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QCD in 2012

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Plan

1 - Introduction
   Non perturbative QCD
   Lattice QCD
   Confinement
   Heavy Ion physics

2 - Asymptotic Freedom
   Perturbative QCD
   Basic concepts and results

3 - QCD in the LHC time
QCD stands as a very solid building block of the SM

The unbroken gauge symmetry of the SM is $\text{SU}(3) \times \text{U}(1)_Q$

For many years the field theory of reference was QED, now QCD is a more complex and intriguing framework

Due to asymptotic freedom, actually QCD is a better defined theory than QED (which has a bad UV limit)

Comparison with experiment is excellent

Steady progress in techniques to extract precise predictions (higher order perturbative, non perturbative, lattice, event generators and simulations)
QCD is an unbroken SU(3) gauge theory with triplet quarks

\[ L = -\frac{1}{4} \sum_{A=1}^{8} F_{\mu\nu}^A F_{\mu\nu}^A + \sum_{j=1}^{n_f} \bar{q}_j (i \bar{D} - m_j) q_j \]

Defs: \[ [t^A, t^B] = i C^{ABC} t^C \quad Tr[t^A t^B] = \frac{1}{2} \delta^{AB} \]

\((C_{ABC}: \text{SU(3) structure constants}, t^A: \text{generator representation})\)

\[ g_\mu = \sum_{A=1}^{8} g_\mu^A t^A \quad (g_\mu^A \text{ is a gluon field}) \]

\[ \overline{D} = D_\mu \gamma^\mu \quad ; \quad D_\mu = \partial_\mu + ie_s g_\mu \quad (D: \text{covariant derivative}) \]

\[ \alpha_s = \frac{e_s^2}{4\pi} \quad (e_s: \text{SU(3) gauge coupling}) \]

\[ F_{\mu\nu}^A = \partial_\mu g_{\nu}^A - \partial_\nu g_{\mu}^A - e_s C_{ABC} g_{\mu}^B g_{\nu}^C \]
QCD is a "simple" theory

\[ L = -\frac{1}{4} \sum_{A=1}^{8} F_{\mu\nu}^A F_{\mu\nu}^A + \sum_{j=1}^{n_f} \bar{q}_j (i\overleftarrow{D} - m_j) q_j \]

but with an extremely rich dynamical content:

- Confinement
- Complex hadron spectrum (light and heavy quarks)
- Spontaneous breaking of (approx.) chiral symm.
- Phase transitions
  
  [Deconfinement (q-g plasma), chiral symmetry restauration,…….]
- Highly non trivial vacuum topology
  
  [Instantons, U(1)\textsubscript{A} symm. breaking, strong CP violation (?)]
- Asymptotic freedom

•••
How do we get predictions from QCD?

• Non perturbative methods
  • Lattice simulations (great continuous progress)
  • Effective lagrangians
    * Chiral lagrangians
    * Heavy quark effective theories
    * Soft Collinear Effective Theory (SCET)
      **********
  • QCD sum rules
  • Potential models (quarkonium)

• Perturbative approach
  Based on asymptotic freedom.
  It still remains the main quantitative connection to experiment.
The main tool for non perturbative QCD in continuous progress

38 years of lattice QCD


QCD potential

QCD coupling const

Phase transition

Flavor physics

Hadron spectrum

Hashimoto
Major progress in recent years

Much more powerful computers now allow for:

- Finer lattice spacing $a \to 0$ (continuum limit)
- Improved lagrangians $[o(a^2)]$
- Larger volume $L = N \cdot a$, larger $N$
  - in most cases, corrections exp. down: $e^{-kV}$
- Smaller quark masses (realistic $\pi$ mass, $m_\pi^2 \to 0$)
  - large $q$ masses numerically simpler:
    - smaller wavelengths need smaller $V$
  - extrapolation guided by resummation of chiral logs
- Unquenching (taking quark loops into account)
Fermions on the lattice

a generic average:

\[
\langle \psi \rangle = \frac{1}{Z} \int D\Lambda D\psi D\bar{\psi} \ [\psi] \exp \left( -S \right) .
\]

By integrating fermions away:

\[
\langle \psi \rangle = \frac{1}{Z} \int D\Lambda [\psi'] \det \mathcal{M} \exp \left( -S_{\text{gauge}} \right)
\]

\[\psi_i \bar{\psi}_j \rightarrow [M^{-1}]_{ij}\]

imaginary time

unquenching: effect of sea

propagator of valence quark in coloured medium

most realistic results for \( n_f=2+1 \) (u,d)+s

QUENCHED

UNQUENCHED
Different methods for fermions on the lattice

<table>
<thead>
<tr>
<th>Method</th>
<th>Speed</th>
<th>Chiral Symm.</th>
<th>Collab.</th>
</tr>
</thead>
<tbody>
<tr>
<td>imp. stagg. (asqtad)</td>
<td>fast</td>
<td>OK</td>
<td>MILC/HPQCD/FNAL</td>
</tr>
<tr>
<td>domain wall</td>
<td>slow</td>
<td>good</td>
<td>RBC/UKQCD</td>
</tr>
<tr>
<td>clover</td>
<td>fast</td>
<td>bad</td>
<td>PACS-CS QCDSF CERN-TOV</td>
</tr>
<tr>
<td>twisted mass</td>
<td>fast</td>
<td>OK</td>
<td>ETMC</td>
</tr>
</tbody>
</table>

A compromise between efficiency and theoretical purity is needed.
Chiral extrapolation

- Lattice simulation is limited in a heavier quark mass region $m_\eta \sim (0.5-1)m_s$.

ChPT predicts the chiral log near the chiral limit.

$c \log(m_\eta/1\text{GeV})$

with a fixed coefficient.

Staggered simulation can push the quark mass much lower.

\[ \langle 0|\partial^\mu A_\mu|\pi \rangle = f_\pi m_\pi^2 \]
The quenched approximation (QA) is superseded: what was rough agreement in QA is now precise with unquenching.
Unquenched lattice simulations reproduce spectrum well

QCD Hadron Spectrum
Plot from A. Kronfeld [1203.1204]

$\pi, \ldots, \Omega$: BMW, MILC, PACS-CS, QCDSF;
$\eta, \eta'$: RBC, UKQCD, Hadron Spectrum ($\omega$);
$D, B$: Fermilab, HPQCD, Mohler-Woloshyn

Note:
$p/\rho \sim 1.2$
not 1.5
as from $3q/2q$

Excellent agreement between different collaborations/lattice formulations
Unquenched lattice simulations reproduce spectrum well

Kuromashi

Here the focus is on strange particles

Wilson $N_f=2+1$
Quark Masses

eg the strange quark

\[ m_u = 2.19(15) \text{ MeV} \]
\[ m_d = 4.67(20) \text{ MeV} \]
\[ m_s = 94(3) \text{ MeV} \]
\[ \frac{m_s}{m_{ud}} = 27.4(4) \]
Lattice is playing an increasingly important role in flavour physics

Lattice inputs (2+1 sea quarks):

\[
\begin{align*}
B_k \\
f_K / f_\pi , f_+ (K \to \pi l \nu) \\
F (B \to D^* l \nu) \\
f_+ (B \to \pi l \nu) \\
\frac{f_{B_s} \sqrt{B_{\bar{B}_s}}}{f_B \sqrt{B_B}}
\end{align*}
\]
Examples
Kaon Mixing

• Summary of Lattice results for $B_K$ from FLAG [1011.4408]

\[ B_K(\overline{\text{MS}}, \ 2 \ \text{GeV}) \]

\[ B_K = 0.536 \pm 0.017, \quad \hat{B}_K = 0.738 \pm 0.020 \]

$(N_f = 2 + 1)$
Confinement: no free coloured particles

\[ V(r) \approx C_F \left[ \frac{\alpha_s(r)}{r} + \ldots + \sigma r \right] \]

- **q-\bar{q} potential:**
  - Has been studied in lattice QCD

- **Short dist.:**
  - The remnant of q is a jet of colourless hadrons

- **Long dist.:**
  - e^+ e^- \rightarrow q \bar{q} \rightarrow hadrons
  - The string breaks up like a magnet.
Lattice QCD offers the most convincing evidence of confinement in quenched approx.

Potential between static quarks on the lattice

V(R,T) = V_0 + \sigma(T)R + C T \ln(2RT)

Potential in units of kT (k=1) as function of R in units 1/T, for different $\beta=1/T$

The linearly rising term slope vanishes at $T_C$
At $T>T_C$ the slope at large $R$ remains zero.

$T_C$ depends on the number of quark flavours.

$T_C \sim 175$ MeV
The QCD phase diagram

Studied on the lattice and probed by colliding heavy ions at AGS, SPS, RHIC, LHC (ALICE, ATLAS, CMS)
Lattice QCD predicts a rapid transition, with correlated deconfinement and chiral restauration.

But not a phase transition, rather a smooth cross over.

\[ \langle L \rangle = \exp(\frac{1}{T} \int_0^1 A_4 d\tau) \]
\[ \sim \exp(-F_Q / T) \]
The order of the phase transition is a function of $m_q$. 

![Graph showing the order of the phase transition as a function of $m_q$. The graph includes a crossover point and a physical point, with regions labeled as 1st order.]
- energy density increases sharply by the latent heat of deconfinement

For $N_f = 2, 2+1$:

\[ T_c \simeq 175 \text{ MeV} \]
\[ \epsilon(T_c) \simeq 0.5 - 1.0 \text{ GeV/fm}^3 \]
Summary of recent results on $T_c$

use $T=0$ scale: $r_0=0.469\text{fm}$  
Karsch LAT’07

$\text{Nf}=2$:

(improved Wilson, $N_t=8, 10$; input: $r_0=0.5\text{ fm}$)
(added $N_t=12$, Lattice’07)  
(rescaled to $r_0$)

Y. Maezawa et al., hep-lat/0702005 (QM’2006)
(improved Wilson, $N_t=4, 6$; input: $m$–$\rho$)  
(no cont. exp. yet)

$\text{Nf}=2=1$:

(improved staggered (asqtad), $N_t=4,6,8$, input $r_1$)  
(rescaled to $r_0$)

(improved staggered ($p4$), $N_t=4,6$; input $r_0$)

(staggered (stout), $N_t=4,6,8,10$; input $fK$)  
(converted to $r_0$)

\[ \text{chiral} \quad \square \quad \text{deconfinement} \quad \bigcirc \quad \text{chiral+deconfinement} \]
Chiral symmetry and its breaking

In the limit \( m_u, m_d \to 0 \) the QCD Lagrangian is \( U(2)_L \times U(2)_R \) symmetric.

But no parity doublets in the hadron spectrum: the symmetry is spontaneously broken by \( \bar{q}q \) condensates.

The 3 pions are the would be Goldstone bosons from the breaking of the axial \( SU(2) \):

\[
U(2)_L \times U(2)_R \rightarrow SU(2)_V \times U(1)_V \times U(1)_A
\]

isospin \( u+d \) baryon number

The quark condensate has been computed on the lattice:

\[
\bar{u}u + \bar{d}d = [234 \pm 4 \pm 17 \text{ MeV}]^3 \quad (\text{MS scheme at 2 GeV})
\]

Fukaya et al 1012.4052
Strong CP violation: possible new physics?

The axial anomaly breaks the singlet axial current

$$\partial_\mu j_5^\mu = \frac{\alpha_s}{4\pi} Tr(F_{\alpha\beta} \tilde{F}^{\alpha\beta})$$

$$\tilde{F}^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$$

As an effect a term is added to the lagrangian

$$\Delta L = \theta \frac{\alpha_s}{4\pi} Tr(F_{\alpha\beta} \tilde{F}^{\alpha\beta})$$

where \(\theta\) arises from the topology of the vacuum in non abelian gauge theories which is far from trivial:

$$\theta = \theta_{\text{instantons}} + \text{Arg Det } m$$

\(m\) quark mass matrix
θ is expected to be o(1). But it would contribute to the neutron electric dipole moment:

\[ d_n(e \cdot cm) \approx 3 \cdot 10^{-16} \theta \]

From experiment: \[ |\theta| \leq 10^{-10} \]

The "strong CP problem" consists in finding an explanation:

- Non renormalisation theorem in SUSY
- An ad hoc symmetry (Peccei-Quinn)
  spont. broken --> axion
- Something not understood on vacuum topology?
CPV in FC channels is dominated by CKM

What in flavour conserv. channels?

present limit on nEDM from Grenoble

$|d_n| < 3 \times 10^{-26}$ e cm (90% cl)
LHC Heavy Ion Experiments (ALICE, ATLAS, CMS)

7 TeV p-p com energy corresponds to $7 \times 82 = 574$ TeV Pb-Pb

Pb has 82 protons and 208 nucleons: $574/208 = 2.76$ TeV

Or 2.76 TeV for each NN pair

From the measured ch. particle multiplicity/unit rapidity $dN_{ch}/d\eta \sim 1600$ (in most central collisions) one estimates:

$\varepsilon_0 \sim 146$ GeV/fm$^3$, $T \sim 640$ MeV $\sim 4$ $T_c$

A review: B. Muller, Schukraft, Wyslouch  ArXiv:1202.3233
Strangeness enhancement for central events [0-20% means most central]

\[ \frac{n_h}{n_l} \sim \exp \left( -\frac{m_h - m_l}{kT} \right) \]

statistical model reproduces yields well (some p deficit?)
Elliptic flow: a tool to study the primeval final state

\[
\frac{dN}{dp_t dy d\phi} = N_0 \left[ 1 + \sum_{i=1}^{6} 2 v_i(y, p_t) \cos(i \phi) \right]
\]

coord. space \hspace{10cm} \text{mom. space} \hspace{10cm} v_{1-6} \text{ now measured}
\[ \frac{dN}{d\phi} \sim \left( 1 + 2v_2 \cos \left[ 2(\phi - \phi_0) \right] \right) + \ldots \]

dominant anisotropy parameter

Hydrodynamic calc’ns depend on \( \eta/s \) (shear viscosity/entropy density).

Luzon (QM12)
η/s can be determined from the $p_T$ or the centrality distributions with compatible results.

Small values of η/s are obtained: 0.07 - 0.43. More precision possible in near future.

For a perfect quantum fluid η/s ~ 1/4π ~ 0.08

On the basis of the AdS/CFT correspondence it is conjectured that this is a lower limit in real QCD

In summary:
the hot dense matter formed is close to a perfect fluid.
EW probes ($W, Z, \gamma$) are not suppressed in the medium

Diagram:
- CMS
- RAA: ratio PbPb/pp
Azimuthal distributions in Au+Au

Near-side: peripheral and central Au+Au similar to p+p

Strong suppression of back-to-back correlations in central Au+Au

Phys Rev Lett 90, 082302
In most central events the energy unbalance of the 2 jets is increased

\[ A_J = \frac{E_T^1 - E_T^2}{E_T^1 + E_T^2} \]
Open charm and beauty (D and B mesons) are suppressed
$J/\psi$ suppression at the start (SPS) was thought to be a clear indicator of colour screening.

Interpretation of data at RHIC and the LHC demands both screening and recombination (late formation of $J/\psi$ from charm quarks in the medium).
b-onium suppression (less affected by recombination)

CMS Preliminary, PbPb $\sqrt{s_{NN}} = 2.76$ TeV

$R_{AA}^{Y(1S)}$ < $R_{AA}^{Y(2S)}$ < $R_{AA}^{Y(1S)}$

Data
- $Y(1S)$
- $Y(2S)$

$L_{int} = 150 \mu b^{-1}$

$|y| < 2.4$

CMS PbPb $\sqrt{s_{NN}} = 2.76$ TeV

Cent. 0-100\%, $|y| < 2.4$

0 < $p_T < 20$ GeV/c

$L_{int} = 7.28 \mu b^{-1}$

$\sigma = 92$ MeV/c$^2$ (fixed to MC)
Conclusion

Heavy Ion collisions have demonstrated the formation of a strongly interacting, hot, near perfect liquid.

The reconstructed temperature and energy density are compatible with what expected for quark-gluon plasma.

Additional properties of this liquid like shear viscosity, equation of state and sound velocity are under continuing study.
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\[ L = -\frac{1}{4} \sum_{A=1}^{8} F_{\mu \nu}^A F_{\mu \nu}^A + \sum_{j=1}^{n_f} \bar{q}_j (i \widehat{D} - m_j) q_j \]

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\[ g_\mu = \sum_{A=1}^{8} g_\mu^A t^A \quad (g_\mu^A \text{ is a gluon field}) \]

\[ \widehat{D} = D_\mu \gamma^\mu \quad ; \quad D_\mu = \partial_\mu + i e_s g_\mu \quad (D: \text{covariant derivative}) \]

\[ \alpha_s = \frac{e_s^2}{4\pi} \quad (e_s: \text{SU}(3) \text{ gauge coupling}) \]

\[ F_{\mu \nu}^A = \partial_\mu g_{\nu}^A - \partial_\nu g_{\mu}^A - e_s C_{ABC} g_{\mu}^B g_{\nu}^C \]
**Physical QCD vertices**

\[-i e_s \gamma^\mu t^A\]

\[p + q + r = 0\]

\[e_s C_{ABC}[g_{\mu\nu}(p-q)_\lambda + \text{perm}]\]

Note: \(e_s^2\)

\[-i e_s^2 \left[ C_{ABF} C_{CDF} (g_{\lambda\nu} g_{\mu\rho} - g_{\lambda\rho} g_{\mu\nu}) + \text{perm} \right]\]
Classical gauge th. lagrangian

Quantisation

Gauge fixing terms
Ghosts

Feynman rules

Perturbation Theory

Infinities

Regularisation
Renormalisation

Cutoff K
Redefinition of m, $\alpha_s$, $Z_i$ (wave funct.n norm'ns)

Perturbative quantum gauge th.
Perturbative QCD and scale invariance

In the QCD lagrangian

\[ L = -\frac{1}{4} \sum_{A=1}^{8} F_{\mu\nu}^{A} F_{\mu\nu}^{A} + \sum_{j=1}^{n_{f}} \bar{q}_{j}(i\overleftarrow{D} - m_{j})q_{j} \]

quark masses are the only parameters with dimensions.

Naively we would expect massless QCD to be scale invariant (dimensionless observables should not depend on the absolute energy scale, but only on ratios of energy variables)

The massless limit should be relevant for the asymptotic large energy limit of processes which are non singular for \( m \to 0 \).
This naïve expectation is false!

For massless QCD the scale symmetry of the classical theory is destroyed by regularisation and renormalisation which introduce a dimensional parameter in the quantum version of the theory ($\Lambda_{\text{QCD}}$).

[When a symmetry of the classical theory is necessarily destroyed by quantisation, regularisation and renormalisation one talks of an "anomaly"]

While massless QCD is finally not scale invariant, the departures from scaling are asymptotically small, logarithmic and computable (in massive QCD there are additional mass corrections suppressed by powers of $m^2/E^2$).
Hard processes

At the "parton" level (q and g) we can apply the asymptotics from massless QCD to processes with the following properties:

• all relevant energy variables are large
  \[ E_i = x_i Q \quad Q \gg m \quad x_i : \text{scaling variables} \]
• no infrared and collinear singularities ("infrared safe")
• finite for \( m \to 0 \) (no mass singularities.)

To satisfy these criteria processes must be sufficiently "inclusive":

• add all final states with massless gluon emission
• add all mass degenerate final states (e.g. q-qbar pairs)
**Bloch-Nordsieck Theorem:**

Infrared singularities cancel between real and virtual diagrams when all resolution indistinguishable final states are added up.

\[
\begin{align*}
\left\langle\right\rangle + \left\langle\right\rangle \quad + \quad \left\langle\right\rangle + \ldots \quad + \quad \left\langle\right\rangle^2 \quad + \\
\left\langle\right\rangle + \left\langle\right\rangle \quad + \quad \left\langle\right\rangle^2
\end{align*}
\]

**Kinoshita-Lee-Nauenberg Theorem:**

Mass singularities are absent if all degenerate states are added up (including collinear qqbar pairs for massless q). If an inclusive final state is taken, only the mass singularities from the initial lines remain.

\(\oplus\) (Will be absorbed inside the initial parton densities)
Note: We compute inclusive rates for partons and take them as equal to rates for hadrons.

Partons and hadrons are considered as two equivalent sets of complete states.

This is called "global duality" and is rather safe in the totally inclusive case. It is less so for distributions, like $d\sigma/dM$ in the invariant mass $M$ ("local duality") where it is reliable only if smeared over a sufficiently large bin of $M$. 

![Diagram of particle interactions](image)
Regularisation and Renormalisation

In general:

• A dimensional "cut off" $K$ is introduced (must be gauge invariant)

• The dependence on the cut-off is eliminated by a redefinition of $m$, $e_s$ and $Z$ using suitable renormalisation conditions.

$$\text{Propagator} = \frac{Z}{p^2 - m^2} + \text{no-pole}$$

Renormalized mass: position of the propagator pole. Wave funct'n renormalization $Z$: residue at the pole.

The renormalized coupling $e_s$ is, for example, defined in terms of a renormalized 3-point vertex at some momenta.
In particular in massless QCD:

If we start with $m_0=0$ the mass is not renormalized because it is protected by a symmetry (chiral symm.) -> $m=0$

The coupling $e_s$ can be defined in terms of the 3-gluon coupling at a scale $-\mu^2$:

$$V_{\text{bare}}(p^2,q^2,r^2) = ZV_{\text{ren}}(p^2,q^2,r^2)$$

($Z=Z_g^{-3/2}$ for $V$ 1PI)

Ward id. guarantee the same result starting from any other vertex

- The scale $\mu$ cannot be zero (infrared singularity)!
- $-\mu^2<0$: no absorptive parts

Similarly $Z_g$ can be defined by the inverse propagator at

$$p^2 = -\mu^2$$

$$P^{-1}_{\text{bare}} = Z_g^{-1}P^{-1}_{\text{ren}}$$
Computing all 1PI diagrams (with cutoff $K$)

\[ p^2 \]

\[ r^2 \]

\[ q^2 \]

\[ + \]

\[ \text{at 1 loop} \]

\[ p^2 = q^2 = r^2 \]

\[ V_{bare} = e_0 \left[ 1 + c\alpha_s \log \frac{K^2}{p^2} + \ldots \right] = \]

\[ = \left[ 1 + c\alpha_s \log \frac{K^2}{-\mu^2} + \ldots \right] e_0 \left[ 1 + c\alpha_s \log \frac{-\mu^2}{p^2} \right] = Z_v^{-1} e_0 \ldots = \]

\[ e_0 = Z_g^{-3/2} Z_v e \]

Note: $V_{bare}$ depends on $K$ but not on $\mu$

\[ \frac{dV_{bare}}{d \log \mu^2} = 0 \]

Both $Z$ and $V_{ren}$ depend on $\mu$
Renormalisation group equation

(We write $\alpha$ for $\alpha$ or $\alpha_s$ in QED or QCD)

In general:

$$G_{\text{Bare}}(K^2, \alpha_0, p_i^2) = Z \cdot G_{\text{ren}}(\mu^2, \alpha, p_i^2)$$

so that:

$$\frac{dG_{\text{Bare}}}{d\log \mu} = \frac{dZ}{d\log \mu} [ZG_{\text{ren}}] = 0$$

or

$$Z \left[ \frac{\partial}{\partial \log \mu} + \frac{\partial \alpha}{\partial \log \mu} \frac{\partial}{\partial \alpha} + \frac{1}{Z} \cdot \frac{\partial Z}{\partial \log \mu} \right] G_{\text{ren}} = 0$$

Finally the RGE can be written as:

$$\left[ \frac{\partial}{\partial \log \mu} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] \cdot G_{\text{ren}} = 0$$

This is a relation among physical quantities (no cutoff $K$)
Consider the RGE:

\[
\left[ \frac{\partial}{\partial \log \mu} \frac{1}{2} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] \cdot G_{\text{ren}} = 0
\]

applied to some hard process at a large scale \( Q : G_{\text{ren}} \rightarrow F(t, \alpha, x_i) \) where \( x_i \) are scaling variables (omitted in the following), and

\[
t = \log \frac{Q^2}{\mu^2}
\]

Assume \( F \) is adimensional, then in the naïve scaling limit \( F \) would be independent of \( t \).

We want to solve the RGE equation:

\[
\left[ -\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] \cdot F(t, \alpha) = 0
\]

with a given boundary cond.: \( F(0, \alpha) \) specified.
Given the general RGE:

\[
\left[ -\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \right] \cdot F(t, \alpha) = 0
\]

The solution, with boundary cond. \( F(0,\alpha) \), is:

\[
F(t,\alpha) = F[0,\alpha(t)] \exp \left( \int_{\alpha}^{\alpha(t)} \frac{\gamma(\alpha')}{\beta(\alpha')} d\alpha' \right)
\]

where the "running coupling" \( \alpha(t) \) is defined by:

\[
t = \int_{\alpha}^{\alpha(t)} \frac{1}{\beta(\alpha')} d\alpha'
\]

Note: at \( t=0 \), \( \alpha(0) = \alpha \). One has:

\[
\frac{\partial}{\partial t} \alpha(t) = \beta(\alpha(t))
\]

The important point is the appearance of the running coupling that determines the asympt. behaviour.
The running coupling

\[ \alpha = \frac{e^2}{4\pi} \quad \bigg\vert \quad t = \log \frac{Q^2}{\mu^2} \]

The running coupling \( \alpha(t) \) is fixed by the beta function:

\[ \frac{\partial \alpha}{\partial \log \mu^2} = \beta(\alpha) \quad \text{or} \quad \frac{\partial}{\partial t} \alpha(t) = \beta(\alpha(t)) \]

The \( \mu \) dependence starts at 1-loop:

Recall: in QCD Ward id. guarantee the same result starting from vertex

\[ \alpha e^3 \]

\[ 2e \frac{\partial e}{\partial \log \mu^2} \alpha e^4 \alpha^2 \]

\[ \beta(\alpha) = \pm b \alpha^2 \left[ 1 + b' \alpha + \ldots \right] \quad (b > 0) \]
By explicit calculation at 1-loop one finds:

**QED:** \[ \beta(\alpha) \sim + b\alpha^2 + ... \]

The sum is over all fermions of charge \( q_e \)

**QCD:** \[ \beta(\alpha) \sim - b\alpha^2 + ... \]

\( b = \sum_i \frac{(N_C Q^2)_i}{3\pi} \)

\( b = \frac{11 N_C - 2 n_f}{12\pi} \)

Here \( N_C = 3 \)

\( t = \log \frac{Q^2}{\mu^2} \)

\( n_f \) is the number of quark flavours

Recall: \[ \frac{\partial}{\partial t} \alpha(t) = \beta(\alpha(t)) \]

If \( \alpha(t) \) is small, we can compute \( b \) in pert. th. The sign in front of \( b \) decides whether: \( \alpha(t) \) increases with \( t \) or \( Q^2 \) (QED) or \( \alpha(t) \) decreases with \( t \) or \( Q^2 \) (QCD).

**QCD** is "asymptotically free". In 4-dim all and only non-abelian gauge theories are asympt. free.
Going back to the equation:

\[
t = \int_\alpha^{\alpha(t)} \frac{1}{\beta(\alpha')} d\alpha' \quad \alpha(0) = \alpha \quad t = 0 \rightarrow Q = \mu
\]

We replace \( \beta(\alpha) \sim \pm b\alpha^2 \), integrate and do a small algebra. We find:

\[
\alpha(t) = \frac{\alpha}{1 + b\alpha t} \sim \alpha(1 - b\alpha t + ...) \quad \text{QCD}
\]

\[
\alpha(t) = \frac{\alpha}{1 - b\alpha t} \sim \alpha(1 + b\alpha t + ...) \quad \text{QED}
\]

In QCD we have:

\[
\alpha(t) = \frac{1}{\alpha + bt} = \frac{1}{b \log \frac{\mu^2}{\Lambda^2} + b \log \frac{Q^2}{\mu^2}} = \frac{1}{b \log \frac{Q^2}{\Lambda^2}}
\]

Note

• \( \alpha \) decreases logarithmically in \( Q^2 \)
• a dimensional parameter \( \Lambda = \Lambda_{\text{QCD}} \) replaces \( \mu \)
\[ \beta(\alpha) \sim \pm b \alpha^2 (1 + b' \alpha + \ldots) \]

In general the pert. coeff.s of \( \beta(\alpha) \) depend on the def. of \( \alpha \), the renorm. scheme etc. But both \( b \) and \( b' \) are indep.

Here is a sketch of the proof:

\[
\frac{d}{d \log \mu} \alpha' = \frac{d}{d \log \mu} \alpha \left(1 + 2k \alpha + \ldots\right) = \pm b \alpha^2 \left(1 + b' \alpha + \ldots\right) \left(1 + 2k \alpha + \ldots\right) = \\
= \pm b \alpha^2 \left(1 + b' \alpha' + \ldots\right) = \beta(\alpha')
\]

QCD:

\[ b' = \frac{153 - 19 n_f}{2 \pi (33 - 2 n_f)} \text{ for } N_C=3 \]

Taking \( b' \) into account:

\[
\alpha(Q^2) = \alpha_0(Q^2) \left[1 - b' \alpha_0(Q^2) \log \log \frac{Q^2}{\Lambda^2} + \ldots\right]
\]
Summarising: the running coupl. $\alpha(Q^2)$ is the crucial quantity:

$$\frac{d\alpha(Q^2)}{d\log Q^2} = \beta[\alpha(Q^2)]$$

$$\beta(\alpha) = -b\alpha^2[1 + b'\alpha + ...] \quad (b>0)$$

$$\alpha(Q^2) = \frac{1}{b\log \frac{Q^2}{\Lambda_{QCD}^2}}(1 + ...)$$

$$b = \frac{11N_C - 2n_f}{12\pi}$$

$\Lambda_{QCD}$ is the scale that breaks scale inv. in massless QCD

$\Lambda_{QCD} = 218\pm24$ MeV ($N_f=5$)

The $\rho$ mass etc are due to $\Lambda_{QCD}$ not to $m_q$

MS$_{\text{bar}}$, $n_f=5$: $\beta(\alpha) \equiv -0.610\alpha^2[1 + 1.261\frac{\alpha}{\pi} + 1.475\left(\frac{\alpha}{\pi}\right)^2 + 9.836\left(\frac{\alpha}{\pi}\right)^3 + ...$]

4th: van Ritbergen, Vermaseren, Larin (1997)

$\sim 50,000$ 4-loop diagrams!!

No hierarchy problem in QCD!

$$\Lambda_{QCD} = M_{Pl}\exp\left(-\frac{1}{2b\alpha(M_{Pl}^2)}\right)$$
Dependence of $\Lambda$ from $n_f$

QED and QCD are theories with decoupling: quarks with mass $m>Q$ do not contribute to the running of $\alpha$ up to the scale $Q$.
So for $2m_c<Q<2m_b$ the relevant asymptotics is for $n_f=4$, while for $2m_b<Q<2m_t$ $n_f=5$.

Going across the $2m_b$ threshold, the $\beta(\alpha)$ coeff.s change, so the $\alpha(t)$ slope changes. But $\alpha(t)$ is continuous so that $\Lambda_4$ and $\Lambda_5$ are different:

$$\alpha(Q^2) \approx \frac{1}{b \log \frac{Q^2}{\Lambda^2}} (1 + \ldots)$$

$$b = \frac{11N_C - 2n_f}{12\pi}$$

From matching $\alpha(Q^2)$

$\Lambda_5 \approx 0.65 \Lambda_4$
Examples of important hard processes

• $e^+e^- \rightarrow \text{hadrons}$
  \[(p+p')^2 = s = Q^2\]

At parton level the final state is

$$q\bar{q} + n \text{ gluons} + n' q\bar{q} \text{ pairs}$$

(i.e. totally inclusive). The conversion of partons into hadrons does not affect the rate (some smearing over a Q bin can be needed for probability 1)

• $l + N \rightarrow l' + \text{hadrons}$
  (Deep Inelastic Scattering: DIS)

$$(k - k')^2 = q^2 = -Q^2$$

$$x = \frac{Q^2}{2(pq)}$$
The simplest application is to the process:
\[ R = \sigma(e^+ e^- \to \text{hadrons}) / \sigma(e^+ e^- \to \mu^+ \mu^-) \]
\[ t = \log Q^2/\mu^2 \rightarrow F(t, \alpha_s) \]

For this process \( \gamma(\alpha) = 0 \): renorm. of charge is the same for quarks and leptons!

**Charge renorm in QED at 1 loop:**

Here is the connected Green 3-p function

Only \( Z_\gamma \) (marked with arrow) survives. \( Z_V^{-1} \) and \( Z_f \) cancel by Ward identity. No \( \alpha_s \) terms (gluon exchange) at 1 loop in the \( \gamma \)-blob \( Z_\gamma \).
\[ R = N_C \sum Q^2 F(t, \alpha_s) = N_C \sum Q^2 [1 + 0(\alpha_s)] \]

The RGE prediction is:

\[ F(t, \alpha_s) = F[0, \alpha_s(t)] \]

with

\[ \alpha_s(t) = \frac{\alpha_s}{1 + b \alpha_s t} \approx \alpha_s \cdot (1 - b \alpha_s t + ...) \]

that is at 2-loops (no \( \alpha_s t, \alpha_s^2 t^2 \) terms, coeff \( \alpha_s^2 t \) fixed...):

\[ F[0, \alpha_s(t)] = 1 + c_1 \alpha_s (1-b \alpha_s t + ...) + c_2 \alpha_s^2 + .... \]

\( c_1 = \frac{1}{\pi} ; \ c_2, c_3, c_4 \) also known (dep. on def. \( \alpha_s \))
In \( \overline{\text{MS}} \) with \( n_f=5 \) for \( e^+e^- \) \( (\alpha_s(Q^2)/\pi) \)

\[
R(Q^2)=3 \sum_f Q_f^2 \left[ 1+\alpha_s+1.4097\alpha_s^2 -12.76709\alpha_s^3 -80.0075\alpha_s^4 + \ldots \right]
\]

**Note:** the sub-leading coeff.s depend on scale choice: if instead of \( Q \) was \( Q/2 \) they would change.

Similar perturbative results at 3-loops exist for \( \Gamma(Z\to\text{hadrons})/\Gamma(Z\to\text{leptons}) \), \( \Gamma(\tau\to\nu\tau+\text{hadrons})/\Gamma(\tau\to\nu\tau+\text{leptons}) \), etc

The pattern of power corrections is controlled by the light-cone operator expansion:

\[
F = \text{pert.} \quad + r_{-2} \frac{m^2}{Q^2} + r_{-4} \frac{\langle 0| Tr[\sum A^A_{\mu\nu}F_{\mu\nu}^A]|0\rangle}{Q^4} + \ldots + r_{-6} \frac{\langle 0| O_6 |0\rangle}{Q^6} + \ldots
\]
Light Cone Operator Product Expansion

\[ R_{e+e^-} \sim \Pi(Q^2) \quad \sigma_{e+e^-} \sim L_{\mu\nu} T^{\mu\nu} \]

\[ T_{\mu\nu} = \sum_n \langle 0|J_\mu^\dagger(0)|n\rangle \langle n|J_\nu(0)|0\rangle (2\pi)^4 \delta^4(q-p_n) = \]

\[ = \int e^{iqx} \langle 0|J_\mu^\dagger(x)J_\nu(0)|0\rangle dx = (-g_{\mu\nu}Q^2 + q_\mu q_\nu) \Pi(Q^2) \]

For \( Q^2 \to \) infinity the \( x^2 \to 0 \) region is dominant. To all orders in pert. th. the OPE can be proven. Schematically, dropping Lorentz indices, near \( x^2 \sim 0 \):

\[ J_\mu^\dagger(x)J(0) \equiv I(x^2) + E(x^2) \sum_{n=0}^{\infty} c_n (x^2) x^{\mu_1} \ldots x^{\mu_n} \cdot O^n_{\mu_1 \ldots \mu_n}(0) + \text{less sing. terms} \]

\[ I(x^2), E(x^2), \ldots, c_n (x^2), \ldots, \text{c-number sing.} \]

\[ O^n: \text{string of local operators.} \]
\( \Sigma(x^2) \) is the sing. of free field th., \( I(x^2) \), \( c_n(x^2) \) contain powers of \( \log(\mu x) \) in interaction. \( I(x^2) \) is the most sing. in \( x^2 \).
Some \( O^n \) are already present in free field th., more appear in interaction.

\( \Pi(Q^2) \) is related to the Fourier transform. Less sing. terms in \( x^2 \) ("higher twist") lead to power suppressed terms in \( 1/Q^2 \).

\[
F = \text{pert.} + r_{-2} \frac{m^2}{Q^2} + r_{-4} \frac{\langle 0 | Tr \left[ \sum F_{\mu \nu}^A F_{\mu \nu}^A \right] | 0 \rangle}{Q^4} + \ldots + r_{-6} \frac{\langle 0 | O_6 | 0 \rangle}{Q^6} + \ldots
\]

Note: \( g_{\mu \nu} g^\mu \) not gauge invariant

The pert. terms come from \( I(x^2) \). Down by \( 1/Q^2 \) are mass terms (e.g. \( m_b^2/Q^2 \)). Dimension 4, 6... operators are suppressed by \( 1/Q^4, 1/Q^6 \) ...
Deep Inelastic Scattering has played a capital role in the development of QCD

\[ I + N \rightarrow I' + X, \quad l = e, \mu, \nu \]

- Many structure functions
- \( F_i(x, Q^2) \): two variables
- Neutral currents, charged currents
- Different beams and targets
- Different polarization

From the beginning: Establishing quarks and gluons as partons

Constructing a field theory of strong int.n.s

and along the years: Quantitative testing of QCD

Totally inclusive

QCD theory of scaling violations crystal clear
(based on ren. group and operator exp.)

Q^2 dependence tested at each x value

Measuring q and g densities in the nucleon

Instrumental to compute all hard processes

Measuring \( \alpha_s \)

Always presenting new challenges, e.g.: Structure functions at small x; heavy flavour structure functions;
polarized parton densities, \( g_1, g_2, h_1 \ldots \); non forward pdf’s

Diffraction
\[ Q^2 = -q^2 = 4EE\sin^2 \frac{\theta}{2} \]

(\(\theta\): 1-1' lab. angle)

\[ m\nu = (p \cdot q) \quad x = \frac{Q^2}{2m\nu} \]

Structure functions \(\sigma = l_{\mu\nu} \cdot W^{\mu\nu}\)

\(l_{\mu\nu}\): leptonic \(W^{\mu\nu}\): hadronic

\[
W_{\mu\nu} = \int e^{iqx} \langle p | J_\mu^\dagger(x) J_\nu(0) | p \rangle dx = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) W_1(Q^2, \nu) + \\
\left( p_\mu - \frac{m\nu}{2} q_\mu \right) \left( p_\nu - \frac{m\nu}{2} q_\nu \right) W_2(Q^2, \nu) + m^2 + \\
-\frac{i}{2m^2} \varepsilon_{\mu\nu\lambda\rho} p^\lambda q^\rho W_3(Q^2, \nu)
\]
Early crucial breakthroughs

- **Approximate Scaling** Bjorken
- **Success of Naive Parton Model** Feynman

From constituent quarks (real? fictitious?) to parton quarks (real!)

- \( R = \frac{\sigma_L}{\sigma_T} \rightarrow 0 \) Spin 1/2 quarks
- \( \sim 50\% \) of momentum carried by neutrals Gluons
- **Quark charges:**
  
  \[
  F = 2F_1 \sim F_2/x \quad \sigma_L \sim 0
  \]

  \[
  F_{\gamma p} = \frac{4}{9} u(x) + \frac{1}{9} d(x) + \ldots
  \]

  \[
  F_{\gamma n} = \frac{4}{9} d(x) + \frac{1}{9} u(x) + \ldots
  \]

  \[
  F_{\nu p} \sim F_{\overline{\nu} n} = 2 d(x) + \ldots
  \]

  \[
  F_{\nu n} \sim F_{\overline{\nu} p} = 2 u(x) + \ldots
  \]

  \[
  F = F(x), \quad u = u(x), \quad d = d(x): \quad \text{naive parton model (scaling)}
  \]

\[
\int (u - \bar{u}) dx = 2
\]

\[
\int (d - \bar{d}) dx = 1
\]

\[
\int (s - \bar{s}) dx = 0
\]
In QCD there are log scaling violations induced by $\alpha_s(Q^2)$.

$$2F_1(x) = \int_x^1 dy \frac{q_0(y)}{y} \sigma_{\text{point}}(\frac{x}{y}) + o\left(\frac{1}{Q^2}\right)$$

Born: $\sigma_{\text{point}} \rightarrow e^2\delta(x/y-1)$

$2F_1 = e^2q_0(x)$

QCD modifies $\sigma_{\text{point}}$ at $o(\alpha_s)$
The result is of the form \( (y>x) \)

\[
\sigma_{\text{point}} = e^2 \left[ \delta \left( \frac{x}{y} - 1 \right) + \frac{\alpha_s}{2\pi} \left( tP \left( \frac{x}{y} \right) + f \left( \frac{x}{y} \right) \right) \right]
\]

The log is from the collinear sing. of the incoming quark leg. In a special gauge, (axial or physical gauge) the dominant real diagram is:

\[
E_k = \frac{yQ}{2x}
\]

\( p_T: \) r transverse mom. \((\text{propag})^2 \sim (1/p_T)^4\)

\( \text{Num} \sim (p_T)^2 \) (helicity non cons. at \( \theta=0 \))

\[
\text{Propag} = \frac{1}{(k-h)^2} = \frac{1}{-2 \cdot E_h \cdot E_k \cdot (1 - \cos \theta)} = \frac{1}{-4E_h E_k \cdot \left( \sin \frac{\theta}{2} \right)^2 \approx \frac{1}{P_T^2}}
\]

\[
\sigma \approx \int \frac{1}{p_T^2} dp_T^2 \approx \log Q^2
\]
We factorise the mass sing. into the quark parton density (non perturbative):

\[
2F_1 = \int dy \frac{q_0(y)}{y} e^{2} \left[ \delta\left(\frac{x}{y} - 1\right) + \frac{\alpha_s}{2\pi} \left( tP\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right) \right) \right]
\]

(All integrals from x to 1)

\[
= \int dy \frac{q_0(y) + \Delta q(y, t)}{y} e^{2} \left[ \delta\left(\frac{x}{y} - 1\right) + \frac{\alpha_s}{2\pi} f\left(\frac{x}{y}\right) \right]
\]

We replace: \( q_0(x) \rightarrow q(x,t) = q_0(x) + \Delta q(x,t) \): effective, \( Q^2 \)-dep. parton density.

\[
\Delta q(x,t) = \frac{\alpha_s}{2\pi} t \int_{x}^{1} dy \frac{q_0(y)}{y} P\left(\frac{x}{y}\right)
\]

According to the RGE, now \( \alpha_s \rightarrow \alpha_s(t) \)

\[
2F_1 = \int dy \frac{q(y, t)}{y} e^{2} \left[ \delta\left(\frac{x}{y} - 1\right) + \frac{\alpha_s(t)}{2\pi} f\left(\frac{x}{y}\right) \right] = e^{2} q(x, t) + o(\alpha_s(t))
\]

\[
\frac{d}{dt} q(x, t) = \frac{\alpha_s(t)}{2\pi} \int dy \frac{q(y, t)}{y} P\left(\frac{x}{y}\right) + o(\alpha_s(t)^2)
\]
The t-evolution eqs. become non diagonal as soon as gluon partons are also included:

The full set becomes

Recall: \[ \int_x^1 dy \frac{q(y, t)}{y} P\left(\frac{x}{y}\right) = [q \otimes P](x, t) \]

\[
\frac{d}{dt} q_i(x, t) = \frac{\alpha_s(t)}{2\pi} [q_i \otimes P_{qq}] + \frac{\alpha_s(t)}{2\pi} [g \otimes P_{qg}]
\]

\[
\frac{d}{dt} g(x, t) = \frac{\alpha_s(t)}{2\pi} \left[ \sum q_i \otimes P_{gq} \right] + \frac{\alpha_s(t)}{2\pi} [g \otimes P_{gg}]
\]

The quark density with fraction \( y \) times the probability of a gluon in a quark with fraction \( x/y \) of the parent long. mom.
The LO form of the splitting functions can be derived directly from the QCD vertices (process indep.: factorisation)

\[ P_{qq}(x) = \frac{4}{3}\left[ \frac{1 + x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right] + o(\alpha_s(t)) \]

Def.:
\[ \int_0^1 \frac{f(x)}{(1-x)_+} dx = \int_0^1 \frac{f(x) - f(1)}{1-x} dx \]

Note quark conserv. fixes the \( \delta \) terms of \( P_{qq} \)

Similarly for \( P_{gg} \) via momentum conservation

\[ P_{gq}(x) = \frac{4}{3}\left[ \frac{1 + (1-x)^2}{x} \right] + o(\alpha_s(t)) \]

\[ P_{qs}(x) = \frac{1}{2}[x^2 + (1-x)^2] + o(\alpha_s(t)) \]

\[ P_{gg}(x) = 6\left[ \frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right] + \frac{33 - 2n_f}{6} \delta(1-x) + o(\alpha_s(t)) \]
Splitting functions

For many years all splitting funct.s P have been known to NLO accuracy: \( \alpha_s P \sim \alpha_s P_1 + \alpha_s^2 P_2 + \ldots \)

GLAP, Floratos et al; Gonzales-Arroyo et al; Curci et al; Furmanski et al

Then the complete, analytic NNLO results have been derived for the first few moments (N<13,14).

Larin, van Ritbergen, Vermaseren+Nogueira

Finally, in 2004, the calculation of the NNLO splitting functions has been totally completed \( \alpha_s P \sim \alpha_s P_1 + \alpha_s^2 P_2 + \alpha_s^3 P_3 + \ldots \)

Moch, Vermaseren, Vogt

A really monumental, fully analytic, computation
The scaling violations are clearly observed and the (N)NLO QCD fits are remarkably good.
Example of NLO QCD evolution fit
This is how the scaling violations appear now after 40 years of DIS measurements.
It took ~40 years to get meaningful data on the longitudinal structure function!!

\[ F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} x^2 \int \frac{dy}{y^3} \left[ \frac{8}{3} F_2(y, Q^2) + \frac{40}{9} y g(y, Q^2)(1 - \frac{x}{y}) \right] \text{ for } n_f=4 \]

Altarelli, Martinelli ’78
Corfou, 9-10 September 2012

QCD in 2012

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Research supported by LHCPhenonet
Plan

1 - Introduction
   Non perturbative QCD
   Lattice QCD
   Confinement
   Heavy Ion physics

2 - Asymptotic Freedom
   Perturbative QCD
   Basic concepts and results

3 - QCD in the LHC time
Measurements of $\alpha_s(m_Z)$

The official compilation due to Bethke is reproduced here:

The agreement among so many different ways of measuring $\alpha_s$ is a strong quantitative test of QCD

$\Lambda^{(5)}_{\overline{MS}} = (213 \pm 9) \text{ MeV}$

New preliminary Bethke value ‘11

$\alpha_s(m_Z) = 0.1183 \pm 0.0010$
However for some entries the stated errors are taken directly from the original works and are not transparent enough (e.g. the lattice determination)

In my opinion one should select few theoretically cleanest processes for measuring $\alpha_s$ and consider all other ways as tests of the theory

Note that in QED $\alpha$ is measured from one single very precise, very clean observable (at present the electron g-2)

The cleanest processes are the totally inclusive ones (no hadronic corrections) with light cone dominance, like $Z$ decay, scaling violations in DIS and perhaps $\tau$ decay (but for $\tau$ the energy scale is low)
The main inclusive methods for $\alpha_s$ at LEP/SLC are:

- inclusive $Z$ decay, $R_l$, $\sigma_l$, $\sigma_h$, $\Gamma_Z$
- inclusive $\tau$ decay

$$R_l, \tau = \frac{\Gamma(Z, \tau \rightarrow \text{hadrons})}{\Gamma(Z, \tau \rightarrow \text{leptons})} \approx R^{EW} (1 + \delta_{QCD} + \delta_{NP})$$

$\delta_{QCD}$ is known to (N)NNLO accuracy:

$$\delta_{QCD} = c_1\left(\frac{\alpha_s(Q)}{\pi}\right) + c_2\left(\frac{\alpha_s(Q)}{\pi}\right)^2 + c_3\left(\frac{\alpha_s(Q)}{\pi}\right)^3 + c_4\left(\frac{\alpha_s(Q)}{\pi}\right)^4 + \ldots$$

$\delta_{NP}$ are power suppressed $(1/Q^2)^n$ terms governed by the OPE.

Here $Q = m_Z$ or $m_\tau$

Clearly the $Z$ case is apriori more reliable because $m_Z >> m_\tau.$
Inclusive Z decays

(assuming the SM, \( m_{\text{exp}} \), \( m_{\text{HeExp}} \)):

\( R_l \) only (traditionally used for no good reason): \( \alpha_s(m_Z) = 0.1226 \pm 0.0038 \)

\( \sigma_l \) is more sensitive to \( \alpha_s \):

\( \alpha_s(m_Z) = 0.1183 \pm 0.0030 \)

Better, one can use all info from \( R_l, \Gamma_Z, \sigma_h, \sigma_l \) ... and in general take \( \alpha_s(m_Z) \) as a parameter to be fitted from the EW precision tests. One obtains (with only \( c_{1-3} \) included):

- LEP1 only: \( \alpha_s(m_Z) = 0.1187 \pm 0.0027 \)
- All EW Data (also \( m_W \) ...): \( \alpha_s(m_Z) = 0.1186 \pm 0.0026 \)

Apriori the main theor. errors are higher QCD orders (\( c_{4...} \)). Error from power corrections very small.

In addition, th. error from possible new physics (eg in Zbb vertex).
Inclusive hadronic Z and $\tau$ decay at $o(\alpha_s^4)$ (NNNLO!!)

Baikov, Chetyrkin, Kuhn ‘08
Baikov, Chetyrkin, Kuhn, Rittinger ‘12

$\sim 20,000$ diagrams

$o(\alpha_s^4)$ terms complete for $\tau$ and Z hadronic decay

For example, Z decay, $R = \Gamma_h/\Gamma_l$

$$R = R_0 \left[ 1 + a_s + 0.76264 \, a_s^2 - 15.49 a_s^3 - 68.2 a_s^4 + \ldots \right]$$

$n_f=5$, $a_s = \alpha_s(m_Z^2)/\pi$

Now no more significant error from higher orders!

Can be used to improve $\alpha_s$ from Z

$$\alpha_s(m_Z^2) = 0.1186 \rightarrow 0.1190 \pm 0.0025$$

Note that the error shown is dominated by the exp. errors.
For example having now fixed $m_H$ does not decrease the error significantly
\( \alpha_s \text{ from } R_\tau \)

\[
R_\tau = \frac{\Gamma(\tau \Rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \Rightarrow \nu_\tau + \text{leptons})}
\]

\( R_\tau \) has a number of advantages that, at least in part, compensate the smallness of \( m_\tau = 1.777 \ \text{GeV} \):

- \( R_\tau \) is even more inclusive than \( R_{e^+e^-}(s) \).

\[
R_\tau = \frac{1}{\pi} \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left( 1 - \frac{s}{m_\tau^2} \right)^2 \text{Im} \Pi_\tau(s)
\]

- one can use analiticity to go to \( |s| = m_\tau^2 \)

\[
R_\tau = \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left( 1 - \frac{s}{m_\tau^2} \right)^2 \Pi_\tau(s)
\]

- factor \((1-s/m_\tau^2)^2\) kills sensitivity to \( \text{Re } s = m_\tau^2 \) (thresholds)
Still the quoted result (by Bethke ’09) looks a bit too precise

Bethke’09 \( \alpha_s(m_Z) = 0.1197 \pm 0.0016 \)

This precision is obtained by taking for granted that corrections suppressed by \( 1/m_\tau^2 \) are negligible.

\[
R_\tau \sim R_\tau^0 [1 + \delta_{\text{pert}} + \delta_{\text{np}}]
\]

This is because in the massless theory:

\[
\delta_{\text{np}} = \frac{\text{ZERO}}{m_\tau^2} + c_4 \cdot \frac{<O_4>}{m_\tau^4} + c_6 \cdot \frac{<O_6>}{m_\tau^6} + \ldots
\]

In fact there are no dim 2 operators (e.g. \( g_\mu g^\mu \) is not gauge invariant) except for light quark \( m^2 \) (\( m \sim \text{few MeV} \) if parton quarks are relevant, \( m \sim \text{few 100 MeV} \) if constituents).

Most people believe that partons are relevant. I am not sure that the gap is not filled by ambiguities of \( o(\Lambda^2/m_\tau^2) \) from \( \delta_{\text{pert}} \).

eg effect of ultraviolet renormalons

GA, Nason, Ridolfi ’95; Chetyrkin, Narison, Zakharov ’98
The scaling violations of non-singlet str. functs. would be ideal: less dependence on input parton densities

$$\frac{d}{dt} \log F(x, t) = \frac{\alpha_s(t)}{2\pi} \int_x^1 dy \frac{F(y, t)}{yF(x, t)} P qq \left( \frac{x}{y} \right)$$

But

• for $F_p - F_n$ exp. errors add up in the difference,

• $F_{3\nu N}$ is not terribly precise

  \((\nu \text{ data only from CCFR, NuTeV})\)

• neglecting sea and glue in $F_2$ for $x > x_0$ decreases the sample, introduces a dependence on $x_0$ and an error from residual singlet terms.
Non singlet electron/muon production

From a recent analysis of eP and eD data, neglecting sea and gluons at $x > 0.3$ (error to be evaluated)

- Non singlet DIS: $\alpha_s(m_Z) = 0.1148 \pm 0.0019$ (exp) +? (NLO)
  $\alpha_s(m_Z) = 0.1134 \pm 0.0020$ (exp) +? (NNLO)

  Bluemlein, Bottcher, Guffanti ‘07

- a rather small central value
- not much difference between NLO and NNLO

According to G. Watt the contribution of singlet to $F_2$ at $x \sim 0.3$ is still $\sim 10\%$
BCDMS data push towards small $\alpha_s$.

According to Watt 162/280 exp points at $x > 0.3$ are dominated by BCDMS.
When one measures $\alpha_s$ from scaling viols. in $F_2$ from e or $\mu$ beams, data are abundant, exp. errors small but:

$$\alpha_s \leftrightarrow \text{gluon correlation} \quad \frac{dF}{d\log Q^2} \sim \alpha_s g$$

There is a strong feedback on $\alpha_s$ of the parametrisation of $g$. A too rigid param’n of gluon may strongly bias $\alpha_s$

The Neural Network approach suppresses $g$ parametrisation errors (The NNPDF Coll. ‘11)

DIS only $\alpha_s(m_Z)=0.1166\pm0.0008\,\text{(exp)} + 0.0009\,\text{(th)}\,\text{(NNLO)}$

Including Tevatron jets may be important to constrain $g$ at large $x$ (and then, via momentum conservation, also at small $x$). But jets rates only known at NLO accuracy

$\uparrow$ With jets and DY $\alpha_s(m_Z)=0.1173\pm0.0007\,\text{(exp)} + 0.0009\,\text{(th)}$
Recent $\alpha_s(m_Z)$ determinations from DIS at NNLO

$$\alpha_s(m_Z) = 0.1129 \pm 0.0014 \text{ (exp)}+?$$

Alekhin, Blumlein, Klein, Moch ‘09

$$\alpha_s(m_Z) = 0.1158 \pm 0.0035 \text{ (exp)}+?$$

Jimenez-Delgado, Reya ‘08

From combined H1+ZEUS data

$$\alpha_s(m_Z) = 0.1147 \pm 0.0012 \text{ (exp)}+?$$

Alekhin, Blumlein, Moch ‘10

Ambiguities:
- Heavy quarks
- $F_L$
- Higher orders

For HERA data the NLO evolution should be improved by a correct treatment of small $x$ effects (negative $g$ at small $x$ and $Q^2$ is a symptom)
Global fit to $\alpha_s$ and PDF

dominated by DIS but not only DIS

$$\alpha_s(m_Z) = 0.1171 \pm 0.0014\text{(exp)} + ? \quad \text{(NNLO)}$$

Martin, Stirling, Thorne, Watt ’09

MRST attribute their larger value of $\alpha_s$ to a more flexible parametrisation of the gluon and claim that the Tevatron jets are needed to fix $g$ at large $x$
In conclusion, for $\alpha_s(m_Z)$ from DIS

Bethke takes $\alpha_s(m_Z) = 0.1142 \pm 0.0023$ from non-singlet and this is what he puts in his average from DIS

recall: $\alpha_s(m_Z) = 0.1134 \pm 0.0020$ (exp) + ? (NNLO)

Bluemlein, Bottcher, Guffanti '07

Problems: neglect singlet at $x>x_0$, small data sample, BCDMS...

From the previous discussion it appears that for singlet there are problems related to the gluon determination and parametrization

$\alpha_s(m_Z)$ tends to slide towards low values if the g problem is not fixed [$\alpha_s(m_Z) \sim 0.113$-0.116]

The NNPDF approach or fixing the g on the Tevatron jets increases $\alpha_s(m_Z)$ [$\alpha_s(m_Z) \sim 0.117$

Still an open problem!

I would take from DIS: $\alpha_s(m_Z) = 0.116 \pm 0.002$ (NNLO)
Summarising

\[ \alpha_s(m_Z) = 0.1190 \pm 0.0025 \quad \text{(NNNLO)} \]

\[ \alpha_s(m_Z) = 0.1197 \pm 0.0016 \pm \? \quad \text{(NNLO)} \]

\[ \alpha_s(m_Z) = 0.116 \pm 0.002 \quad \text{(NNLO)} \]

Combining Z decay and DIS

\[ \boxed{\alpha_s(m_Z) = 0.1172 \pm 0.0016} \quad \text{my choice} \]

Adding the \( \tau \) (optimistically forgetting the extra th error)

\[ \alpha_s(m_Z) = 0.1184 \pm 0.0011 \]

Compare with Bethke \[ \alpha_s(m_Z) = 0.1183 \pm 0.0010 \]
The basic experimental set ups:
• no initial hadron (....LEP, ILC, CLIC)
• 1 hadron (....HERA, .... LHeC)
• 2 hadrons (....SppS, Tevatron, LHC)

Progress in particle physics needs their continuous interplay to take full advantage of their complementarity
Parton densities extracted from DIS are used to compute hard processes, via the Factorisation Theorem (FT):

\[ \sigma(s) = \sum_{A,B} \int dx_1 dx_2 p_A(x_1, Q^2) p_B(x_2, Q^2) \hat{\sigma}_{AB}(x_1, x_2, s, Q^2) \]

For example, at hadron colliders

- Very stringent tests of QCD
- Feedback on constraining parton densities

Is the FT proven?
In pert. theory up to NNLO has been explicitly checked to hold.
At all orders detailed studies only for DY

Collins, Soper, Sterman ‘85,’88
A large amount of theoretical work was devoted to directly prepare the interpretation of LHC experiments

• New and improved generators for event simulation
• Resummations
• New techniques for advanced QCD and EW calculations
• Calculations for signals, backgrounds and interpretation

e.g. the top quark FB asymmetry at the Tevatron has generated much work (axi-gluons, FC Z’…)
QCD event simulation  A big boost in view of the LHC

General algorithms for computer NLO calculations
the dipole  Catani, Seymour,....  FKS formalisms  Frixione, Kunszt, Signer Kosower....
the antenna pattern  Beyond general purpose HERWIG PYTHIA, SHERPA

Matching matrix elements and parton showers
LO ME: ALPGEN, MadGraph, MLM, (L)-CKKW
NLO ME: MC@NLO
POWHEG, MENLOPS

Mangano.....

Perturbative (+ resumm.s)
\[
d\sigma = A\alpha_s^N \left[ 1 + (c_{1,1}L + c_{1,0})\alpha_s \\
+ (c_{2,2}L^2 + c_{2,1}L + c_{2,0})\alpha_s^2 + \ldots \right]
\]
L= large log eg L=log(p_T/m)

Complementary virtues:
the hard skeleton plus the shower development and hadronization

On going progress in automatisation

Parton showers
collinear emissions factorize
\[
d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \times \frac{\alpha_s}{2\pi} \frac{dt}{t} P_{qg}(z)dz\frac{d\varphi}{2\pi}
\]
\[
t = (p_q + p_g)^2 \rightarrow 0
\]
Resummation of large logs

Beyond the RGE $[\log Q^2/\mu^2]$ there are often other large logs $L$

Examples of $L$:

$\log p_T^2/Q^2$ in $p_T$ distrib.'ns for $W$, $H$ (Sudakov logs)
$\log 1/x$ for small $x$ structure functions in DIS
$\log 1/(1-x)$ Thrust distributions, large $x$ in DIS.......

When $\alpha_s(Q^2)L^2$ or $\alpha_s(Q^2)L$ are large, the sequences
$(\alpha_s(Q^2)L^1$ or $2)^n$ have to be resummed (the LL or NLL coefficients can often be computed to all orders).

Leading logarithmic
Important recent work on jet recombination algorithms

G. Salam et al

SISCone, anti-$k_T$

Cacciari
Zanderighi

It is essential that a correct jet finding is implemented by LHC experiments for an optimal matching of theory and experiment
Singlet splitting function at small $x$

The problem of correctly including BFKL at small $x$ has been solved by Ciafaloni, Colferai, Salam, Stasto (CCSS) and Altarelli, Ball, Forte (ABF).

Momentum cons. + symmetry + running coupling effect

→ soft simple pole in anom. dim

• BFKL sharp rise tamed

• resummed result close to NLO in HERA region

• new expansion stable

Makes the ground solid for LHC predictions (eg b production)
QCD for LHC: very difficult calculations needed

New powerful techniques for loop calculations

Basic idea: Loops can be fully reconstructed from their unitarity cuts

First proposed by Bern, Dixon, Kosower ‘93-‘97
Revived by Britto, Cachazo, Feng ’04
Perfected by Ossola, Papadopoulos, Pittau ’06

Generalized d-dimension unitarity
K. Ellis, Giele, Kunszt, Melnikov ‘08-’09

A review:
One-loop calculations in quantum field theory:
from Feynman diagrams to unitarity cuts
K. Ellis, Kunszt, Melnikov, Zanderighi ArXiv: 1105.4319
The Industrial Age of NLO

- In recent years, much reference to “NLO revolution”
- development of new wave of tools in anticipation of LHC
- especially numerical techniques: straightforward generation of new results for complicated final states
- 2011-12: time for putting these revolutionary ideas to work
Examples of recent NLO calculations in pp collisions

- \texttt{ttbb} Bredenstein et al ‘09–’10, Bevilacqua et al ‘09
- \texttt{ttW} K. Ellis, Campbell ’12
- \texttt{W+3jets} Berger et al ‘09, R.K.Ellis, Melnikov, Zanderighi ‘09,
- \texttt{Z,γ* +3jets} Berger et al ‘10
- \texttt{WW+2jets} Melia et al ‘10–’11, Jager, Zanderighi ‘12
- \texttt{WWbb} Denner et al ‘10
- \texttt{tt+2jets} Bevilacqua et al ‘10–’11
- \texttt{bbbb, jjjj} Greiner et al ‘11, Bern et al ‘11
- \texttt{W, Z+4jets} Berger et al ‘11, Bern et al ‘12; \texttt{W+5jets} Bern et al ‘12

And the Higgs cross section and distributions are known to NNLO Harlander, Kilgore ‘02; Anastasiou, Melnikov ‘02; Ravindran et al ‘03; Anastasiou, Melnikov, Petriello ‘04, Bozzi et al ‘07

A terrific amount of work by QCD theorists for LHC
Parton densities extracted from DIS (with feedback from other hard processes) are available for further use.

Fig. 19: Parton distributions by the MRST group.
NNPDF: R. Ball et al ‘08

xΣ

xg

Neural Network pdf
greater dependence on parametrization.
a large ensemble of pdf allowed

Uncertainties larger than for
CTEQ, MRST, Alekhin
in unmeasured region

M. Ubiali

J. Rojo
Jet Production in $pp$ or $pp^{\text{bar}}$ interactions

$p_1p_2$-> jet +X: all scalar products large
$(p_1+p_2)^2=s$; $(p_{1,2}-\text{jet})^2=t,u$ also large -> the jet must be at large $p_T$

NLO QCD fits
no free parameters except exp. norm’n
Note: many orders of magnitude!
W, Z and Drell-Yan lepton pair production at hadron colliders.

\[ P \rightarrow P_A + P_B \sim \gamma, W, Z \]

\[ O(1): \text{Drell, Yan}; O(\alpha_s): \text{Altarelli, K.Ellis, Martinelli; Kubar-Andre, Paige}; O(\alpha_s^2): \text{Hamberg, van Neerven, Matsuura+Zijestra} \]
1979

The K-factor paper

The first NLO calculation in QCD
The prediction for $\sigma B_{W,Z}$ is obtained using parton densities from DIS, the measured $\Lambda$ and Br. ratios from the EW theory.

$p_T$ distribution has also been a classic laboratory...

---

see later
An important task: preparing the optimal pdf’s for the LHC

Dedicated groups
MSTW, CTEQ, NNPDF, HERAPDF,.....
tt^{\bar{b}} cross section known to NNLO plus resummation of soft Coulomb effects

The mass dependence of $\sigma$ can measure $m_{\text{top}}$

Beneke et al '11, '12
Ahrens et al '11

Barnreuther, Czakon, Mitov '12
The Higgs cross sections and distributions are at the center of the stage now

see for a review

Handbook of LHC Higgs cross sections
Dittmaier, Mariotti, Passarino and Tanaka editors
ArXiv 1101.0593,1201.3084
Higgs production via g+g → H

Very important for the LHC

Effective lagrangian (m_t -> infinity)

\[ \mathcal{L} = C_1 H G^{\mu\nu} G_{\mu\nu} \]

C_1 known to \( \alpha_s^4 \)

Chetyrkin, Kniehl, Steinhauser’97

NLO corr.s computed with effective lagrangian

Dawson
Djouadi, Spira, Graudenz, Zerwas

AND the full theory

Djouadi, Spira, Graudenz, Zerwas

They agree very well
More recently the NNLO calculation was completed (analytic)

Catani, de Florian, Grazzini ’01.
Harlander, Kilgore ’01, ’02
Anastasiou, Melnikov’02
Ravindran, Smith, van Neerven ’03

Also NLO $y$ and $p_T$ distributions have been computed

De Florian, Grazzini, Kunszt ‘99
Glosser, Schmidt’02
Anastasiou, Melnikov, Petriello’05
Ravindran, Smith, van Neerven’06

Recent progress:
Resummation of large partonic-energy logs

DeMarzani, Ball, Del Duca, Forte, Vicini’08
# Higgs-related advances

## Signals

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Higgs $p_T$ distribution: $[\log(p_T/m_H)]^n$ resummed

Bozzi, Catani, De Florian, Grazzini’03-’08

Figure 7. Resummed pQCD prediction for the Higgs transverse momentum distribution at the LHC, from Bozzi et al. [25]
~28 years ago at CERN we computed the $W$ and $Z$ \( p_T \) distribution in QCD

GA, K.Ellis, M. Greco, G.Martinelli '84

Here all relevant ingredients were first assembled and matched. Later mainly refinements were added.
The avantgarde of contemporary QCD research

N=4 SUSY QCD and AdS/CFT correspondence

N=4 SUSY QCD has $\beta(\alpha) = 0$ and is loop finite

In limit $N_c \to \infty$ with $\lambda = g^2 N_c$ fixed, planar diagrams are dominant

The large $\lambda$ limit corresponds by AdS/CFT duality to the weakly coupled string (gravity) theory on AdS$_5 \times$S$_5$

There is progress towards a solution of planar N=4 SUSY QCD amplitudes

N = 8 Supergravity, related to N = 4 SUSY Yang-Mills, has been proven finite up to 4 loops. It could possibly lead to a finite field theory of gravity in 4 dimensions

see L. Dixon talk at ICHEP ’12
Conclusion

QCD is a non abelian unbroken gauge quantum field theory of fundamental physical relevance

Its physics content is very large and our knowledge esp. in the non perturbative domain is still very limited but progress both from experiment (HERA, Tevatron, RHIC, LHC) and from theory is continuous

Very good agreement with experiment
EXTRA
**Why SU(N_C=3) Colour?**

**Observed:** hadrons

**Colour singlets**

**The group must:**

• admit complex reprs.: q different from q*

(qq must not be a singlet) SU(n>2), SO(4n+2), E(6)

(ln SU(3)q~3, q*~3*, 3X3=6+3*, 3X3*=8+1)

• allow a totally antisymm. qqq singlet

(qqq is totally symm. in space [s-wave], spin and SU(3) [SU(6) ~56])

(ln SU(3) ε_{abc}q^aq^bq^c ~ 1)
All observed hadrons are colourless composites of quarks:

- **Baryons**: $qqq$
- **Mesons**: $q\bar{q}$

For example:
- Proton $p$: $uud$
- Pion $\pi^+$: $ud$

**Colour is essential for Fermi statistics**

The state $\Delta^{++}$ with spin 3/2

\[
\begin{align*}
    u \uparrow u \uparrow u \uparrow
\end{align*}
\]

is symmetric in space and spin but antisymmetric in colour and for explaining the observed spectrum.

For example: the "decuplet"

\[
\begin{align*}
    &\Delta & ddd & ddu & duu & uuu \\
    &\Sigma & dds & dus & uus & \\
    &\Xi & dss & uss & \\
    &\Omega & sss
\end{align*}
\]

**Hadron spectroscopy**
Confinement explains why the nuclear forces are short range while massless gluon exchange would be long range:

Nucleons are colour singlets: they can only exchange colour singlets (pions not gluons)

\[ V \sim \exp(-m_\pi r)/r \]

The range of nuclear forces is determined by the pion mass:
\[ r \sim m_\pi^{-1} \sim 10^{-13} \text{ cm} \]
SU(\(N_C\)): many processes measure \(N_C\)

**Examples:**

- \(R = \sigma(e^+ e^- \rightarrow \text{hadrons}) / \sigma(e^+ e^- \rightarrow \mu^+ \mu^-)\)

**Above \(b\bar{b}\) threshold and below \(m_Z\):**

\[
R = N_C \cdot \left[ 3 \cdot \frac{1}{9} + 2 \cdot \frac{4}{9} \right] = \left( N_C \cdot \frac{11}{9} \right)
\]

(Computable small rad corr's neglected)

\(\text{d}, \text{s}, \text{b} \quad \text{u}, \text{c}\)

\(e^- \quad \gamma \quad q \quad q^*\)

\(q = \text{d}, \text{s}, \text{b}, \text{u}, \text{c}\)

\(11/3\)
\[ B(\tau \to e\nu\nu) \sim \frac{1}{2 + N_C} \]

\( N_C = 3 \rightarrow B \approx 20\% \)

Exp.: \( B \approx 18\% \)

Tree level!

\[ f = e, \mu, u \]

\[ B(W \to e\nu) \sim \frac{1}{3 + 2 \cdot N_C} \]

\( N_C = 3 \rightarrow B \approx 11\% \)

Exp.: \( B \approx 10.7\% \)

\[ f = e, \mu, \tau, u, c \]

\[ \Gamma(\pi^0 \to \gamma\gamma) \sim \left( \frac{N_C}{3} \right)^2 \frac{\alpha^2 m_{\pi^0}^3}{32\pi^3 f_{\pi}^2} = (7.73 \pm 0.04) \left( \frac{N_C}{3} \right)^2 \text{eV} \]

Exp.: \((7.7\pm0.5)\text{ eV} \)

\[ f_\pi = (130.7\pm0.37)\text{MeV} \]

(PDG2000)
Infrared and collinear safety

\[
\text{Propagator} = \frac{1}{(p+k)^2 - m^2} = \frac{1}{2(pk)} = \\
= \frac{1}{2E_k E_p} \cdot \frac{1}{1 - \beta_p \cos \theta}
\]

- \(E_k \to 0\) \quad \text{infrared singularity.}

- For \(m \to 0\), \(\beta_p = \sqrt{1 - \frac{m^2}{E_p^2}} \to 1\) and \(1 - \beta_p \cos \theta\) vanishes at \(\cos \theta = 1\)

\text{collinear (mass) singularity}
Summarising: we started from the massless classical theory and we ended up with QCD where an energy scale $\Lambda=\Lambda_{QCD}$ appears.

$\Lambda$ depends on the def. of $\alpha_s$ (i.e. the reg. procedure, the ren. scheme…) and on the number of excited flavours $n_f$.

Definition of $\alpha_s$

We have introduced the ren. coupling $\alpha_s$ in terms of the 3-g ren. vertex at $p^2=-\mu^2$ (momentum subtraction). The value of $\alpha_s$ (hence $\Lambda$) in this scheme depends on $\mu$.

But the most common def. of $\alpha_s$ is in the framework of dimensional reg.

Dim. reg. is a gauge and Lorentz inv. reg. that is most simply implemented in calculations. It consists in formulating the theory in $d<4$ space-time dimensions.
Dimensional Regularisation (DR)

Rewrite the theory in d (integer) dim. Expression of diagrams also OK for any d.

\[ g_{\mu\nu} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \quad (d \times d) \]

\[ k^\mu = (k^0, k^1, \ldots, k^{d-1}) \]

Dirac \( \gamma^\mu f(d)\gamma^\nu f(d) \)

\[ \text{Tr} \, \gamma^\mu \gamma^\nu = f(d) g^{\mu\nu} \]

For d<4 loop integrals less divergent.

\[ I = \int \frac{1}{k^2 (p-k)^2} \, d^d k \]

The coupling carries dimensions: \( e_d = \mu^\varepsilon \epsilon \)

(d=4-2\varepsilon; this is how a mass scale enters!)

\[ \psi \propto m^{\frac{d-1}{2}} \]

\[ \phi, A_\nu \propto m^{\frac{d-2}{2}} \]

\[ e \propto m^{\frac{4-d}{2}} \]
The formal expression of loop integrals can be written for all d. For example:

\[
\int \frac{1}{(k^2 - m^2)^2} \frac{d^d k}{(2\pi)^d} = \frac{\Gamma\left(2 - \frac{d}{2}\right)(-m^2)^{\frac{d}{2} - 2}}{(4\pi)^{\frac{d}{2}}}
\]

For \(d=4-2\varepsilon\) we can expand, using:

\[
\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + 0(\varepsilon) \quad \gamma_E = 0.5772...
\]

For some quantity we obtain from diagrams

\[
G = 1 + \alpha \left( \frac{\mu^2}{-p^2} \right)^\varepsilon \left[ B\left(\frac{1}{\varepsilon} + \log(4\pi) - \gamma_E\right) + A + 0(\varepsilon) \right]
\]

In MS we write this as (diagram by diagram):

\[
Z = 1 + \alpha \left[ B\left(\frac{1}{\varepsilon} + \log(4\pi) - \gamma_E\right) \right]
\]

\[
G = Z G_R
\]

\[
G_R = 1 + \alpha \left[ B \log \frac{\mu^2}{-p^2} + A \right]
\]
Consider first the case $\gamma(\alpha)=0$.
This is not unphysical: it occurs for $R_{e^+e^-}$.
Recall that $\gamma(\alpha)=d\log Z/d\log \mu^2$. It is zero because QCD corr's cannot
renormalise the electric charge (or the proton and positron
charges would be different)

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial \alpha}\right] \cdot F(t, \alpha) = 0$$

The solution is $F[0,\alpha(t)]$, where the "running coupling" $\alpha(t)$ is
defined by:

$$t = \int_\alpha^{\alpha(t)} \frac{1}{\beta(\alpha')} d\alpha'$$

Take $d/dt$ and $d/d\alpha$ of both sides:

$$\begin{align*}
(d/dt) & \quad 1 = \frac{1}{\beta(\alpha(t))} \frac{\partial}{\partial t} \alpha(t) \\
(d/d\alpha) & \quad 0 = -\frac{1}{\beta(\alpha)} + \frac{1}{\beta(\alpha(t))} \frac{\partial}{\partial \alpha} \alpha(t)
\end{align*}$$
We have found

\[ \frac{\partial}{\partial t} \alpha(t) = \beta(\alpha(t)) \quad ; \quad \frac{\partial}{\partial \alpha} \alpha(t) = \frac{\beta(\alpha(t))}{\beta(\alpha)} \]

Using these eqs. we check that

\[ F(t, \alpha) = F[0, \alpha(t)] \]

is the solution (note that \( \alpha(0) = \alpha \), so that the boundary cond. is satisfied)

\[ \left[ -\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} \right] \cdot F(0, \alpha(t)) = 0 \]

With \( F' = dF(0, \alpha)/d\alpha \), we have:

\[ \left[ -\frac{\partial}{\partial t} \alpha(t) + \beta(\alpha) \frac{\partial}{\partial \alpha} \alpha(t) \right] F' = \]

\[ = \left[ -\beta(\alpha(t)) + \beta(\alpha) \cdot \frac{\beta(\alpha(t))}{\beta(\alpha)} \right] F' = 0 \]
Similarly for the more general equation:

\[-\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma(\alpha) \cdot F(t, \alpha) = 0\]

The solution is:

\[F[0, \alpha(t)] \exp \int_{\alpha}^{\alpha(t)} \frac{\gamma(\alpha')}{\beta(\alpha')} d\alpha'\]

as can be easily checked given that:
- the differential operator applied to \(F[0, \alpha(t)]\) vanishes
- the exponential is by itself a solution of the complete equation.

**Summary:** The important point is the appearance of the running coupling that determines the asymptotic behaviour.
\( \gamma\text{-N cross-section} \)

\[
\frac{d\sigma}{dQ^2\,d\nu} = \frac{4\pi\alpha^2E'}{Q^4E} \left[ 2\left(\sin\frac{\theta}{2}\right)^2 W_1 + \left(\cos\frac{\theta}{2}\right)^2 W_2 \right]
\]

\( \nu\text{-N (\nu\text{-N}) cross-section} \)

\[
\frac{d\sigma^\nu, \bar{\nu}}{dQ^2\,d\nu} = \frac{G_F^2E'}{2\pi E} \left(\frac{m_W^2}{Q^2 + m_W^2}\right)^2 \left[ 2\left(\sin\frac{\theta}{2}\right)^2 W_1^\nu + \left(\cos\frac{\theta}{2}\right)^2 W_2^\nu + \frac{E + E'}{m} \left(\sin\frac{\theta}{2}\right)^2 W_3^\nu \right]
\]

Scaling limit: \( Q^2 \gg m^2 \quad \text{x fixed} \)

\[
\begin{align*}
\ mW_1(Q^2,\nu) &\to F_1(x) & \text{Bjorken} \\
\nu W_2(Q^2,\nu) &\to F_2(x) \\
\nu W_3(Q^2,\nu) &\to F_3(x)
\end{align*}
\]
In the scaling limit the following relations with the cross sections of the fixed-helicity gauge bosons ($\gamma, W^{\pm}$) hold:

$\sigma_L$: longitudinal $\rightarrow$ helicity = 0
$\sigma_{RH}$: right-handed $\rightarrow$ helicity = +1
$\sigma_{LH}$: left-handed $\rightarrow$ helicity = -1

$[\sigma_T = \sigma_{RH} + \sigma_{LH} : \text{transverse}]$

$$\sigma_L = \frac{2\pi}{s} \left[ \frac{F_2(x)}{2x} - F_1(x) \right]$$

$$\sigma_{RH} = \frac{2\pi}{s} \left[ F_1(x) + \frac{1}{2} F_3(x) \right]$$

$$\sigma_{LH} = \frac{2\pi}{s} \left[ F_1(x) - \frac{1}{2} F_3(x) \right]$$
“Naïve” parton model

Bjorken & Feynman language: The virtual $\gamma$ sees the quark partons inside the proton as quasi-free because the QCD interaction time (Lorentz dilated) is much longer than $\tau_\gamma \sim 1/Q$

Breit frame: in this frame $E\gamma=0$:

$q = (0;0, 0, Q)$
$p = \left(\frac{Q}{2x};0, 0, -\frac{Q}{2x}\right)$

Note: $x=Q^2/2(pq)$

Take a parton with 4-mom $p_q = yp$. Since $E\gamma=0$, the quark momentum is reversed: $y=x$.

$\sigma_{\text{point}} \sim e^2 \delta(x/y-1)$
$2F_1 = e^2q_0(x)$

Spin 1/2 partons: $\sigma_{l}=0$
Spin 0 partons: $\sigma_{t}=0$
The calculation (in a nut shell)

- Calculate anomalous dimensions (Mellin moments of splitting functions)
  \[\gamma_{ij}^{(n)}(N) = - \int_0^1 dx \ x^{N-1} \ P_{ij}^{(n)}(x)\]

- **One-loop** Feynman diagrams
  \(\gamma_{ij}^{(0)} / P_{ij}^{(0)}\)
  (pencil + paper)

- **Two-loop** Feynman diagrams
  \(\gamma_{ij}^{(1)} / P_{ij}^{(1)}\)
  (simple computer algebra)

- **Three-loop** Feynman diagrams
  \(\gamma_{ij}^{(2)} / P_{ij}^{(2)}\)
  (cutting edge technology \(\rightarrow\) computer algebra system FORM Vermaseren '89-'04)
NLO singlet splitting functions

\[ P_{\bar{q}s}^{(0)}(x) = 0 \]
\[ P_{\bar{q}s}^{(0)}(x) = 2n_f p_{\bar{q}s}(x) \]
\[ P_{\bar{q}s}^{(0)}(x) = 2C_F p_{\bar{q}s}(x) \]
\[ P_{\bar{s}s}^{(0)}(x) = C_A \left( 4p_{\bar{q}s}(x) + \frac{11}{3} \delta(1-x) \right) - \frac{2}{3} n_f \delta(1-x) \]

\[ P_{\bar{s}s}^{(1)}(x) = 4C_F n_f \left( \frac{20}{9} \frac{1}{x} - 2 + 6x - 4H_0 + x^2 \left[ \frac{8}{3} H_0 - \frac{56}{9} \right] + (1+x) \left[ 5H_0 - 2H_{0,0} \right] \right) \]
\[ P_{\bar{s}s}^{(1)}(x) = 4C_A n_f \left( \frac{20}{9} \frac{1}{x} - 2 + 25x - 2p_{\bar{q}s}(\bar{x})H_{1,0} - 2p_{\bar{q}s}(x)H_{1,1} + x^2 \left[ \frac{44}{3} \frac{1}{x} - \frac{218}{9} \right] + 4(1-x) \left[ H_{0,0} - 2H_0 + xH_1 \right] - 4\zeta_2 x - 6H_{0,0} + 9H_0 \right) + 4C_F n_f \left( 2p_{\bar{q}s}(x) \left[ H_{1,0} + H_{1,1} + H_2 - \zeta_2 \right] + 4x^2 \left[ H_0 + H_{0,0} + \frac{5}{2} \right] + 2(1-x) \left[ H_0 + H_{0,0} - 2xH_1 + \frac{29}{4} \right] - \frac{15}{2} \right) \right) \]
\[ P_{\bar{s}s}^{(1)}(x) = 4C_A C_F \left( \frac{1}{x} + 2p_{\bar{q}s}(x) \left[ H_{1,0} + H_{1,1} + H_2 - \frac{11}{6} H_1 \right] - x^2 \left[ \frac{8}{3} H_0 - \frac{44}{9} \right] + 4\zeta_2 - 2 - 7H_0 + 2H_{0,0} - 2H_1 x + (1+x) \left[ 2H_{0,0} - 5H_0 + \frac{37}{9} \right] - 2p_{\bar{q}s}(\bar{x})H_{1,0} \right) - 4C_F n_f \left( \frac{2}{3} x \right) \]
\[ - \frac{p_{\bar{q}s}(x)}{3} \left[ \frac{1}{3} H_1 - \frac{10}{9} \right] \right) + 4C_F \left( p_{\bar{q}s}(x) \left[ 3H_1 - 2H_{1,1} \right] + (1+x) \left[ H_{0,0} - \frac{7}{2} + \frac{7}{2} H_0 \right] - 3H_{0,0} + 1 - \frac{2}{3} H_0 + 2H_1 x \right) \]

\[ P_{\bar{s}s}^{(1)}(x) = 4C_A n_f \left( 1 - x - \frac{10}{9} P_{\bar{q}s}(x) - \frac{13}{9} \left( \frac{1}{x} - x^2 \right) - \frac{2}{3} (1+x) H_0 - \frac{2}{3} \delta(1-x) \right) + 4C_A \left( \frac{27}{9} \right) + (1+x) \left[ \frac{11}{3} H_0 + 8H_{0,0} - \frac{27}{2} \right] + 2p_{\bar{q}s}(\bar{x}) \left[ H_{0,0} - 2H_{1,0} - \zeta_2 \right] - \frac{67}{9} \left( \frac{1}{x} - x^2 \right) - 12H_0 \]
\[ - \frac{44}{3} x^2 H_0 + 2p_{\bar{q}s}(x) \left[ \frac{67}{18} - \zeta_2 + H_{0,0} + 2H_{1,0} + 2H_2 \right] + 8(1-x) \left[ \frac{8}{3} + 3\zeta_3 \right] \right) + 4C_F n_f \left( \frac{2}{3} x + \frac{2}{3} x^2 - 12 + (1+x) \left[ 4 - 5H_0 - 2H_{0,0} \right] - \frac{1}{2} \delta(1-x) \right). \]
NNLO singlet splitting functions
- Exact result, estimates from fixed moments and leading small-$x$ term
- Splitting function $P_{gq}^{(2)}$ (left) and $P_{gg}^{(2)}$ (right)
\[
\frac{d}{dt} q(x, t) = \frac{\alpha_s(t)}{2\pi} \int_x^1 \frac{dy}{y} q(y, t) P\left(\frac{x}{y}\right) = \frac{\alpha_s(t)}{2\pi} [q \otimes P](x, t)
\]

(Mellin) Moments:

\[q_n = \int_0^1 q(x)x^{n-1} \, dx \quad \quad \quad P_n = \int_0^1 P(x)x^{n-1} \, dx\]

Taking moments of both sides

\[\frac{d}{dt} q_n(t) = \frac{\alpha_s(t)}{2\pi} \cdot P_n \cdot q_n(t)\]

A much simpler equation!

Proof: \[
\int_0^1 dx x^{n-1} \int_x^1 dy \frac{q(y, t)}{y} P\left(\frac{x}{y}\right) = \int_0^1 dy \frac{q(y, t)}{y} y^n \int_0^1 dz z^{n-1} P(z)
\]

PDF or structure function moments \(M_n(t, \alpha_s)\) obey RGE (\(q_n\) is a particular case).

\[
\left[-\frac{\partial}{\partial t} + \beta(\alpha) \frac{\partial}{\partial \alpha} + \gamma_n(\alpha)\right] \cdot M_n(t, \alpha) = 0
\]
RGE general solution:

\[ M_n(t, \alpha_s) = c_n(0, \alpha_s(t)) \exp^{\int_{\alpha_s}^{\alpha_s(t)} \frac{\gamma_n(\alpha')}{\beta(\alpha')} d\alpha'} \cdot O_n(\alpha_s) \]

In lowest order, applied to \( q_n \), we have:

\[ \gamma_n(\alpha) \approx \frac{P_n}{2\pi} \alpha + ... \]
\[ \beta(\alpha) \approx -b \alpha^2 + ... \]

\[ q_n(t) = q_n(0) \exp^{\int_{\alpha_s}^{\alpha_s(t)} \frac{\gamma_n(\alpha')}{\beta(\alpha')} d\alpha'} \approx \left[ \frac{\alpha_s}{\alpha_s(t)} \right]^{\frac{P_n}{2\pi b}} q_n(0) \]

This is exactly the solution of

\[ \frac{d}{dt} q_n(t) = \frac{\alpha_s(t)}{2\pi} \cdot P_n \cdot q_n(t) \]

with boundary cond. at \( t=0: q_n(0) \)

Gross, Wilczek; Politzer
Scaling violations in DIS

The scaling violations are clearly observed and the (N)NLO QCD fits are remarkably good.

These fits provide

- an impressive set of QCD tests
- measurements of $q(x, Q^2)$, $g(x, Q^2)$
- measurements of $\alpha_s(Q^2)$

\[
\begin{align*}
\frac{\partial q_i(x, Q^2)}{\partial \log Q^2} &= \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left\{ P_{q_i q_j}(y, \alpha_s) q_j\left(\frac{x}{y}, Q^2\right) + P_{q_i g}(y, \alpha_s) g\left(\frac{x}{y}, Q^2\right) \right\} \\
\frac{\partial g(x, Q^2)}{\partial \log Q^2} &= \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} \left\{ P_{g q_j}(y, \alpha_s) q_j\left(\frac{x}{y}, Q^2\right) + P_{g g}(y, \alpha_s) g\left(\frac{x}{y}, Q^2\right) \right\}
\end{align*}
\]

GLAP
HERA is a main source of information on pdf’s for LHC
Different fits to same DIS data are comparable

HERA LHC Workshop ’06

$\frac{xd_V}{Q^2 = 20 GeV^2}$

$xg$

$Q^2 = 20 GeV^2$
But differ from those obtained from all the data

This shows that extrapolation from one data set to another is dangerous
Fantastic technical skill!!

Essential for the LHC