STRING AMPLITUDES with COHERENT VERTEX OPERATORS and FIXED LOOP MOMENTA

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(CORFU XVIII Eurostrings 2012)

* with E. Copeland & P. Saffin (in progress)

Objectives

- To obtain better understanding of **string interactions** for **massive strings** (beyond leading Regge trajectories)

Applications:

→ fundamental cosmic strings

→ black holes

→ AdS/CFT *

* along lines of Minahan (?) JHEP 1207 (2012) 187

Objectives II

.. In particular, would like to compute

Here! **power per unit solid angle** in direction $\hat{\mathbb{P}}^\mu$ from macroscopic fundamental strings (coherent vertex operators!):

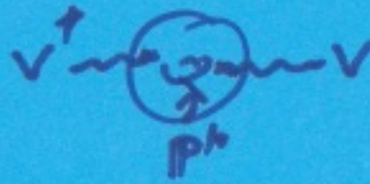
Power

$$\boxed{\frac{dP}{d\Omega_{S^{D-2}}} = \frac{1}{(2\pi)^D}\,\frac{2\pi\alpha'}{L}\int d\mathbb{P}^0\,\mathbb{P}^0\,|\underline{\mathbb{P}}|^{D-2}\,\mathrm{Im}\,\mathcal{M}(\mathbb{P})}$$

D non-compact spacetime dimensions

length of macroscopic string, $L = 4\pi\sqrt{\alpha'}\,\ell$

Loop momenta



Classical Strings

.. Will study embeddings of the form: $\boxed{X : \underset{\substack{\uparrow \\ \text{worldsheet}}}{\Sigma} \to \underbrace{\mathbb{R}^{D-1,1} \times T^{26-D}}_{\text{spacetime}}}$

Burden solutions*

$$X^{\pm}(z,\bar{z}) = -ip^{+} \ln|z|^2$$

$$X^{i}(z,\bar{z}) = \frac{i}{n}\left(\lambda_n^i z^{-n} - \lambda_n^{*i} z^{n}\right) + \frac{i}{m}\left(\bar{\lambda}_m^i \bar{z}^{-m} - \bar{\lambda}_m^{*i} \bar{z}^{m}\right) \qquad (i = 1, \ldots, D-2) \quad (\mathbb{R}^{D-2})$$

$$X^{a}(z,\bar{z}) = 0 \qquad (a = D, \ldots, 25) \quad (T^{26-D})$$

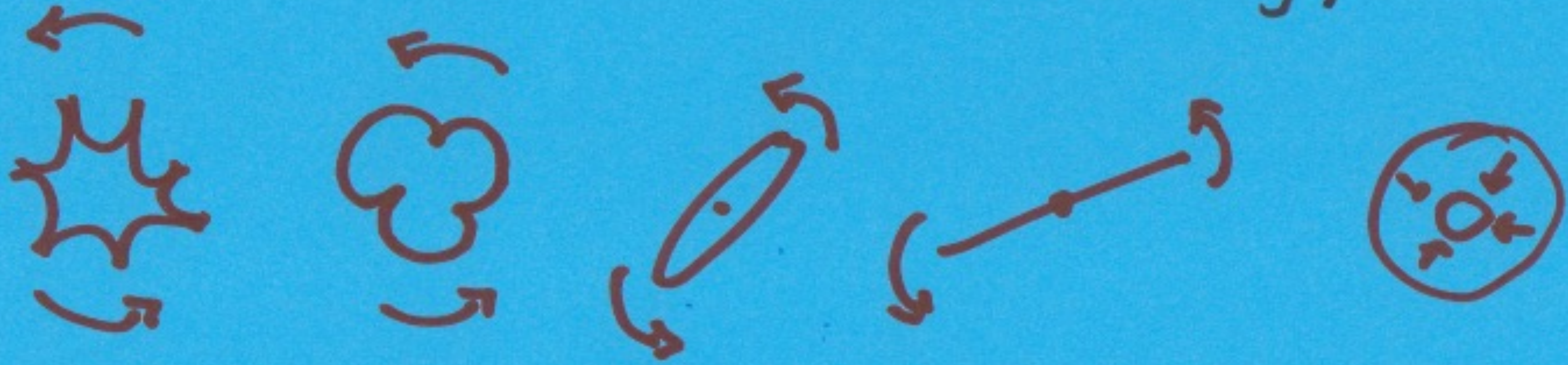
with:

$$\lambda_n \cdot \lambda_n = \bar{\lambda}_m \cdot \bar{\lambda}_m = 0 \quad \& \quad p^{+} = |\lambda_n| = |\bar{\lambda}_m| = \underbrace{\ell}_{\text{dimensionless string length}}$$

* Burden, PL 164B (1985) 277

Classical Strings II

... that is, hypocycloids, epicycloids, pulsating ellipses, folded string, ...



... classical strings generically exhibit cusps! Cusps produce very strong gravitational radiation ...

STRING THEORY COMPUTATION

of $\frac{dP}{d\Omega}$ for Burden

class of solutions

$\longrightarrow$

State of the Art

o **Decay Rates of massive strings:**

(mass eigenstates)

Green & Veneziano.

Wilkinson, Mitchel & Turok 89–90

Manes

Iengo & Russo

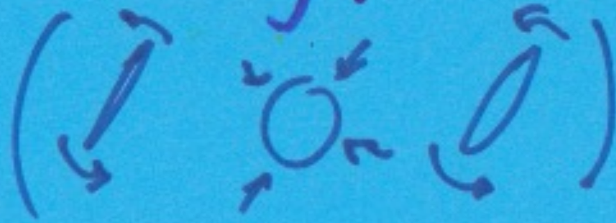
Chialva, Iengo & Russo.

Gutperle & Krym

D'Hoker & Phong

⋮ ⋮

... leading Regge trajectory only!

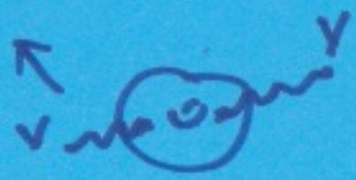


... higher harmonics neglected!

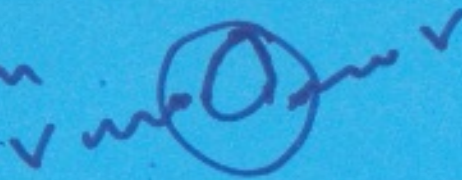
Decay Rates & Power

- Unitarity, $SS^{\dagger} = \mathbb{1}$, implies the power, P, is:

$$\boxed{\frac{dP}{d\Omega_{S^{D-2}}} = \frac{1}{(2\pi)^D}\,\frac{2\pi\alpha'}{L}\int d\mathbb{P}^0\,\mathbb{P}^0\,|\underline{\mathbb{P}}|^{D-2}\,\mathrm{Im}\,\mathcal{M}_{n,m}(\mathbb{P})}$$



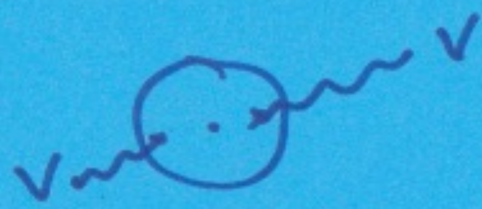
$\sim \int d\mathbb{P}^0 \ldots \Big[$ Im



pinched A-cycle

focus here

+ Im



pinched B-cycle

+ Im

$\Big]$

tadpole

Field theory

A-cycle, B-cycle } Im

tadpole }

+

Starting Point:

$= \int d^D p$

• 2-Point S-Matrix element: $S_{fi} = \delta_{fi} + A_{fi}$,

$$A_{fi} = \frac{1}{2} \int d^D p \int d(\text{moduli}) \int \mathcal{D}(\text{ghosts})\, e^{-S_G}$$

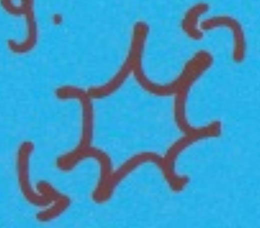
$$\times \int \mathcal{D}X\, e^{-S_X}\, \delta^D\!\left(p^\mu - \frac{1}{2\pi\alpha'} \oint_A \partial X^\mu - \bar{\partial} X^\mu\right) V^\dagger V$$

(c) ghosts

(b) Polyakov

(a) coherent vertex operators

(e.g.

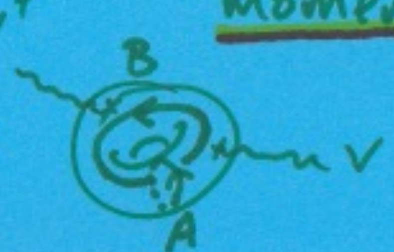


or

)

fixed A-cycle loop momenta

(T^2 :



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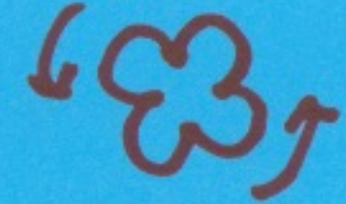
Why Coherent Vertex Operators?

- All mass eigenstates can be extracted from coherent states
- Very easy to vary size & energy of states in amplitudes*
- Easier combinatorics!

* natural setup for cosmic strings & B.H. studies

Coherent Vertex Operators*



$$V(z,\bar{z}) = \mathcal{N}\int ds\; \exp\left\{\frac{1}{n}\, e^{ins}\, \lambda_n\cdot P_n\, e^{-inq\cdot X(z)}\right\} e^{ip\cdot X(z)}$$

$$\times \exp\left\{\frac{1}{m}\, e^{-ims}\, \bar{\lambda}_m\cdot \bar{P}_m\, e^{-imq\cdot X(\bar{z})}\right\} e^{ip\cdot X(\bar{z})}$$

with:

$$\begin{cases} P_n^i \equiv \displaystyle\sum_{\ell=1}^{n} \frac{i}{(\ell-1)!}\, \partial^\ell X^i\, S_{n-\ell}(a_s), \\[2ex] S_{n-\ell}(a_s) \equiv \displaystyle\oint \frac{dw}{2\pi i w}\, w^{-(n-\ell)}\, e^{\sum_{s=1}^{n} a_s w^s} \\[2ex] a_s \equiv -\dfrac{1}{s!}\, inq\cdot\partial^s X \end{cases}$$

quantum analogue of Burden solutions !

complete Bell polyonomials

* D.S., M. Hindmarsh; PRD 84, PRL 106 (2011)

Fixed-Loop Momenta X-Path Integral

$$\int DX e^{-S_X} \delta^{Dh}\left(\mathbb{P}_I^\mu - \frac{1}{2\pi\alpha'}\oint_{A_I} \partial X^\mu - \bar\partial X^\mu\right) D_1 X^{\mu_1} \cdots D_I X^{\mu_I} e^{i\int J\cdot X}$$

$$= (2\pi)^D \delta^D\left(\int J\right) f(R, g_s, \Omega_{IJ})$$

$$\times \sum_{\#} \left(-\eta DD \ln|E|^2\right)^{\#} \left(i2\pi\alpha' \mathbb{P} D J_m \int \omega - i\int J D \ln|E|^2\right)^{\#}$$

$$\times \sum \left|\exp\left(i\frac{\pi\alpha'}{2}\mathbb{P}\cdot\Omega\mathbb{P} + i\pi\alpha' \mathbb{P}\cdot\int J\int\omega\right)\right|^2$$

$$\times \exp\left(\frac{\alpha'}{4}\iint' J\cdot J' \ln|E|^2\right)$$

($E(z,z')$: Prime form, Ω_{IJ} : Period matrix)

Ghost Path Integral

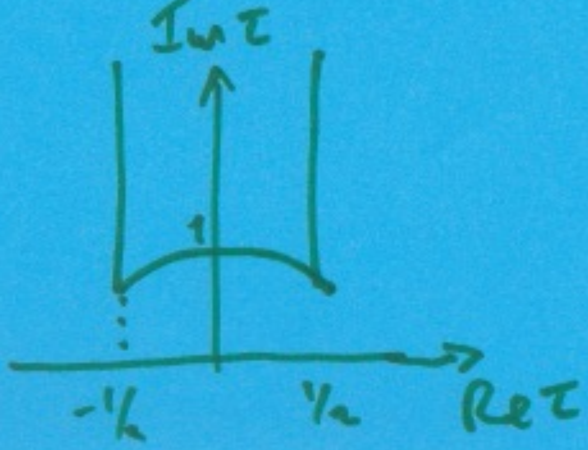
$$\int \mathcal{D}(\text{ghosts})\, e^{-S_G} =$$

$$= \int \mathcal{D}(b\bar{b})\, \mathcal{D}(c\bar{c})\, e^{-S_G}\, |(\mu, b)|^2\, c(w)\, \bar{c}(\bar{w})$$

$$= \det{}' \Delta_{(-1)}\, |(\mu, \phi)|^2\, |\psi(w)|^2$$

$$= |\eta(\tau)|^4$$

genus $h=1$ result
(eta function)



Full 2-Point Amplitude

Combining above ingredients, we find:

$$\mathcal{A}_{fi} = i(2\pi)^D \delta^D(\delta J)\, \frac{1}{\sqrt{2}\,\nu_{D-1}}\, \frac{2\pi\alpha'}{L}$$

chiral half F

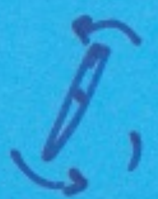
$$\underbrace{\times \int d^D\mathbb{P} \sum_{\text{inst.}} \frac{1}{2} \int d^2\tau \int d^2 z \int ds\, F_n(\mathbb{P})\, \overline{F}_m(\overline{\mathbb{P}})}_{\equiv\, \mathcal{M}_{n,m}(\mathbb{P})}$$

$\equiv \mathcal{M}_{n,m}(\mathbb{P})$: invariant amplitude

Note: chiral splitting !*

* D'Hoker & Phong; CMP 125 (1989) 469

Full 2-Point Amplitude II

... e.g., when $\boxed{(n,m)=(1,1)}$ (i.e. , , ...):

$$\mathcal{M}_{1,1}(\mathbb{P}) = \frac{1}{2}\int d^2\tau \int d^2 z \int ds\, \mathcal{F}_1(\mathbb{P})\, \bar{\mathcal{F}}_1(\bar{\mathbb{P}}),$$

with:

$$\mathcal{F}_1(\mathbb{P}) = \frac{g_D}{I_0(2\ell^2)}\, \eta(\tau)^{-24} e^{\pi i \tau \mathbb{P}^2} E^{-p^2} e^{-2\pi i \mathbb{P}\cdot p z}$$

$$\times \exp\left(e^{is}|\lambda|^2 e^{2\pi i \mathbb{P}\cdot q z} E^2 \partial_z^2 \ln E\right)$$

$$\times I_0\left(2 e^{is/2} |\mathbb{P}\cdot\lambda| e^{\pi i \mathbb{P}\cdot q z} (2\pi E)\right)$$

Modified Bessel function

Full 2-Point Amplitude III

... and for general harmonics (n,m), (i.e. , , ...)

$$F_n(\mathbb{P}) = g_D C\, \eta(\tau)^{-24} e^{\pi i \tau \mathbb{P}^2 - 2\pi i \mathbb{P}\cdot p z} E^{-p^2}$$

$$\times \exp\left(e^{ins}\frac{|\lambda n|^2}{n^2} e^{2\pi i \mathbb{P}\cdot n q z} E^{2n}\sum_{r,t=1}^{n}\frac{S_{n-r}}{(r-1)!}\frac{S_{n-t}}{(t-1)!}\partial_z^{r+t}\ln E\right)$$

$$\times I_0\left(2e^{\frac{ins}{2}}\frac{|\mathbb{P}\cdot\lambda n|}{n} e^{\pi i \mathbb{P}\cdot nqz}\, 2\pi E^n S_{n-1}\right)$$ ← modified Bessel

with $\boxed{S_{n-r}(a_s)}$ a complete Bell polynomial

with:

$$a_s = -\frac{n}{s!}\partial_z^s\left(-\ln|E|^2 + 4\pi\, \mathbb{P}\cdot q\, \mathrm{Im}\,z\right)$$

Normalisation & Unitarity

- Fix overall normalisation by unitarity,

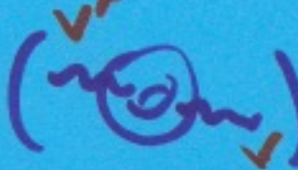
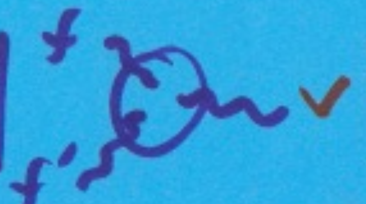
$$SS^\dagger = \mathbb{1},$$

which in terms of $\mathcal{M}_{n,m}(\mathbb{P})$ reads:

$$2\,\mathrm{Im}\,\mathcal{M}^{(1\text{-loop})}_{n,m}(\mathbb{P}) = \frac{1}{2} \sum\!\!\!\!\!\!\int_{\substack{m^2, M^2 \\ (f,f')}} \delta(\mathbb{P}^2 + m^2)\,\delta\big((k-\mathbb{P})^2 + M^2\big)$$

$$\times\,\theta(\mathbb{P}^0)\,\theta(k^0 - \mathbb{P}^0)\,\left|\mathcal{M}^{(\text{tree})}_{ff';nm}\right|^2$$

↖ quantizes spectrum
e.g. ... $\omega_n \sim \frac{4\pi n}{L}$, $n = 1, 2, \dots$
(gravitons)

$\Big($ Im $\Big($  $\Big) \sim \sum_{f,f'} \delta(\;)\,\delta(\;)\,\Big|$  $\Big|^2 \Big)$

Example: $(n,m)=(1,1)$

... We find the leading behaviour:

$$\frac{dP_n}{d\Omega_{S^{D-2}}} \simeq \frac{8\pi G_D \mu^2}{(2\pi)^{D-4}} \frac{1}{(n-1)!^4} \left(\frac{n}{\sqrt{2}} \sqrt{|\hat{P}\cdot\hat{\lambda}|\,|\hat{P}\cdot\hat{\tilde{\lambda}}|} \right)^{4n-4}$$

$$\times \frac{n^2}{4} \left(\frac{4\pi n}{L} \right)^{D-4} B_n(\hat{P}^{D-1})$$

with

$$\begin{cases} B_n(\hat{P}^{D-1})\Big|_{IR} \simeq 1 + O(\omega n/E) \\ B_n(\hat{P}^{D-1})\Big|_{UV} \simeq f_D(\omega n/E, \hat{P}^{D-1})\, e^{-\left(1-\frac{1}{e^2}\right)\frac{L^2}{8\pi^2\alpha'}}, \end{cases}$$

Conclusions

- 1^{st} computation of coherent state amplitudes
  - ressumations involving Bessel functions
  - leading L dependence (in IR) agrees with classical* and mass eigenstate† computations, but additional exponential suppression at high energies (UV)

numerical

† Iengo & Russo, Chialva .. Gutperle & Krym, ...
* Burden 85, Vilenkin, Vachaspati 85

Conclusions II

- Equations capture a rich set of amplitudes corresponding (classically) to radiation from hypocycloids, epicycloids, with or without cusps

...

- Computed a very general set of genus-h correlators from which arbitrary amplitudes may be extracted

fixed A-cycle momenta

Outlook

Q: To what extent do classical comp. reproduce quantum comp.? (cusps, back reaction, ...)

Q: Can we tell experimentalists what to search for? (cosmic string context, ..) (pulsar timing exps.).

Q: Study B.H.? (interactions, Hawking rad, ...)

$$\vdots \left(r \sim \sqrt{\frac{1}{n^2} + \frac{1}{m^2}}\, L \,\Big|_{\substack{n,m \to \infty \\ L \text{ fixed large}}}, \quad E = \frac{L}{2\pi\alpha'}, \right)$$

Classical Gravitational Radiation

.. The general $D=4$ expression for the total power per solid angle was computed by Burden '85

E.g., when $(n,m)=(1,1)$, $D=4$, $\omega_n = \frac{4\pi n}{L}$,

Classical power

$$\boxed{\frac{dP}{d\Omega_{S^2}} = 8\pi G\mu^2 \sum_{n=1}^{\infty} n^2 \Big[\big(\cot^2\theta\, J_n^2(n\sin\theta) + J_n'^2(n\sin\theta)\big)^2 + (1+\cos\psi)^2 \cot^2\theta\, J_n^2(n\sin\theta)\, J_n'^2(n\sin\theta) \Big]}$$

∞ for ($\psi = 0$: ... $\psi = \pi$: ...) !

Bessel functions

string theory Analogue?

Decay Rates & Power

- Unitarity, $SS^\dagger = 1$, implies:

$$\Gamma = \frac{2\pi\alpha'}{L} \int \frac{d^D \mathbb{P}}{(2\pi)^D} \, \mathrm{Im}\, \mathcal{M}_{n,m}(\mathbb{P})$$

(decay Rates)

and:

$$\frac{dP}{d\Omega_{S^{D-2}}} = \frac{1}{(2\pi)^D} \frac{2\pi\alpha'}{L} \int d\mathbb{P}^0 \, \mathbb{P}^0 \, |\underline{\mathbb{P}}|^{D-2} \, \mathrm{Im}\, \mathcal{M}_{n,m}(\mathbb{P})$$

classical
$\Updownarrow$
quantum

(total power per unit angle in direction $\hat{\mathbb{P}}^i$)

Imaginary Part

Imaginary part of all fixed-loop momenta amplitudes is determinde from:

$$\mathrm{Im} \int_{-1/2}^{1/2} d\tau_1 \int_{f}^{\infty} d\tau_2 \int_0^1 d\sigma^2\, \tau_2\, e^{-2\pi\tau_2(1-\sigma^2)(p^2+m^2)}$$

$$\times\, e^{-2\pi\tau_2\sigma^2((k-p)^2+M^2)}$$

$$= \frac{1}{2(2\pi)^2}\, \delta(p^2+m^2)\, \delta((k-p)^2+M^2)\, \theta(p^0)\, \theta(k^0-p^0)$$

(e.g., for gravitons ($m^2=0$): $p^2=0$ $\qquad$ $\omega_n = \frac{4\pi n}{L}\left(1-Q+Q\frac{2N-2}{2d^2-2}\right)^{-1}$

e.g. $D=4$, $Q=\frac{1}{2}(1+\sin\theta)$,)