6\log r)\,,\\
\hat{\chi} &= \hat{\chi}_{(0)} + r^2\hat{\chi}_{(2)} + r^4\log r\hat{\chi}_{(4,1)} + r^4\hat{\chi}_{(4)} + \mathcal{O}(r^6\log r)\,.
\end{aligned}$$

The coefficients $\{\hat{h}_{(0)\hat{a}\hat{b}}, \hat{h}_{(4)\hat{a}\hat{b}}, \hat{\phi}_{(0)}, \hat{\phi}_{(4)}, \hat{\chi}_{(0)}, \hat{\chi}_{(4)}\}$ contain the data from which all other coefficients are determined together with the trace $\hat{h}^{\hat{a}}_{(4)\hat{a}}$ and divergence $\nabla^{\hat{a}}_{(0)}\hat{h}_{(4)\hat{a}\hat{b}}$.

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$$\begin{aligned}
S_{\text{ct}} &= \int d^4x\sqrt{-\hat{h}}\left(-3 - \frac{1}{4}\hat{Q} + \hat{\mathcal{A}}\log r\right)\,,\\
8\hat{\mathcal{A}} &= \hat{Q}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}} - \frac{1}{3}\hat{Q}^2 + \frac{1}{2}\left(\Box^{(\hat{h})}\hat{\phi} - e^{2\hat{\phi}}(\partial\hat{\chi})^2\right)^2 + \frac{1}{2}e^{2\hat{\phi}}\left(\Box^{(\hat{h})}\hat{\chi} + 2\partial_{\hat{a}}\hat{\phi}\partial^{\hat{a}}\hat{\chi}\right)^2\,,\\
\hat{Q} &= \hat{h}^{\hat{a}\hat{b}}\hat{Q}_{\hat{a}\hat{b}}\,, \qquad \hat{Q}_{\hat{a}\hat{b}} = \hat{R}_{(\hat{h})\hat{a}\hat{b}} - \frac{1}{2}\partial_{\hat{a}}\hat{\phi}\partial_{\hat{b}}\hat{\phi} - \frac{1}{2}e^{2\hat{\phi}}\partial_{\hat{a}}\hat{\chi}\partial_{\hat{b}}\hat{\chi}\,.
\end{aligned}$$

- The action is $SL(2,\mathbb{R})$ invariant
- $\langle T_{\hat{a}\hat{b}}\rangle = -\dfrac{2}{\sqrt{-\hat{h}_{(0)}}}\dfrac{\delta S_{\text{ren}}^{\text{on-shell}}}{\delta\hat{h}_{(0)}^{\hat{a}\hat{b}}} = \hat{t}_{\hat{a}\hat{b}}$
- $\hat{h}_{(0)}^{\hat{a}\hat{b}}\hat{t}_{\hat{a}\hat{b}} = \hat{\mathcal{A}}_{(0)} = \lim_{r\to 0} r^{-4}\hat{\mathcal{A}}$

# The Scherk-Schwarz reduction

We want to perform a Scheck-Schwarz reduction on a circle $u \sim u + 2\pi L$. We split the coordinates $x^{\hat{\mu}} \to (x^\mu, u)$ and take the ansatz $(k \neq 0)$

$$\begin{aligned} d\hat{s}^2 &= ds^2 + e^{2\Phi}\left(du + A_\mu dx^\mu\right)^2 \\ \hat{\phi} &= \phi \\ \hat{\chi} &= \chi + ku \end{aligned}$$

The 4D theory :

$$S_{\text{bulk}} = \int d^4x\sqrt{-g}\, e^{\Phi}\left(R - \frac{1}{4}e^{2\Phi}F^2 - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{k^2}{2}e^{2\phi}B^2 - V\right),$$

$$B = A - \frac{d\chi}{k}, \quad F = dB\,, \quad V = \frac{k^2}{2}e^{-2\Phi+2\phi} - 12\,.$$

- is not in Einstein frame $(ds_E^2 = e^{\Phi}ds^2)$, but the radial gauge is preserved.
- admits a $z = 2$ Lifshitz solution (reduction of 5D $z = 0$ Schrödinger).

## The boundary conditions

The $z = 2$ Lifshitz solution seen from a 4D perspective :

$$
\begin{aligned}
ds^2 &= -e^{-2\Phi}\frac{dt^2}{r^4} + \frac{1}{r^2}\left(dx^2 + dy^2\right) + \frac{dr^2}{r^2}\,, \\
B &= -e^{-2\Phi}\frac{dt}{r^2}\,,\ \phi = \text{cst}\,,\ \Phi = \phi + \ln\left(\frac{k}{2}\right)\,.
\end{aligned}
$$

The $z = 2$ Lifshitz solution seen from a 5D perspective :

$$
\begin{aligned}
&\hat{\phi}_{(0)} = \text{cst}\ , \qquad \hat{\chi}_{(0)} = ku + \text{cst}\,, \\
&\hat{t}_{\hat{a}\hat{b}} = 0\,, \qquad \hat{\phi}_{(4)} = 0\,, \qquad \hat{\chi}_{(4)} = 0\,, \\
&\hat{h}_{(0)\hat{a}\hat{b}} = \text{conformally flat and admits a hypersurface orthogonal null Killing vector } \partial_u\ ,
\end{aligned}
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Contrast with $z = 2$ AlLif seen from a 5D perspective :

$$
\begin{aligned}
\hat{\phi}_{(0)} &= \mathsf{cst}\ , \qquad \hat{\chi}_{(0)} = ku + \mathsf{cst} + \chi_{(0)}(x^a)\,, \\
\hat{t}_{\hat{a}\hat{b}} &= 0\,, \qquad \hat{\phi}_{(4)} = 0\,, \qquad \hat{\chi}_{(4)} = 0\,, \\
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Contrast with $z = 2$ AlLif seen from a 5D perspective :

$$\begin{aligned} \hat{\phi}_{(0)} &= \text{cst}\,, \qquad \hat{\chi}_{(0)} = ku + \text{cst} + \chi_{(0)}(x^a)\,, \\ \hat{t}_{\hat{a}\hat{b}} &= 0\,, \qquad \hat{\phi}_{(4)} = 0\,, \qquad \hat{\chi}_{(4)} = 0\,, \\ \hat{h}_{(0)\hat{a}\hat{b}} &= \text{conformally flat and admits a hypersurface orthogonal null Killing vector } \partial_u\,, \end{aligned}$$

- In 5D : Same conditions have been observed on the AdS boundary metric to produce a $\text{Sch}_{z=2}$ space in FG coordinate [Hartong, Rollier, 2012]
- In 4D : Exact agreement with $z = 2$ AlLif as defined in [Ross, 2011]

- $\partial_u$ must be a Killing vector of the 5D metric

- For the pure $z=2$ Lifshitz solution $\phi = \text{cte}$ and $\Phi - \phi = \ln\left(\frac{k}{2}\right)$.

  We have in general

$$e^{2\Phi} = \hat{h}_{uu} = \frac{1}{r^2}\hat{h}_{(0)uu} + \hat{h}_{(2)uu} + \dots\,, \quad \hat{h}_{(2)uu} = -\frac{\hat{R}_{(0)uu}}{2} + \frac{k^2 e^{2\hat{\phi}_{(0)}}}{4}.$$

  In order to maintain the 4D asymptotics

$$\Phi \sim \mathcal{O}(r^0) \quad \text{and} \quad \Phi - \phi|_{r=0} = \Phi_{(0)} - \phi_{(0)} = \ln\left(\frac{k}{2}\right)\,.$$

  we must require $\hat{h}_{(0)uu} = \hat{R}_{(0)uu} = 0$.

  - $\partial_u$ is a boundary HSO null Killing vector (Raychauduhuri equation).

- We additionally need $\Phi_{(0)} = \text{cte}$ in order to not violate the $z=2$ Lifshitz asymptotics by going to a radial gauge in an Einstein frame.

  - $\Phi_{(0)} = \text{cte} \Rightarrow \hat{\phi}_{(0)} = \text{cte}$

## Scherk-Schwarz reduction and results

First we make $\partial_u$ a manifest HSO null Killing vector of the AlAdS boundary metric

$$\hat{h}_{(0)\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = 2H_{(0)}dudt + \Pi_{(0)ij}\left(dx^i + H_{(0)}N^i_{(0)}dt\right)\left(dx^j + H_{(0)}N^j_{(0)}dt\right)$$

Then we reduce according to

$$\begin{aligned} d\hat{s}^2 &= \frac{dr^2}{r^2} + \hat{h}_{\hat{a}\hat{b}}dx^{\hat{a}}dx^{\hat{b}} = \frac{dr^2}{r^2} + h_{ab}dx^a dx^b + e^{2\Phi}\left(du + A_a dx^a\right)^2 \,, \\ \hat{\phi} &= \phi\,, \qquad \hat{\chi} \;=\; \chi + ku\,. \end{aligned}$$

We read off the 4D Fefferman-Graham expansion for $z=2$ AlLif

$$\begin{aligned} h_{tt} &= -\frac{H^2_{(0)}e^{-2\Phi_{(0)}}}{r^4} + \frac{\log r h_{(2,1)tt}}{r^2} + \frac{h_{(2)tt}}{r^2} + \mathcal{O}((\log r)^2)\,, \\ h_{ti} &= \frac{h_{(0)ti}}{r^2} + \log r h_{(2,1)ti} + h_{(2)ti} + \mathcal{O}(r^2(\log r)^2)\,, \\ h_{ij} &= \frac{\Pi_{(0)ij}}{r^2} + h_{(2)ij} + r^2\log r h_{(4,1)ij} + r^2 h_{(4)ij} + \mathcal{O}(r^4(\log r)^2)\,, \end{aligned}$$

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$$\hat{h}_{(0)\hat{a}\hat{b}} dx^{\hat{a}} dx^{\hat{b}} = 2H_{(0)} du dt + \Pi_{(0)ij} \left( dx^i + H_{(0)} N^i_{(0)} dt \right) \left( dx^j + H_{(0)} N^j_{(0)} dt \right)$$

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We read off the 4D Fefferman-Graham expansion for $z = 2$ AlLif

$$\begin{aligned} B_r &= r B_{(0)r} + \mathcal{O}(r^3 \log r) \,, \\ B_t &= \frac{1}{r^2} H_{(0)} e^{-2\Phi_{(0)}} + \mathcal{O}(\log r) \\ B_i &= r^2 \log r B_{(2,1)i} + r^2 B_{(2)i} + \mathcal{O}(r^4 (\log r)^2) \,, \\ \Phi &= \Phi_{(0)} + r^2 \log r \Phi_{(2,1)} + r^2 \Phi_{(2)} + \mathcal{O}(r^4 (\log r)^2) \,, \end{aligned}$$

- The 5D data can be mapped to 4D data from which we can express all other 4D coefficients.

$$\{\hat{h}_{(0)\hat{a}\hat{b}}, \hat{t}_{\hat{a}\hat{b}}, \hat{\phi}_{(0)}, \hat{\phi}_{(4)}, \hat{\chi}_{(0)}, \hat{\chi}_{(4)}\}$$
$$\downarrow$$
$$\{H_{(0)}, h_{(0)it}, \Pi_{(0)ij}, \Phi_{(0)}, h_{(2)tt}, \Phi_{(2)}, B_{(2)i}, B_{(4)t}, h_{(6)tt}, h_{(4)ti}, h_{(4)ij}, \phi_{(4)}, B_{(2)r}\}$$

- The constraints on the trace and divergence of $\hat{t}_{\hat{a}\hat{b}}$ translate in 4D constraints among $\{\Phi_{(2)}, B_{(2)i}, B_{(4)t}, h_{(6)tt}, h_{(4)ti}, h_{(4)ij}\}$.

By reduction we automatically obtain the relevant counterterms

$$\begin{aligned} S_{\text{ct}} &= \int d^3x\sqrt{-h}e^{\Phi}\left[-3 - \frac{1}{4}\left(R_{(h)} - \frac{1}{4}e^{2\Phi}F^2 - \frac{1}{2}(\partial\phi)^2 - \frac{k^2}{2}e^{2\phi}B^2 \right.\right. \\ &\quad \left.\left. -\frac{k^2}{2}e^{2\phi-2\Phi}\right) + \log r\left(\mathcal{A}^{(0)} + \mathcal{A}^{(2)} + \mathcal{A}^{(4)}\right)\right], \end{aligned}$$

where $\mathcal{A}^{(n)}$ is $n$th order in derivatives.

- In 5D the conformal anomaly is induced by diffeomorphisms (PBH) acting on the boundary metric as a conformal rescaling $\hat{h}_{(0)\hat{a}\hat{b}} \to \Omega^2 \hat{h}_{(0)\hat{a}\hat{b}}$.

  From a 4D point of view, these are the anisotropic conformal rescalings of [Horava, Melby-Thompson, 2009]

$$h_{(0)tt} \to \Omega^4 h_{(0)tt}\,, \quad h_{(0)ti} \to \Omega^2 h_{(0)ti}\,, \quad \Pi_{(0)ij} \to \Omega^2 \Pi_{(0)ij}\,.$$

- For AlLif$_{z=2}$ the associated anomaly is (with $\chi_{(0)} = \text{cte}$)

$$\int d^3x\sqrt{-h}e^{\Phi}\left(\mathcal{A}^{(0)} + \mathcal{A}^{(2)} + \mathcal{A}^{(4)}\right)\Big|_{\text{on-shell}} =$$

$$\int dtd^2xH_{(0)}\sqrt{\Pi_{(0)}}\left[C_1\left(4K_{(0)ij}K^{ij}_{(0)} - 2K^2_{(0)}\right) + C_2\left(\mathcal{R}_{(0)} + D^i_{(0)}\partial_i \ln(H_{(0)})\right)^2\right]$$

  - Invariance under anisotropic rescalings

  The central charges defined in [Baggio, de Boer, Holsheimer, 2011] are

$$C_1 = \frac{l^2_{\text{Lif}}}{64\pi G_4} = 6C_2$$

$$K_{(0)ij} = \frac{e^{\Phi_{(0)}}}{2H_{(0)}}(\partial_t\Pi_{(0)ij} - D_{(0)i}h_{(0)tj} - D_{(0)j}h_{(0)ti})$$

$$l^2_{\text{Lif}} = l^2_{\text{AdS}}e^{\Phi_{(0)}}$$

# Remarks and extensions

The massive vector models with / without the scalars :

- Using asymptotically constant scalars we can compare our results with the ones obtained by [Ross, 2011], [Mann, McNees, 2011], [Griffin, Horava, Melby-Thompson, 2011], [Baggio, de Boer, Holsheimer, 2011].
  - Agreement for the local counterterms and $C_1$ but $C_2 \neq 0$ is new.
- On-shell the AlLif$_{z=2}$ anomaly forms an action of the Horava-Lifshitz type for $z = 2$ conformal gravity in $2 + 1$ dimensions with nonzero potential.

Some future directions :

- Compute the Lifshitz one-point functions
- Compare the Lifshitz boundary stress tensor in this setting to [Ross, Saremi, 2009]
- Better understand the dual point of view : some DLCQ of $\mathcal{N} = 4$ SYM leading to a Lifshitz Chern-Simons gauge theory. [Balasubramanian, McGreevy, 2011]