HIGHER SPIN GRAVITY AND EXACT HOLOGRAPHY

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Introduction:

- $\text{AdS}_{d+1}/\text{CFT}_d$: extra dimension
- Higher Spin / Vector Model correspondence:
  Exactly solvable, Renormalizable, ...
- Higher Spin Gravity: Vasiliev '80 - '92
  Gravity ($s = 2$) + HS gauge fields ($s = 3, 4, \ldots$)
  (supersymmetric version)
- These fields sit on the leading Regge trajectory (which contains the graviton) of the string spectrum (on AdS with $\lambda = 0$)
- Tensionless limit of String Theory: $\frac{R_{AdS}}{\ell_s} \sim \lambda^{\frac{1}{4}} \to 0$
  [Sundborg '94 '01 ; Witten '01]

Klebanov-Polyakov conjecture: '02

AdS_{d+1}

Higher Spin Theory with even spins $s = 0, 2, 4, \dots$

CFT_d

Vector Model with $O(N)$ symmetry

- Earlier work by Sezgin & Sundell '02
- All integer spins $\Rightarrow U(N)$ vector model
- This correspondence works for $d \geq 3$
- Two fixed points ($d = 3$): two quantization/boundary conditions

$$\begin{cases} \Delta = 1 & \text{free O(N) model}: \quad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \\ \Delta = 2 & \text{critical O(N) model}: \mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{N}(\phi \cdot \phi)^2 \end{cases}$$

- Substantial evidences of the conjecture are provided by the agreement of three-point functions [Giombi & Yin: '09 '10]

AdS_3/CFT_2: [Gaberdiel & Gopakumar '10]

AdS_3

Massless HS fields
+ massive scalar with
$M^2 = -1 + \lambda^2$

CFT_2

$W_{N,k}$ minimal models:

$$\frac{su(N)_k \oplus su(N)_1}{su(N+1)_{k+1}}$$

- 't Hooft limit: $N, k \to \infty$; $0 \le \lambda \equiv \frac{N}{N+k} \le 1$ fixed
  $\lambda = 0$: free fermion ; $\lambda = 1$: free boson
- Central charge: $c \sim N \Rightarrow$ Vector-like Model
- Matching the spectrum: Partition functions
  [Gaberdiel, Gipakumar, Hartman, Raju '11]
- Matching the symmetry: $\mathcal{W}$ symmetry (Triality)

This talk: part I

- Presents a direct constriction of AdS_4 HS gravity from $3d$ field theory [Das & Jevicki: '03; Koch, KJ, Jevicki, Rodrigues: '10]
- The construction is based on the notion of bi-local fields:

$$\Phi_c(x,y) = \sum_{i=1}^{N} \phi^i(x) \cdot \phi^i(y) \qquad O(N) \text{ singlet}$$

- Represents a direct change of variables

$$Z = \int [d\phi^a(x)] e^{-S[\phi]} = \int \prod_{x,y} d\Phi_c(x,y) e^{-S_c[\Phi_c]}$$

- $S_c[\Phi_c]$ is an effective action and is exact: reproduces all $O(N)$-invariant correlators

$$\langle \phi(x_1)\cdot\phi(y_1)\ \phi(x_2)\cdot\phi(y_2)\cdots\phi(x_n)\cdot\phi(y_n)\rangle$$

- This formulation is seen to give a bulk description of HS theory (with extra dimension and interactions)

This talk: part II

- HS theory in (Euclidean) 3d has black hole solutions:
  BTZ black hole $\Longrightarrow$ Higher Spin Black Hole
- BH entropy was calculated from thermodynamics: [Kraus & Perlmutter '11]

$$\ln Z_{BH}(\hat{\tau}, \alpha, \hat{\bar{\tau}}, \bar{\alpha}) = \frac{i\pi c}{12\hat{\tau}} \left[1 - \frac{4}{3}\frac{\alpha^2}{\hat{\tau}^4} + \frac{400}{27}\frac{\lambda^2 - 7}{\lambda^2 - 4}\frac{\alpha^4}{\hat{\tau}^8} - \frac{1600}{27}\frac{5\lambda^4 - 85\lambda^2 + 377}{(\lambda^2 - 4)^2}\frac{\alpha^6}{\hat{\tau}^{12}} + \cdots \right] + \text{rightmoving}$$

  where $\hat{\tau}$ is the modular parameter of the torus, α is the chemical potential of the spin-3 current, and λ indicates the bulk symmetry algebra: hs[λ].
- Validity of the calculation: large c and high temperature

$$\hat{\tau} \sim \frac{1}{T_H} \to 0, \quad \alpha \to 0 \qquad \text{and} \qquad \frac{\alpha}{\hat{\tau}^2} \quad \text{fixed}$$

- Reproduce the BH entropy (free energy) purely from CFT: $\mathcal{W}$-symmetry

CFT_3: the vector model

- N-component scalar field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{g}{4}(\phi \cdot \phi)^2, \qquad a = 1, ..., N$$

- Two fixed points: $g = 0$ (UV); $g \neq 0$ (IR)
- Conformal currents: [Makeenko: '81]

$$J_{\mu_1 \ldots \mu_s} = \sum_{k=0}^{s} (-1)^k (\#)(\#) \partial_{\mu_1} \cdots \partial_{\mu_k} \phi^a \; \partial_{\mu_{k+1}} \cdots \partial_{\mu_s} \phi^a - traces$$

- The currents represent boundary duals of AdS_4 HS fields

$$J_{\mu_1 \mu_2 \cdots \mu_s}(x) \Longleftrightarrow \mathcal{H}_{\hat{\mu}_1 \hat{\mu}_2 \cdots \hat{\mu}_s}(x, z \to 0)$$

where z is the AdS direction.

Bi-local representation:

- The construction is based on the bi-local field:

$$\Phi(x,y) \equiv \phi(x)\cdot\phi(y) = \sum_{a=1}^{N} \phi^a(x)\phi^a(y)$$

- The collective action evaluates the complete $O(N)$ invariant partition function [Jevicki & Sakita '80]

$$Z = \int [d\phi^a(x)] e^{-S[\phi]} = \int \prod_{x,y} d\Phi(x,y) e^{-S_c[\Phi]}$$

$$S_c[\Phi] = \mathrm{Tr}[-(\partial_x^2 + \partial_y^2)\Phi(x,y)] + \boxed{\frac{N}{2}\mathrm{Tr}\ln\Phi}$$

where the trace is defined as $\mathrm{Tr}B = \int d^3x\, B(x,x)$.

- Origin of the $\ln\Phi$ interaction: Jacobian

$$\int d\vec{\phi} e^{-S[\phi]} \rightarrow \int d\Phi \det\left|\frac{\partial\phi^a(x)}{\partial\Phi(x_1,x_2)}\right| e^{-S[\Phi]}$$

Large N expansion:

- Reproduces all the nontrivial $O(N)$-invariant correlators

$$\langle \phi(x_1)\cdot\phi(y_1)\ \phi(x_2)\cdot\phi(y_2)\cdots\phi(x_n)\cdot\phi(y_n)\rangle$$

- The collective action is nonlinear
- AdS_4 HS gravity coupling constant

$$g = \frac{1}{\sqrt{N}}$$

- Expanding around the background $\Phi = \Phi_0 + \frac{1}{\sqrt{N}}\eta$ gives rise to an infinite number of interaction vertices

$$S_c = S[\Phi_0] + \mathrm{Tr}[\Phi_0^{-1}\eta\Phi_0^{-1}\eta] + \frac{g}{4}\eta^2 + \sum_{n\geq 3} N^{1-n/2}\mathrm{Tr}B^n, \qquad B \equiv \Phi_0^{-1}\eta$$

- Represents covariant-type gauge of the vector model

Physical gauge: time-like gauge

- The bi-local field $\Phi_c(x, y)$ has a one-time description: $x^+ = y^+ = t$

$$\Psi(t; \vec{x}, \vec{y}) = \sum_a \phi^a(t, \vec{x}) \cdot \phi^a(t, \vec{y})$$

[Jevicki, KJ, Ye: '11]
with the conjugate momenta

$$\Pi(\vec{x}, \vec{y}) = -i \frac{\delta}{\delta\Psi(\vec{x}, \vec{y})}$$

- The Hamiltonian is given by [Jevicki & Sakita '80]

$$H = 2\mathrm{Tr}(\Pi\Psi\Pi) + \frac{1}{2}\int [-\nabla_x^2 \Psi(\tilde{x}, \tilde{y})|_{\tilde{x}=\tilde{y}}] + \boxed{\frac{N}{8}\mathrm{Tr}\Psi^{-1}}$$

where we have set the coupling constant $g = 0$.

$1/N$ expansion:

- The above Hamiltonian has a natural $1/N$ expansion:

$$\Psi = \Psi_0 + \frac{1}{\sqrt{N}}\eta, \qquad \Pi = \sqrt{N}\pi$$

- The first few orders of Hamiltonian

$$\begin{aligned} H^{(2)} &= 2\mathrm{Tr}(\pi\Psi_0\pi) + \frac{1}{8}\mathrm{Tr}(\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}) \\ H^{(3)} &= \frac{2}{\sqrt{N}}\mathrm{Tr}(\pi\eta\pi) - \frac{1}{8\sqrt{N}}\mathrm{Tr}(\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}) \\ H^{(4)} &= \frac{1}{8N}\mathrm{Tr}(\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}) \end{aligned}$$

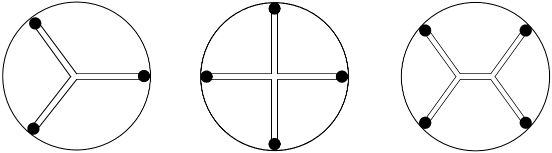
- Scattering amplitude: the collective S-matrix

$$S = \lim \prod_i (E_i^2 - (|\vec{k}_i| + |\vec{k}_{i'}|)^2)\langle\tilde{\Psi}(E_1, \vec{k}_1, \vec{k}_{1'})\tilde{\Psi}(E_2, \vec{k}_2, \vec{k}_{2'})\cdots\rangle$$

['t Hooft "A two-dimensional model for mesons" 1974]

S=1: [de Mello Koch, Jevicki, KJ, Rodrigues & Ye '12]

- The three-point ($S_3 = 0$) and four-point ($S_4 = 0$) amplitudes vanish



$$S_3 = \langle 0|\alpha_{\vec{p}_3\vec{p}_{3'}}\, T\exp\left[-i\int_{-\infty}^{\infty} dt\, H^{(3)}(t)\right]\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}\alpha^\dagger_{\vec{p}_1\vec{p}_{1'}}|0\rangle = 0$$

- The vanishing of S_4 is due to genuine cancellations
- This signals the working of Coleman-Mandula theorem
- Maldacena and Zhiboedov ['11] have shown that the existence of higher-spin currents implies that the CFT correlators (nonzero!) are given by free fields (bosons or fermions)
- It is very important to understand these results from bulk point of view

Fronsdal's equation: Free Higher Spin Theory in AdS

- Massless spin-s gauge fields can be described by totally symmetric tensors $h_{\mu_1 \cdots \mu_s}$ subject to the double traceless condition $h^{\rho}{}_{\rho}{}^{\eta}{}_{\eta\mu_5\ldots\mu_s} = 0$ which becomes nontrivial for $s \geq 4$.
- The gauge invariant equation of motion: [Fronsdal '78]

$$\nabla_\rho \nabla^\rho h_{\mu_1\ldots\mu_s} - s \nabla_\rho \nabla_{\mu_1} h^{\rho}{}_{\mu_2\ldots\mu_s} + \frac{1}{2} s(s-1) \nabla_{\mu_1} \nabla_{\mu_2} h^{\rho}{}_{\rho\mu_3\ldots\mu_s} + 2(s-1)(s+d-3) h_{\mu_1\ldots\mu_s} = 0$$

- Gauge symmetry

$$\delta_\Lambda h^{\mu_1\cdots\mu_s} = \nabla^{\mu_1} \Lambda^{\mu_2\cdots\mu_s}$$

  where the gauge parameter is single-traceless:

$$g_{\mu_2\mu_3} \Lambda^{\mu_2\cdots\mu_s} = 0$$

- Light-cone gauge fixing: Metsaev '99

Explicitly:

- $SO(2,3)$ isometry generators (10) in the conformal form:

$$
\begin{aligned}
\hat{p}^- &= -\frac{p^x p^x + p^z p^z}{2p^+}, \\
\hat{m}^{+-} &= t\hat{p}^- - x^- p^+, \\
\hat{m}^{-x} &= x^- p^x - x\hat{p}^- + \frac{p^\theta p^z}{p^+}, \\
\hat{d} &= t\hat{p}^- + x^- p^+ + xp^x + zp^z + d_a, \\
\hat{k}^- &= -\frac{1}{2}(x^2 + z^2)\hat{p}^- + x^-(x^- p^+ + xp^x + zp^z + d_a) \\
&\quad + \frac{1}{p^+}\left((xp^z - zp^x)p^\theta + (p^\theta)^2\right), \\
\hat{k}^+ &= t^2\hat{p}^- + t(xp^x + zp^z + d_a) - \frac{1}{2}(x^2 + z^2)p^+, \\
&\quad \ldots
\end{aligned}
$$

- They operate in the (AdS+HS) space: θ is the HS coordinate

$$\Phi(x^+ = t;\, x^-, x, z;\, \theta)$$

For the bi-local fields:

- The conformal generators (10): 3d conformal group

$$
\begin{aligned}
\hat{p}^- &= p_1^- + p_2^- = -\Big(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\Big), \\
\hat{m}^{+-} &= t\hat{p}^- - x_1^- p_1^+ - x_2^- p_2^+, \\
\hat{m}^{+i} &= t\hat{p}^i - x_1^i p_1^+ - x_2^i p_2^+, \\
\hat{d} &= t\hat{p}^- + x_1^- p_1^+ + x_2^- p_2^+ + x_1^i p_1^i + x_2^i p_2^i + 2d_\phi, \\
\hat{k}^- &= x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} + x_1^-(x_1^- p_1^+ + x_1^i p_1^i + d_\phi) \\
&\quad + x_2^-(x_2^- p_2^+ + x_2^i p_2^i + d_\phi), \\
\hat{k}^+ &= t^2 \hat{p}^- + t(x_1^i p_1^i + x_2^i p_2^i + 2d_\phi) - \frac{1}{2} x_1^i x_1^i p_1^+ - \frac{1}{2} x_2^i x_2^i p_2^+, \\
&\cdots
\end{aligned}
$$

- They operate in the 5d dipole space:

$$
\Psi(x_1^+ = x_2^+ = t;\, x_1^-, x_1;\, x_2^-, x_2)
$$

Operator AdS/CFT

AdS_4/CFT_3 correspondence:

$$CFT_3 : \textit{collective bilocal fields} \quad \Longleftrightarrow \quad AdS_4 : \textit{higher spin fields}$$

$$\Psi(x^+; x_1^-, x_1; x_2^-, x_2) \quad \Longleftrightarrow \quad \Phi(x^+; x^-, x, z; \theta)$$

- Same number of dimensions

$$1+2+2 \qquad = \qquad 1+3+1$$

- Representation of the conformal group $SO(2,3)$
- Clear from analysis of the two representations that one does not have a coordinate transformation

Solution: canonical transformation

- Identifying the generators of the dipole with the generators of HS:

$$
\begin{aligned}
x^- &= \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}, \qquad z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} \\
p^+ &= p_1^+ + p_2^+ \\
x &= \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+}, \qquad p^z = \sqrt{\frac{p_2^+}{p_1^+}}p_1 - \sqrt{\frac{p_1^+}{p_2^+}}p_2 \\
p^x &= p_1 + p_2 \\
\theta &= 2\arctan\sqrt{p_2^+/p_1^+} \\
p^\theta &= \sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{x_1 - x_2}{2}\left(\sqrt{\frac{p_2^+}{p_1^+}}p_1 + \sqrt{\frac{p_1^+}{p_2^+}}p_2\right)
\end{aligned}
$$

- 10 equations of $2 \times 4 = 8$ canonical variables
- All the Poisson brackets are satisfied:

$$\{x^-, p^+\} = \{x, p\} = \{\theta, p^\theta\} = \{z, p^z\} = 1$$

- Possible to lift to quantum version

Generalization to higher dimensions

[Jevicki, KJ & Ye '11]

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}$$

$$p^+ = p_1^+ + p_2^+$$

$$x^i = \frac{x_1^i p_1^+ + x_2^i p_2^+}{p_1^+ + p_2^+}$$

$$p^i = p_1^i + p_2^i$$

$$z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} \sqrt{(x_1^i - x_2^i)^2}$$

$$p^z = \frac{x_1^j - x_2^j}{\sqrt{(x_1^i - x_2^i)^2}} \left(p_1^j \sqrt{\frac{p_2^+}{p_1^+}} - p_2^j \sqrt{\frac{p_1^+}{p_2^+}} \right)$$

$$\cdots$$

From bi-local field to HS field:

- Changing to AdS variables using an inverse transform gives the AdS HS field in terms of the bi-local one

$$\Phi(x^-, x, z, \theta) = \int dp^+ dp^x dp^z e^{i(x^- p^+ + xp^x + zp^z)}$$
$$\int dp_1^+ dp_2^+ dp_1 dp_2 \delta(p_1^+ + p_2^+ - p^+)\delta(p_1 + p_2 - p^x)$$
$$\delta\left(p_1\sqrt{p_2^+/p_1^+} - p_2\sqrt{p_1^+/p_2^+} - p^z\right)$$
$$\delta\left(2\arctan\sqrt{p_2^+/p_1^+} - \theta\right)\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)$$

where $\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)$ is a Fourier transform of the bilocal field

$$\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2) = \int e^{-i(x_1^- p^+ + x_2^- p_2^+ + x_1 p_1 + x_2 p_2)}\Psi(x_1^-, x_2^-, x_1, x_2)$$

Checking the $z = 0$ projection

- One can check our identification of the extra AdS coordinate z by evaluating the $z = 0$ limit
- At $z = 0$, the integral transformation simplifies to:

$$\begin{aligned} \Phi(x^-, x, z=0, \theta) &= \int dp_1^+ dp_2^+ e^{ix^-(p_1^+ + p_2^+)} \\ & \quad \delta(\theta - 2\tan^{-1}\sqrt{p_2^+/p_1^+})\tilde{\Psi}(p_1^+, p_2^+, x, x) \end{aligned}$$

- Fourier expand the delta function and perform the integrals (which give the derivatives), for a particular spin, we found the conformal currents:

$$\mathcal{J}^s = \sum_{k=0}^{s} \frac{(-1)^k \Gamma(s+1/2)\Gamma(s+1/2)}{k!(s-k)!\Gamma(s-k+1/2)\Gamma(k+1/2)} (\partial_+)^k \phi \, (\partial_+)^{s-k} \phi$$

Full nonlinear theory: Vasiliev theory in AdS_4

- Master fields:

$$\begin{aligned} W &= W_\mu(x^\nu|y^\alpha, \bar{y}^{\dot{\alpha}}, z^\beta, \bar{z}^{\dot{\beta}})dx^\mu \\ S &= S_\alpha(x|y, \bar{y}, z, \bar{z})dz^\alpha + S_{\dot{\alpha}}(x|y, \bar{y}, z, \bar{z})d\bar{z}^{\dot{\alpha}} \\ B &= B(x|y, \bar{y}, z, \bar{z}) \end{aligned}$$

- Nonlinear equations of motion:

$$\begin{aligned} &d_x W + W * W = 0 \\ &d_Z W + d_x S + [W, S]_* = 0 \\ &d_Z S + S * S = B * K\, dz^2 + B * \bar{K} d\bar{z}^2 \\ &d_x B + W * B - B * \pi(W) = 0 \\ &d_Z B + S * B - B * \pi(S) = 0 \end{aligned}$$

where $K = e^{z^\alpha y_\alpha}$ is the Kleinian and π is the ‘parity’ operator

$$\pi(f(y, \bar{y}, z, \bar{z})) = f(-y, \bar{y}, -z, \bar{z}) = K * f(y, \bar{y}, z, \bar{z}) * K$$

- No action principle yet

Gauge symmetry

- Star product: associative but non-commutative

$$f(y,z) * g(y,z) = \int d^2u d^2v e^{u^\alpha v_\alpha} f(y+u, z+u) g(y+v, z-v)$$

- Gauge transformations:

$$\begin{aligned} \delta W &= d_x \epsilon + [W, \epsilon]_*, \\ \delta S &= d_Z \epsilon + [S, \epsilon]_*, \\ \delta B &= B * \pi(\epsilon) - \epsilon * B, \end{aligned}$$

- Compact form:

$$\begin{aligned} dA + A * A &= B * (K dz^2 + \bar{K} d\bar{z}^2) \\ dB + A * B - B * \pi(A) &= 0 \end{aligned}$$

where the gauge field is

$$A = W_\mu dx^\mu + S_\alpha dz^\alpha + S_{\dot\alpha} d\bar{z}^{\dot\alpha}$$

Linearization of Vasiliev's theory

- AdS background: only the gravitational fields ($s = 2$) pick up a nonzero background

$$\begin{aligned} &W = W_0, \qquad S = 0, \qquad B = 0, \\ &W_0 = w_0^L + e_0, \\ &w_0^L = \frac{dx^i}{8z}\left[(\sigma^{iz})_{\alpha\beta} y^\alpha y^\beta + (\sigma^{iz})_{\dot\alpha\dot\beta} \bar y^{\dot\alpha} \bar y^{\dot\beta}\right], \\ &e_0 = \frac{dx_\mu}{4z} \sigma^\mu_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta}. \end{aligned}$$

- The $z, \bar z$ spinors are totally auxiliary $\Rightarrow$ Compact EOMs
- The physical degrees of freedom are fully contained in

$$W(x; y, \bar y) = \sum_{n,m} dx^\nu W^{(n,m)}_{\nu,\alpha_1\dots\alpha_n\dot\beta_1\dots\dot\beta_m} y^{\alpha_1} \cdots y^{\alpha_n} \bar y^{\dot\beta_1} \cdots \bar y^{\dot\beta_m}$$

  where the spin is related to $n + m = 2(s-1)$
- The linearized equations reduce to Fronsdal's equation after identifying

$$h_{\mu_1\dots\mu_s} = W^{(s-1,s-1)}_{\mu_1,\alpha_1\dots\alpha_{s-1}\dot\beta_1\dots\dot\beta_{s-1}} \sigma^{\alpha_1\dot\beta_1}_{\mu_2} \cdots \sigma^{\alpha_{s-1}\dot\beta_{s-1}}_{\mu_s} + \textit{symmetrization}$$

A symmetric gauge

- Bi-local field theory is symmetric: $3d + 3d$

$$\Psi(x_\mu^1, x_\mu^2)$$

- There is a symmetric gauge in Vasiliev's theory: $4d + 4d$

$$F(y, \bar{y}; z, \bar{z})$$

- Solving the zero curvature equation $dW + W * W = 0$ using the pure gauge solution

$$W_\mu = g^{-1} * \partial_\mu g$$

- After a gauge transformation (+ solving two more equations), all the spacetime dependence of the master fields is gone, one ends up with the following equations

$$d_Z S + S * S = B * (K dz^2 + \bar{K} d\bar{z}^2)$$
$$d_Z B + S * B - B * \pi(S) = 0$$

$W = 0$ gauge

- In components, one has five independent equations

$$F_{z^1\bar{z}^2} = \partial_1 \bar{S}_2 - \bar{\partial}_2 S_1 + [S_1, \bar{S}_2]_* = 0 \quad (1)$$
$$F_{z^2\bar{z}^1} = \partial_2 \bar{S}_1 - \bar{\partial}_1 S_2 + [S_2, \bar{S}_1]_* = 0 \quad (2)$$
$$F_{z^1\bar{z}^1} = \partial_1 \bar{S}_1 - \bar{\partial}_1 S_1 + [S_1, \bar{S}_1]_* = 0 \quad (3)$$
$$F_{z^2\bar{z}^2} = \partial_2 \bar{S}_2 - \bar{\partial}_2 S_2 + [S_2, \bar{S}_2]_* = 0 \quad (4)$$
$$F_{z^1 z^2} * K = F_{\bar{z}^1\bar{z}^2} * \bar{K} \quad (5)$$

- The last equation is the reality condition of the B field.
- Analog with self-dual Yang-Mills

Higher Spin	SDYM
$F_{z^1\bar{z}^2} = 0$	$F_{yz} = 0$
$F_{z^2\bar{z}^1} = 0$	$F_{\bar{y}\bar{z}} = 0$
$F_{z^1\bar{z}^1} + F_{z^2\bar{z}^2} = 0$	$F_{y\bar{y}} + F_{z\bar{z}} = 0$
$F_{z^1\bar{z}^1} - F_{z^2\bar{z}^2} = 0$	
$F_{z^1 z^2} * K = F_{\bar{z}^1\bar{z}^2} * \bar{K}$	

where $z^1 = y, \bar{z}^1 = \bar{y}, z^2 = \bar{z}, \bar{z}^2 = z$

An ansatz

- Using the ansatz

$$S_1 = M^{-1} * \partial_1 M, \qquad S_2 = \bar{M}^{-1} * \partial_2 \bar{M}$$
$$\bar{S}_1 = \bar{M}^{-1} * \bar{\partial}_1 \bar{M}, \qquad \bar{S}_2 = M^{-1} * \bar{\partial}_2 M$$

- $F_{1\dot{2}}$ and $F_{2\dot{1}}$ are solved automatically, $F_{1\dot{1}}$ and $F_{2\dot{2}}$ become

$$\begin{aligned} \bar{\partial}_1(J^{-1} * \partial_1 J) &= 0 \qquad (Ia) \\ \partial_2(J^{-1} * \bar{\partial}_2 J) &= 0 \qquad (Ib) \end{aligned}$$

where $J = M * \bar{M}^{-1}$ is a (residual) gauge invariant quantity.

- The last equation (5) becomes

$$\partial_2(J^{-1} * \partial_1 J) * K + \bar{\partial}_1(J^{-1} * \bar{\partial}_2 J) * \bar{K} = 0 \qquad (II)$$

Comments:

- We now have equations for a single scalar field:

$$J(y_\alpha, \bar{y}_{\dot{\alpha}}, z_\alpha, \bar{z}_{\dot{\alpha}})$$

- Equation (Ia, Ib) can be thought of as constraints giving the reduction:

$$4 + 4 \rightarrow 3 + 3$$

- Equation (II) represents an equation of motion
- One can expect that an action can be written down for this system
- Closest in form to the covariant version of collective field equation of motion
- There exists a non-linear transformation between the J field and the bilocal collective field Ψ (work in progress)

Chern-Simons formulation of HS in 3d: [Blencowe '89]

- It is consistent to truncation the infinite number of HS fields to finite N.
- Einstein-Hilbert action can be re-expressed in terms of CS theory with gauge group: $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$
  [Achúcarro & Townsend '86 ; Witten '88]
- HS theory (with maximal spin N) can be written in terms of CS theory with gauge group $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$:

$$S_{HS} = S_{cs}[A] = \frac{k_{cs}}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \qquad k_{cs} = \frac{\ell}{4G}$$

  where the gauge fields are

$$A = (j_\mu^a T_a + \cdots + j_\mu^{a_1 \cdots a_{N-1}} T_{a_1 \cdots a_{N-1}}) dx^\mu$$

- In the infinite spin case ($N \to \infty$), the gauge group is $hs[\lambda] \times hs[\lambda]$, where λ is a deformation parameter of the HS theory (background field $B_0 = \lambda$)

Asymptotic symmetry: W-symmetry

- Brown-Henneaux procedure of Pure gravity (in AdS):

$$sl(2) \Longrightarrow Virasoro\ symmetry$$

- Extension to finite N: [Campoleoni, Fredenhagen, Pfenninger & Theisen '10]

$$sl(N) \Longrightarrow W_N\ symmetry$$

- Infinite case: [Henneaux & Rey '10 ; Gaberdiel & Hartman '11]

$$hs[\lambda] \Longrightarrow \mathcal{W}_\infty[\lambda]\ symmetry$$

- They all have the same central charge:

$$c = 6k_{cs} = \frac{3\ell}{2G_N}$$

BTZ black holes

- The metric:

$$ds^2 = d\rho^2 + \frac{2\pi}{k}\left(\mathcal{L}(dx^+)^2 + \bar{\mathcal{L}}(dx^-)^2\right) - \left(e^{2\rho} + \frac{4\pi^2}{k^2}\mathcal{L}\bar{\mathcal{L}}e^{-2\rho}\right)dx^+dx^-$$

where $x^\pm = t \pm \phi$, $\phi \cong \phi + 2\pi$ and

$$\mathcal{L} = \frac{M\ell - J}{4\pi}, \qquad \bar{\mathcal{L}} = \frac{M\ell + J}{4\pi}$$

with M the mass and J the angular momentum.

- In terms of the connections:

$$\begin{aligned} A &= (e^\rho L_1 - \frac{2\pi}{k}e^{-\rho}\mathcal{L}L_{-1})dx^+ + L_0 d\rho \\ \bar{A} &= -(e^\rho L_{-1} - \frac{2\pi}{k}\bar{\mathcal{L}}e^{-\rho}L_1)dx^- - L_0 d\rho \end{aligned}$$

where $L_{0,\pm 1}$ are the SL(2) generators.

Higher Spin Black Holes

- In ordinary gravity, the black hole horizon (and singularities) are diffeomorphism invariant
- Higher spin gauge symmetry $>$ diffeomorphism
- It is not obvious how to define a black hole in higher spin gravity because neither the Riemann tensor (Ricci scalar) nor the causal structure of the metric are gauge invariant
- In Euclidean signature, the problem is simpler because a black hole is simply a smooth classical solution with torus boundary conditions
- This definition has been used to construct explicit black hole solutions carrying higher spin charge
  [Gutperle & Kraus '11 ;
  Ammon, Gutperle, Kraus & Perlmutter '11 ;
  Castro, Hijano, Lepage-Jutier & Maloney '11]

Explicit solutions :

- $SL(3)$: [Gutperle & Kraus '11]

$$
\begin{aligned}
A =& L_0 d\rho + \left(e^{\rho} L_1 - \frac{2\pi}{k}\mathcal{L} e^{-\rho} L_{-1} + \frac{\pi}{2k\sigma}\mathcal{W} e^{-2\rho} W_{-2}\right) dx^+ \\
&+ \frac{\alpha}{\bar{\tau}}\left(e^{2\rho} W_2 - \frac{4\pi}{k}\mathcal{L} W_0 + \frac{4\pi^2}{k^2}\mathcal{L}^2 e^{-2\rho} W_{-2} + \frac{4\pi}{k}\mathcal{W} e^{-\rho} L_{-1}\right) dx^-
\end{aligned}
$$

  where α is the chemical potential of the spin-3 current.

- $hs[\lambda]$: [Kraus & Perlmutter '11]

$$
\begin{aligned}
A =& b^{-1} a b + b^{-1} db, \qquad b = e^{\rho V_0^2} \\
a_+ =& V_1^2 - \frac{2\pi \mathcal{L}}{k} - N(\lambda)\frac{\pi \mathcal{W}}{2k} V_{-2}^3 + J \\
a_- =& \frac{\alpha}{\bar{\tau}} N(\lambda)\left(a_+ * a_+ - \frac{2\pi \mathcal{L}}{3k}(\lambda^2 - 1)\right)
\end{aligned}
$$

  where $N(\lambda)$ is a normalization factor, V_m^s are the $hs[\lambda]$ generators, and J contains infinite higher-spin fields: $J = J_4 V_{-3}^4 + J_5 V_{-4}^5 + \cdots$

The partition function

- Smoothness of the Euclidean horizon (absence of a conical singularity) $\Longrightarrow$ Holonomy conditions:

$$\mathrm{Tr}(w^n) = \mathrm{Tr}(w^n_{BTZ}), \qquad n = 2, 3, \cdots$$

where the holonomy matrix is $w = 2\pi(\tau A_+ - \bar{\tau} A_-)$

$\Longrightarrow$ Integrability condition $\Longrightarrow$ first law of thermodynamics: $S \neq A/4G$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathcal{W}}{\partial \tau} \Longrightarrow \tau = \frac{i}{4\pi^2}\frac{\partial S}{\partial \mathcal{L}}, \quad \alpha = \frac{i}{4\pi^2}\frac{\partial S}{\partial \mathcal{W}}$$

- Calculation of the free energy / partition function:

$$\ln Z = S + 4\pi^2 i(\tau\mathcal{L} + \alpha\mathcal{W} - \bar{\tau}\bar{\mathcal{L}} - \bar{\alpha}\bar{\mathcal{W}}) \Rightarrow \mathcal{L} = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \tau}, \mathcal{W} = -\frac{i}{4\pi^2}\frac{\partial \ln Z}{\partial \alpha}$$

- Free energy $\sim$ HS CS action (not gauge invariant)?
  [Banados, Canto & Theisen '12]

The gravity result:

- Free energy: [Kraus & Perlmutter '11]

$$\ln Z_{BH}(\hat{\tau}, \alpha, \hat{\bar{\tau}}, \bar{\alpha}) = \frac{i\pi c}{12\hat{\tau}} \left[1 - \frac{4}{3}\frac{\alpha^2}{\hat{\tau}^4} + \frac{400}{27}\frac{\lambda^2 - 7}{\lambda^2 - 4}\frac{\alpha^4}{\hat{\tau}^8} - \frac{1600}{27}\frac{5\lambda^4 - 85\lambda^2 + 377}{(\lambda^2 - 4)^2}\frac{\alpha^6}{\hat{\tau}^{12}} + \cdots \right] + \text{rightmoving}$$

where $\hat{\tau}$ is the modular parameter of the torus, α is the chemical potential of the spin-3 current, and λ indicates the bulk symmetry algebra: hs[λ].

- Validity of the calculation: large c and high temperature

$$\hat{\tau} \sim \frac{1}{T_H} \to 0, \quad \alpha \to 0 \qquad \text{and} \qquad \frac{\alpha}{\hat{\tau}^2} \quad \text{fixed}$$

- From the CFT point of view:

$$Z_{CFT}(\hat{\tau}, \alpha) = \mathrm{Tr}_i \left(\hat{q}^{L_0 - \frac{c}{24}} y^{W_0} \right)$$

where $\hat{q} = e^{2\pi i \hat{\tau}}$, $y = e^{2\pi i \alpha}$ and the trace is sum over all the representations.

The general strategy:

- Perform a modular (S)-transformation ($\hat{\tau} = -1/\tau$, $q = e^{2\pi i \tau} \to 0$), the answer will be dominated by the vacuum state
- First expand the partition function in terms of the chemical potential:

$$Z_{CFT}(\hat{\tau}, \alpha) = \mathrm{Tr}_i \left(\hat{q}^{L_0 - \frac{c}{24}} \right) + \frac{(2\pi i)^2 \alpha^2}{2!} \mathrm{Tr}_i \left(W_0^2 \hat{q}^{L_0 - \frac{c}{24}} \right) + \frac{(2\pi i)^4 \alpha^4}{4!} \mathrm{Tr}_i \left(W_0^4 \hat{q}^{L_0 - \frac{c}{24}} \right) + \cdots$$

- Then apply S-transformation to each individual term
- The α-independent term gives BTZ result: the Cardy's formula

$$\mathrm{Tr}_i \left(\hat{q}^{L_0 - \frac{c}{24}} \right) = \sum_{ij} S_{ij} \mathrm{Tr}_j \left(q^{L_0 - \frac{c}{24}} \right) \sim \left(\sum_i S_{i0} \right) q^{-\frac{c}{24}}$$

$$\Longrightarrow \ln Z = -\frac{i\pi c}{12} \tau = \frac{i\pi c}{12\hat{\tau}} \qquad (\hat{\tau} \to 0)$$

- The odd powers of W_0 terms are subleading because: (1) only the vacuum representation is needed; (2) only the leading c (large) terms are compared.

The comparison

- Under S-transformation:

$$\mathrm{Tr}(W_0^2\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_i S_{i0}\Big[\mathrm{Tr}(W_0^2\,q^{L_0-\frac{c}{24}}) + \cdots + \#_2(\lambda,\tau)\mathrm{Tr}_0(q^{L_0-\frac{c}{24}})\Big]$$

$$\mathrm{Tr}(W_0^4\,\hat{q}^{L_0-\frac{c}{24}}) = \sum_i S_{i0}\Big[\mathrm{Tr}(W_0^4\,q^{L_0-\frac{c}{24}}) + \cdots + \#_4(\lambda,\tau)\mathrm{Tr}_0(q^{L_0-\frac{c}{24}})\Big]$$

- Collect the contributing terms

$$Z = \sum_i S_{i0}\Big[1 + \#_2(\lambda,\tau) + \#_4(\lambda,\tau) + \cdots\Big]q^{-\frac{c}{24}}$$

$$\sim q^{-\frac{c}{24}}\Big[1 + \#_2(\lambda,\tau) + \#_4(\lambda,\tau) + \cdots\Big]$$

- Exponentiating the gravity result

$$Z_{BH} = q^{-\frac{c}{24}}\Big[1 + \frac{i\pi c}{9}\alpha^2\tau^5 - \frac{100 i\pi c}{81}\frac{\lambda^2-7}{\lambda^2-4}\alpha^4\tau^9 + \cdots\Big]$$

Torus amplitude

- The torus amplitude is defined by

$$F_i((a^1, z_1), \ldots, (a^n, z_n); q) = z_1^{h_1} \cdots z_n^{h_n} \mathrm{Tr}_i \left(V(a^1, z_1) \cdots V(a^n, z_n) q^{L_0 - \frac{c}{24}} \right)$$

  where h_j are the conformal dimensions of the chiral field a^j: $L_0 a^j = h_j a^j$.
- In our case, the chiral fields are the higher spin fields (in the $\mathcal{W}_\infty[\lambda]$ algebra)
- These functions are periodic under the transformations

$$z_j \mapsto e^{2\pi i} z_j, \qquad z_j \mapsto q\, z_j$$

  and hence the name 'tours amplitude'
- We are interested in the modular transformation properties of the traces with insertion of zero modes
- Expanding the vertex operators as $V(a, z) = \sum a_m z^{-m-h}$, the zero modes can be extracted via the contour integrals

$$\mathrm{Tr}(a_0^1 \cdots a_0^n q^{L_0 - \frac{c}{24}}) = \frac{1}{(2\pi i)^n} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_n}{z_n} F((a^1, z_1), \ldots, (a^n, z_n); q)$$

Modular transformation of the torus amplitude

- Under a modular transformation, the functions F_i transform as

$$F_i\Big((a^1, z_1), \ldots, (a^n, z_n); \frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^{\sum_l h_l} \times \sum_j M_{ij} F_j\left((a^1, z_1^{c\tau+d}), \ldots, (a^n, z_n^{c\tau+d}); \tau\right)$$

where $M_{ij} \equiv M_{ij}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a representation of the modular group, i.e. a constant matrix for each modular transformation. [Zhu '96]

- In particular, for the S-transformation $\tau \mapsto -1/\tau$, we have

$$F_i\Big((a^1, z_1), \ldots, (a^n, z_n); -\frac{1}{\tau}\Big) = \tau^{\sum_l h_l} \sum_j S_{ij}\, F_j((a^1, z_1^{\tau}), \ldots, (a^n, z_n^{\tau}); \tau)$$

The final formula ?

- Under the modular transformation, the trace with insertion of zero modes transforms as

$$\begin{aligned}\mathrm{Tr}_r(a_0^1 \cdots a_0^n \, \hat{q}^{L_0 - \frac{c}{24}}) =& \frac{1}{(2\pi i)^n} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_n}{z_n} \\ & \tau^{\sum_l h_l} \sum_s S_{rs} \, F_s((a^1, z_1^\tau), \ldots, (a^n, z_n^\tau); \tau) \\ =& \frac{1}{(2\pi i)^n} \tau^{-n+\sum_j h_j} \sum_s S_{rs} \\ & \int_1^q \frac{d\tilde{z}_1}{\tilde{z}_1} \cdots \int_1^q \frac{d\tilde{z}_n}{\tilde{z}_n} F_s((a^1, \tilde{z}_1), \ldots, (a^n, \tilde{z}_n); \tau)\end{aligned}$$

  where we did a change of variables $z^\tau \to \tilde{z}$.
- Still one needs to compute the integral
- The computation simplifies at $q \to 0$ and large c: the dominant contribution will come from the vacuum ($s = 0$).

Torus recursion relations: [Zhu '96]

$$F\left((a^1, z_1), (a^2, z_2), \ldots, (a^n, z_n); q\right) = F\left(a_0^1, (a^2, z_2), \ldots, (a^n, z_n); q\right)$$
$$+ \sum_{j=2}^{n} \sum_{m=0}^{\infty} \mathcal{P}_{m+1}\left(\frac{z_j}{z_1}, q\right) \times F\left((a^2, z_2), \ldots, (a^1[m]a^j, z_j), \ldots, (a^n, z_n); q\right)$$

where $\mathcal{P}$ is the Weierstrass function and the bracketed modes are defined via

$$a[m] = (2\pi i)^{-m-1} \sum_{i \geq m} c(h_a, i, m) a_{-h_a+1+i}$$

The coefficients $c(h_a, i, m)$ are found by the expansion:

$$(\ln(1+z))^n (1+z)^{h-1} = \sum_{j \geq n} c(h, j, n) z^j.$$

For insertion of W fields: $h = 3$

$$W[1] = (2\pi i)^{-2}\left(W_{-1} + \frac{3}{2}W_0 + \frac{1}{3}W_1 - \frac{1}{12}W_2 + \frac{1}{30}W_3 + \cdots\right)$$

Two-point function

$$Z^{(2)} \equiv \frac{(2\pi i\alpha)^2}{2!}\,\mathrm{Tr}\big(W_0 W_0 \hat{q}^{L_0-\frac{c}{24}}\big)$$
$$\approx \frac{\alpha^2\tau^4}{2}\int_1^q \frac{dz_1}{z_1}\int_1^q \frac{dz_2}{z_2} F\big((W,z_1),(W,z_2);\tau\big)$$

Applying the recursion relation, we find

$$F\big((W,z_1),(W,z_2);\tau\big) = z_2^3\,\mathrm{Tr}\big(W_0 W(z_2)\, q^{L_0-\frac{c}{24}}\big) + \sum_m \mathcal{P}_{m+1}\left(\frac{z_2}{z_1}\right) F\big((W[m]W,z_2);\tau\big)$$

Only the $m=1$ term will contribute $W(z)=V(W_{-3}\Omega,z),\ V(\Omega,z)=1$

$$Z^{(2)} \approx \frac{1}{2} q^{-\frac{c}{24}}(2\pi i)^3\alpha^2\tau^5\langle W[1]W_{-3}\rangle \approx \frac{1}{2}q^{-\frac{c}{24}}(2\pi i)\alpha^2\tau^5\frac{1}{30}\langle W_3 W_{-3}\rangle$$

The central charge term:

$$[W_3,W_{-3}] \sim \frac{5N_3 c}{6} \Rightarrow Z^{(2)} \approx \frac{i\pi c}{36}N_3\alpha^2\tau^5 q^{-\frac{c}{24}}$$

Normalization

- The constant

$$N_3 = \frac{16}{5}\sigma^2(\lambda^2 - 4)$$

- Using the WW OPE

$$W(z)W(0) \sim \frac{10c}{3}\frac{1}{z^6} + \cdots$$

- The normalization constant

$$\sigma^2 = \frac{5}{4(\lambda^2 - 4)} \Rightarrow N_3 = 4$$

- The agreement of the two-point result

$$Z^{(2)} \approx \frac{i\pi c}{9}\alpha^2\tau^5 q^{-c/24}$$

Extension to higher points:

$$
\begin{aligned}
Z^{(4)} &\equiv \frac{(2\pi i\alpha)^4}{4!}\,\mathrm{Tr}\big(W_0 W_0 W_0 W_0 \hat{q}^{L_0-\frac{c}{24}}\big) \\
&\approx \frac{\alpha^4\tau^8}{4!}\int F\big((W,z_1),(W,z_2),(W,z_3),(W,z_4);\tau\big) \\
&\approx -q^{-c/24}\,2\pi ic\,\frac{2}{27}\left(5N_3^2-7N_4\right)\alpha^4\tau^9 \\
&\approx -q^{-c/24}\,\frac{100i\pi c}{81}\,\frac{\lambda^2-7}{\lambda^2-4}\alpha^4\tau^9 \\
Z^{(6)} &\approx q^{-c/24}\,2\pi ic\,\left(\frac{17N_3^3}{648}-\frac{581N_3N_4}{9720}+\frac{497N_4^2}{12150N_3}+\frac{101N_5}{2160}\right)\alpha^6\tau^{13} \\
&\approx q^{-c/24}\,\frac{400i\pi c}{81}\,\frac{5\lambda^4-85\lambda^2+377}{(\lambda^2-4)^2}\alpha^6\tau^{13}
\end{aligned}
$$

They agree with the gravity result.

$\mathcal{W}_\infty[\lambda]$ commutation relations

$$
\begin{aligned}
[W_m, W_n] =& 2(m-n)U_{m+n} + \frac{N_3}{12}(m-n)(2m^2+2n^2-mn-8)L_{m+n} \\
&+ \frac{cN_3}{144}m(m^2-1)(m^2-4)\delta_{m+n,0} + \frac{8N_3}{c}(m-n)\Lambda^{(4)}_{m+n} \\
[W_m, U_n] =& (3m-2n)X_{m+n} + \frac{N_4}{15N_3}(n^3-5m^3-3mn^2+5m^2n-9n+17m)W_{m+n} \\
&- \frac{24N_4}{15cN_3}(7+17m-9n)\Lambda^{(5)}_{m+n} + \frac{84N_4}{15cN_3}\Theta^{(6)}_{m+n} \\
[W_m, X_n] =& (4m-2n)Y_{m+n} - \frac{N_5}{56N_4}(28m^3-21m^2n+9mn^2-2n^3-88m+32n)U_{m+n} \\
&+ \frac{42N_5}{5cN_3^2}(2m-n)\Lambda^{(6)}_{m+n} + \cdots \\
[U_m, U_n] =& 3(m-n)Y_{m+n} + n_{44}(m-n)(-7+m^2-mn+n^2)U_{m+n} \\
&- \frac{N_4}{360}(m-n)(108-39m^2+3m^4+20mn-2m^3n-39n^2 \\
&+ 4m^2n^2-2mn^3+3n^4)L_{m+n} - (m-n)\frac{N_4 n_q}{cN_3^2}\Lambda^{(6)}_{m+n} \\
&- \frac{cN_4}{4320}m(m^2-1)(m^2-4)(m^2-9)\delta_{m+n,0}
\end{aligned}
$$

Summary and Outlook

- An one-to-one mapping between HS_4 theory and CFT_3 was established (at the quadratic level)
- We reproduced the higher spin corrections to the black hole entropy from calculating correlation functions of $\mathcal{W}$-currents on the torus
- This gives a detailed/different check that $\mathcal{W}_\infty[\lambda]$ is indeed the correct symmetry algebra of the dual CFT
- Future directions:
  - Action principle for HS in $d > 3$
  - Non-abelian HS theory
  - Quantization of HS theory: renormalizable?
  - Loop corrections to the correlation functions
  - Higher point correlation functions
  - Higher order computation of the BH entropy