Quantum field theory in four dimensions

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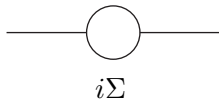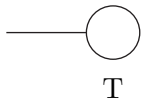
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1. UV infinities (without renormalization)
2. The FDR integral (with examples)
3. Renormalization
4. Infrared and Collinear divergences
5. Conclusions

Making a theory finite *without renormalization*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{M^2}{2}\Phi^2 - M\frac{\lambda}{3!}\Phi^3$$

Has 2 infinities

$$\text{T} \qquad\qquad\qquad i\Sigma$$

Shift the field: $\Phi \rightarrow \Phi + v$

$$\xrightarrow{\quad p \quad} = \frac{i}{p^2 - M^2} \qquad\qquad \prec = -i\lambda M$$

$$\longrightarrow\!\otimes = -i\left(M^2 v + \frac{\lambda M}{2}v^2\right) \quad \longrightarrow\!\otimes\!\longrightarrow = -i\lambda M v$$
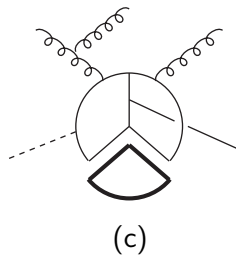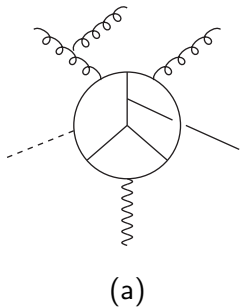
Impose to tadpoles

$$v = \lambda M \frac{I_{inf}}{2}$$

$$I_{inf} = i\mu_R^{-\epsilon} \int d^n q \frac{1}{(q^2 - \mu^2)^2}$$

$$i\,\bar{\Sigma}\big|_{inf} = \text{———}\bigcirc\text{———} \quad + \quad \text{———}\otimes\text{———} \quad \big|_{inf} = 0$$

and the theory is finite!

Vacuum inside high frequencies

(a)

(b)

(c)

Representing *vacuum configurations*

A rank-r one-loop example

$$\mu_R^{-\epsilon} \int d^n q \frac{q_{\alpha_1} \cdots q_{\alpha_r}}{(q^2 - \mu^2)^j}$$

A scalar two-loop case

$$\mu_R^{-2\epsilon} \int d^n q_1 d^n q_2 \frac{1}{(q_1^2 - \mu^2)^{j_1}(q_2^2 - \mu^2)^{j_2}((q_1 + q_2)^2 - \mu^2)^{j_3}}$$

with $\mu \to 0$

They contain no physical information!

$\Rightarrow$ Ingredient n. 1 of the FDR integral

Extracting *vacuum configurations*

By partial fraction decomposition:

$$D = (q^2 - M^2) \xrightarrow{To\ avoid\ IR} \bar{D} = D - \mu^2 \qquad (\bar{q}^2 = q^2 - \mu^2)$$

$$\frac{1}{\bar{D}} = \frac{1}{\bar{q}^2}\left(1 + \frac{M^2}{\bar{D}}\right)$$

$$\frac{1}{\bar{D}} = \frac{1}{\bar{q}^2} + M^4\frac{1}{\bar{q}^4} + \frac{M^4}{\bar{D}\bar{q}^4}$$

$$\frac{1}{\bar{D}^3} = \frac{1}{\bar{q}^6} + M^2\left(\frac{1}{\bar{D}^3\bar{q}^2} + \frac{1}{\bar{D}^2\bar{q}^4} + \frac{1}{\bar{D}\bar{q}^6}\right)$$

$\Rightarrow$ Ingredient n. 2 of the FDR integral

Subtracting *vacuum configurations*

Last ingredient of the FDR integral:

$$I_{\ell-loop}^{\mathrm{DR}} = \mu_R^{-\ell\epsilon} \int \prod_{i=1}^{\ell} d^n q_i \, J(\{q_\alpha, q^2, \not q\})$$

$$J(\{q_\alpha, \bar{q}^2, \not{\bar q}\}) = J_V(\{q_\alpha, \bar{q}^2, \not{\bar q}\}) + J_F(\{q_\alpha, \bar{q}^2, \not{\bar q}\})$$

$$I_{\ell-loop}^{\mathrm{FDR}} = \int \prod_{i=1}^{\ell} [d^4 q_i] \, J(\{q_\alpha, \bar{q}^2, \not{\bar q}\}) \equiv \lim_{\mu \to 0} \int \prod_{i=1}^{\ell} d^4 q_i J_F(\{q_\alpha, \bar{q}^2, \not{\bar q}\}) \Bigg|_{\mu=\mu_R}$$

$$\text{The symbol } \int [d^4 q] \text{ means:}$$

1. use partial fraction to move all divergences in vacuum integrands, treating $\bar{q}^2$ and $\bar{\slashed{q}}$ globally;

2. drop all divergent vacuum terms from the integrand;

3. integrate over $d^4 q$;

4. take the limit $\mu \to 0$, until a logarithmic dependence on μ is reached;

5. compute the result in $\mu = \mu_R$.

Properties of the FDR Integral

1. *The FDR integral is a physical quantity in which all high frequencies giving rise to unphysical vacuum configurations either do not contribute or are fully subtracted*

2. *It is invariant under any shift of the integration variables*

3. *It is gauge invariant*

4. *It does not depend any more on the initial cut-off μ*

In fact, only log divergent integrals give a contribution to J_V, when $\mu \to 0$

$$J_V \sim K + \sum_{i=1}^{\ell} a_i \ln^i(\mu/\mu_R)$$

One-loop examples

- $$\int [d^4q] \frac{q^\alpha q^\beta}{\bar{D}^3} = M^2 \lim_{\mu \to 0} \int d^4q \, q^\alpha q^\beta \left(\frac{1}{\bar{D}^3 \bar{q}^2} + \frac{1}{\bar{D}^2 \bar{q}^4} + \frac{1}{\bar{D} \bar{q}^6} \right) \Bigg|_{\mu = \mu_R}$$

$$= \frac{g^{\alpha\beta}}{4} I_0^{\text{FDR}} \qquad \text{where}$$

- $$I_0^{\text{FDR}} = \int [d^4q] \frac{1}{\bar{D}^2} = \lim_{\mu \to 0} M^2 \int d^4q \left(\frac{1}{\bar{D}^2 \bar{q}^2} + \frac{1}{\bar{D} \bar{q}^4} \right) \Bigg|_{\mu = \mu_R}$$

$$= -i\pi^2 \ln \frac{M^2}{\mu_R^2} \qquad \text{and gauge invariance holds}$$

- $$\int [d^4q] \frac{\bar{q}^2 - M^2}{\bar{D}^3} = \int [d^4q] \frac{1}{\bar{D}^2} \qquad \left(\int [d^4q] \frac{\mu^2}{\bar{D}^3} = \frac{i\pi^2}{2} \right)$$

Two-loop example

- $I^{\text{FDR}} \quad = \quad \displaystyle\int [d^4 q_1][d^4 q_2] \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_{12}}$ with

$$\bar{D}_1 \quad = \quad \bar{q}_1^2 - m_1^2, \qquad \bar{D}_2 = \bar{q}_2^2 - m_2^2, \qquad \bar{D}_{12} = \bar{q}_{12}^2 - m_{12}^2$$

$$
\begin{aligned}
I^{\text{FDR}} \quad \equiv \quad & \lim_{\mu \to 0} \int d^4 q_1 \int d^4 q_2 \left(\frac{m_1^2 m_2^2}{(\bar{D}_1 \bar{q}_1^2)(\bar{D}_2 \bar{q}_2^2)\bar{q}_{12}^2} + \frac{m_1^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2)\bar{q}_2^2 (\bar{D}_{12}\bar{q}_{12}^2)} + \frac{m_2^2 m_{12}^2}{\bar{q}_1^2 (\bar{D}_2 \bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)} \right. \\
& - m_1^4 \frac{q_1^2 + 2(q_1 \cdot q_2)}{(\bar{D}_1 \bar{q}_1^4)\bar{q}_2^4 \bar{q}_{12}^2} - m_2^4 \frac{q_2^2 + 2(q_1 \cdot q_2)}{\bar{q}_1^4 (\bar{D}_2 \bar{q}_2^4)\bar{q}_{12}^2} - m_{12}^4 \frac{q_{12}^2 - 2(q_1 \cdot q_2)}{\bar{q}_1^4 \bar{q}_2^2 (\bar{D}_{12}\bar{q}_{12}^4)} \\
& \left. + \frac{m_1^2 m_2^2 m_{12}^2}{(\bar{D}_1 \bar{q}_1^2)(\bar{D}_2 \bar{q}_2^2)(\bar{D}_{12}\bar{q}_{12}^2)} \right) \bigg|_{\mu = \mu_R}
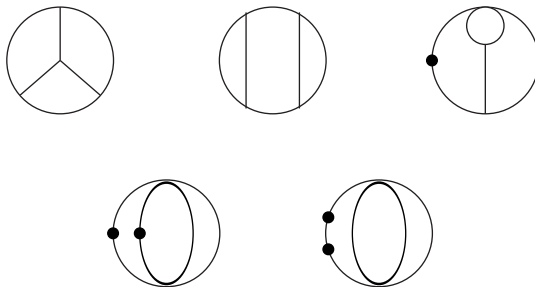\end{aligned}
$$

One- and two-loop correspondences

$\dfrac{1}{\epsilon}$ subtr. after integration $\leftrightarrow$ $\dfrac{1}{\bar{q}^4}$ subtr. before integration

$\dfrac{1}{\epsilon^2}$ subtr. after integration $\leftrightarrow$ $\dfrac{1}{\bar{q}_1^4 \bar{q}_2^2 \bar{q}_{12}^2}$ subtr. before integration
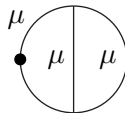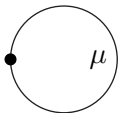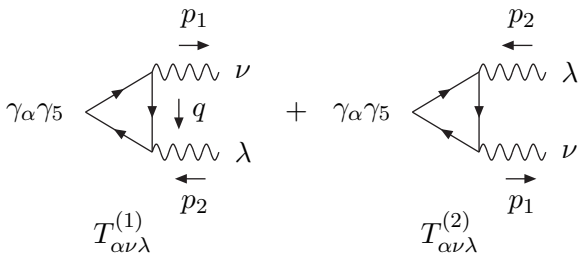
Beyond two-loop

Five irreducible three-loop topologies give rise to logarithmically divergent subtraction vacuum diagrams

The FDR gauge invariance guarantees the correctness of the procedure

The ABJ anomaly

$$p^\alpha T_{\alpha\nu\lambda} = -i\frac{e^2}{4\pi^4}\text{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1] \int [d^4q]\, \mu^2 \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2}$$

$$p^\alpha T_{\alpha\nu\lambda} = \frac{e^2}{8\pi^2}\text{Tr}[\gamma_5 \not{p}_2 \gamma_\lambda \gamma_\nu \not{p}_1]$$

Renormalization I

- By gauge invariance $G_{\ell-loop}^{\text{FDR}}(\mu_R) = G_{\ell-loop}^{\text{DR}}(\mu_R)$

   with no need to add counterterms to the Lagrangian

- Furthermore

$$G_{\ell-loop}^{\text{FDR}}(\mu_R) = \sum_{i=0}^{\ell} a_i^{\text{DR}} \log^i(\mu_R) + \text{R}^{\text{DR}}(\{p, M\}) + \text{R}_0$$

   BOTH in renormalizable and non-renormalizable theories

- μ_R disappears in renormalizable theories, when fixing parameters in terms of observables

   possibility to fix μ_R with an extra measurement ALSO in non-renormalizable theories (Quantum Gravity) ⇒ Predictivity?

Renormalization II
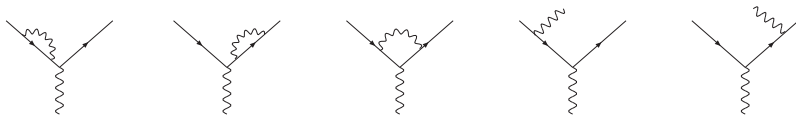
The important points to keep in mind are that:

1. the Lagrangian is left untouched;

2. gauge invariance is not broken;

3. four-dimensionality is kept;

4. one additional measurement is enough to fix μ_R at any perturbative order.

Conceptual jump:

> *Could one just* DEFINE *differently the loop integrals?*

By doing so, the mechanism nature uses to wipe the infinities out, either by reabsortion into the vacuum, or via renormalization, becomes less relevant.

QED corrections to massless $\Gamma(Z \to f\bar{f})$ in FDR

$$\Gamma_V(Z \to f\bar{f}) = \Gamma_0(Z \to f\bar{f})\frac{\alpha}{\pi}\left[-\frac{1}{2}\ln^2\left(\frac{\mu^2}{s}\right) - \frac{3}{2}\ln\left(\frac{\mu^2}{s}\right) + \frac{7}{18}\pi^2 + \frac{\pi}{2\sqrt{3}} - \frac{7}{2}\right]$$

$$\Gamma_R(Z \to f\bar{f}) = \Gamma_0(Z \to f\bar{f})\frac{\alpha}{\pi}\left[\frac{1}{2}\ln^2\left(\frac{\mu^2}{s}\right) + \frac{3}{2}\ln\left(\frac{\mu^2}{s}\right) - \frac{7}{18}\pi^2 - \frac{\pi}{2\sqrt{3}} + \frac{17}{4}\right]$$

$$\Gamma(Z \to f\bar{f}) = \Gamma_0(Z \to f\bar{f})\left(1 + \frac{3}{4}\frac{\alpha}{\pi}\right)$$

$$\sum_{pol} u(p)\bar{u}(p) = \not{p} + \mu\,, \quad \sum_{pol} \epsilon_\alpha(p)\epsilon_\beta^*(p) = -g_{\alpha\beta} + \frac{p_\alpha p_\beta}{\mu^2}$$

Conclusions

1. FDR discriminates between observable physics and unobservable infinities occurring at large values of the integration momenta

2. This allows one to define, in a mathematical consistent way and at any order in the perturbative expansion, the UV divergent integrals appearing in quantum field theories as four-dimensional integrals over the physical spectrum only

3. The physics of renormalizable theories is reproduced and *the possibility for the non-renormalizable theories to become predictive could be opened*

4. Infrared and collinear divergences can also be naturally accommodated

5. FDR looks promising for practical calculations too:
   - *No counterterms need to be added to the Lagrangian*
   - *Both virtual and real contributions are kept in four dimensions*