Reduction at the integrand level beyond NLO

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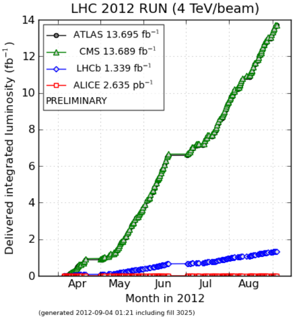
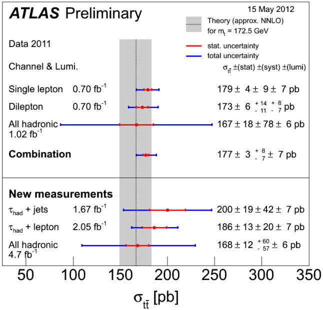
LHC@NNLO

- LHC outstanding performance calls for serious theoretical work



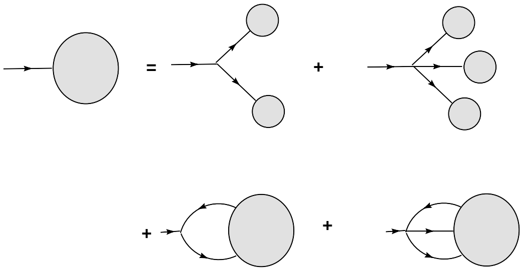


- Precision becomes very important for most of the processes
- Fixed-order calculations need to advance
- Capitalize over previous achievements

Dyson-Schwinger Recursive Equations

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].



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C. G. Papadopoulos, Comput. Phys. Commun. **137** (2001) 247 [arXiv:hep-ph/0007335].

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- Including all SM, in both unitary and F-gauge, masses, CKM, unstable particle widths, complex mass scheme, etc.
- For QCD color connection representation: revival of the 't Hooft ideas ('71) in the modern era. Citation Alert !

HELAC Color treatment

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \rightarrow \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} A_\sigma$$

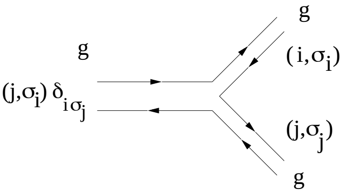
gluons $\rightarrow (i,j)$, quark $\rightarrow (i,0)$, anti-quark $\rightarrow (0,j)$, other $\rightarrow (0,0)$

$$\sum_{\{i\},\{j\}} |\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}|^2$$

$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

$$\mathcal{C}_{\sigma,\sigma\prime} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} \delta_{i_{\sigma'_1},j_1} \delta_{i_{\sigma'_2},j_2} \ldots \delta_{i_{\sigma'_k},j_k} = N_c^{m(\sigma,\sigma\prime)}$$

Color-connection Feynman Rules



HELAC tree order current version

- 2007 HELAC: http://helac-phegas.web.cern.ch/helac-phegas/

  A. Cafarella, C. G. Papadopoulos and M. Worek, Comput. Phys. Commun. **180** (2009) 1941 [arXiv:0710.2427 [hep-ph]].

HELAC tree order current version

- 2007 `HELAC`: http://helac-phegas.web.cern.ch/helac-phegas/
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- Very easy to use: just edit the `user.inp` file and then execute the command `./run.sh`

```
# Compulsory information
colpar 1          # colliding particles: 1=pp, 2=ppbar, 3=e+e-
inist  35 35      # initial state; enter 0 to sum over initial states
finst  35 35      # final state
energy 14000      # collision energy (GeV)

# For reference, here is the particle numbering:
# ve e u d vm mu c s vt ta t  b  a  z  w+ w- g  h chi f+ f- jet
# 1  2 3 4  5  6 7 8  9 10 11 12 31 32 33 34 35 41 42 43 44  100
# The respective antiparticles have a minus sign (for example: positron is -2)
# A jet in the final state is denoted by the number 100

# Enter here your additional commands if you wish to alterate the default values
```

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- Including kt-reweight for jet matching
- Latest: $W + 5$ jets at LHC

NLO

- Tree-order integrated over m-body phase space + SF in D=4
- Virtual corrections integrated over m-body phase space + SF in D=4
- Real corrections subtracted integrated over (m+1)-body phase space + SF in D=4
- I- integrated over m-body phase space + SF in D=4
- KP- integrated over m-body phase space + SF integrated over SF, in D=4

The HELAC-NLO adventure

- 2006 OPP: The method that enables us to think seriously about NLO calculations.

Based on previous work by Bern, Dixon, Kosower, Britto, Cachazo, Feng.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725** (2005) 275 [arXiv:hep-th/0412103].

Complete framework: numerical (fast) & algebraic (stable)

G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763** (2007) 147 [arXiv:hep-ph/0609007].

The HELAC-NLO adventure

$$\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}$$

Algebra & Integrals

$$A \to \frac{N(q)}{\prod D_i}$$

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\ &+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\ &+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \end{aligned}$$

Solving for known values of the loop momentum q

R_1: the rational terms from the reduction itself

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \to$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \qquad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$
\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)
$$

The "Extra Integrals" are of the form

$$I^{(n;2\ell)}_{s;\mu_1\cdots\mu_r} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)},$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,,\, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) \;=\; b(ij)+\tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) \;=\; c(ijk)+\tilde{q}^2 c^{(2)}(ijk)\,.$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} \;=\; -\frac{i\pi^2}{2}\left[m_i^2+m_j^2-\frac{(p_i-p_j)^2}{3}\right]+\mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} \;=\; -\frac{i\pi^2}{2}+\mathcal{O}(\epsilon)\,, \quad \int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} \;=\; -\frac{i\pi^2}{6}+\mathcal{O}(\epsilon)\,.$$

Furthermore, by defining

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3;\tilde{q}^2) + \tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \,,$$

the following expansion holds

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q)\,,$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)}\,.$$

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$\begin{aligned} \mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\ &- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \end{aligned}$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

UV-behavior: only up to 4-vertices (beyond one loop ?).

Alternatives: GKMZ-approach, Blackhat

- 2006 `OPP`: The method that enables us to think seriously about NLO calculations.

- 2007 `CutTools`: Reduction at the integrand level + rational terms R_1

  G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0803** (2008) 042 [arXiv:0711.3596 [hep-ph]].

  G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0805** (2008) 004 [arXiv:0802.1876 [hep-ph]].

The HELAC-NLO adventure

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- OneLOop: One-loop scalar integrals in dimensional regularization (UV+IR) including complex masses

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- 2008 HELAC-1LOOP: Based on HELAC to produce virtual one-loop amplitudes

A. van Hameren, C. G. Papadopoulos and R. Pittau, JHEP **0909** (2009) 106 [arXiv:0903.4665 [hep-ph]].

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- `OneLOop`: One-loop scalar integrals in dimensional regularization (UV+IR) including complex masses
- 2008 `HELAC-1LOOP`: Based on `HELAC` to produce virtual one-loop amplitudes
- 2009 `HELAC-Dipoles`: Based on `HELAC` to *automatically* produce Catani-Seymour dipoles, I-operator, KP-operator, arbitrary masses

M. Czakon, C. G. Papadopoulos and M. Worek, JHEP **0908** (2009) 085 [arXiv:0905.0883 [hep-ph]].

HELAC-NLO

- Complete software for NLO-QCD at LHC:
  LO: highly automated
  Virtual: very efficient
  Real: KP- and I-operator contributions also very efficient,
  Real-subtracted: quite efficient taking into account the current theoretical developments

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- Speed, stability, efficiency issues under control
  Improvements in PS for real corrections under investigation (KALEU from A. van Hameren). Towards an order of magnitude improvement !

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- Speed, stability, efficiency issues under control
  Improvements in PS for real corrections under investigation (KALEU from A. van Hameren). Towards an order of magnitude improvement !
- Provide NLO calculator for all processes $2 \to n$ with 6-7 particles attached to the loop
  Providing a library of preconfigured setups for all (6-7 particles attached to the loop, up to 12 particles overall) processes of interest for LHC

HELAC-NLO

- $pp \to t\bar{t}b\bar{b}$: proof-of-concept

  G. Bevilacqua, M. Czakon, C. G. Papadopoulos, R. Pittau and M. Worek, JHEP **0909** (2009) 109 [arXiv:0907.4723 [hep-ph]].

  M. Worek, JHEP **1202** (2012) 043 [arXiv:1112.4325 [hep-ph]].

- $pp \to t\bar{t} + j + j$: one of the most advanced

  G. Bevilacqua, M. Czakon, C. G. Papadopoulos and M. Worek, Phys. Rev. Lett. **104** (2010) 162002 [arXiv:1002.4009 [hep-ph]].

  G. Bevilacqua, M. Czakon, C. G. Papadopoulos and M. Worek, Phys. Rev. D **84** (2011) 114017 [arXiv:1108.2851 [hep-ph]].

- $pp \to W^+W^-b\bar{b}$ including $(t\bar{t})$: finite width and complex masses

  G. Bevilacqua, M. Czakon, A. van Hameren, C. G. Papadopoulos and M. Worek, JHEP **1102** (2011) 083 [arXiv:1012.4230 [hep-ph]].

- $pp \to t\bar{t}t\bar{t}$: BSM phenomenology

  G. Bevilacqua and M. Worek, JHEP **1207** (2012) 111 [arXiv:1206.3064 [hep-ph]].

- $pp \to t\bar{t} + j \oplus$ PS : interfacing with Parton Shower at NLO (`POWHEG`)

  A. Kardos, C. Papadopoulos and Z. Trocsanyi, Phys. Lett. B **705** (2011) 76 [arXiv:1101.2672 [hep-ph]].

- $pp \to t\bar{t} + H \oplus$ PS

  M. V. Garzelli, A. Kardos, C. G. Papadopoulos and Z. Trocsanyi, Europhys. Lett. **96** (2011) 11001 [arXiv:1108.0387 [hep-ph]].

- $pp \to t\bar{t} + Z/W^{\pm} \oplus$ PS

  M. V. Garzelli, A. Kardos, C. G. Papadopoulos and Z. Trocsanyi, arXiv:1208.2665 [hep-ph].

What do we need at NNLO ?

- Virtual: $n-$particle two-loop amplitudes
- Real-Virtual: $n+1-$particle real times one-loop
- Real-Real: $n+2-$particle real

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- Real-Virtual: $n+1-$particle real times one-loop
- Real-Real: $n+2-$particle real

Recent work on two-loop amplitudes

P. Mastrolia and G. Ossola, JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

D. A. Kosower and K. J. Larsen, Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

H. Johansson, D. A. Kosower and K. J. Larsen, arXiv:1208.1754 [hep-th].

S. Badger, H. Frellesvig and Y. Zhang, JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, JHEP **1208** (2012) 065 [arXiv:1207.2976 [hep-ph]].

Two-loop amplitudes

- Reduction at the integrand level
- Master Integrals

Two-loop amplitudes

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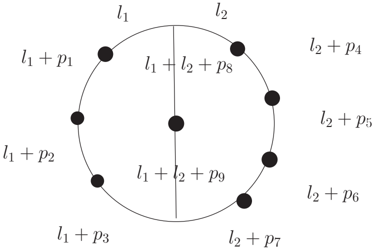
- Master Integrals

- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].



$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k)$$

Two-loop amplitudes

The general strategy consists in finding function $x_j \equiv x_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \ .$$

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Let us go a step back at one loop

$$1 = T_1(q) D_1 + T_2(q) D_2 + \cdots + T_n(q) D_n$$

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Let us go a step back at one loop

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Constant terms: $T_j(q) = x_j$

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$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \quad .$$

Let us go a step back at one loop

$$1 = T_1(q)D_1 + T_2(q)D_2 + \cdots + T_n(q)D_n$$

Constant terms: $T_j(q) = x_j$

$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j p_j{}^\mu + \sum_{j=1}^{n} x_j \mu_j = 1 \quad .$$

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j p_j{}^\mu = 0 \quad , \sum_{j=1}^{n} x_j \mu_j = 1$$

Two-loop amplitudes

The general strategy consists in finding function $x_j \equiv x_j(l_1, l_2)$

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- solution exists for $n = 6$ $d = 4$

Two-loop amplitudes

Linear terms $T(q) = P_1(q)$, count tensor structures:

$$1 \;,\; q^\mu \;,\; q^\mu q^\nu \;,\; q^2 q^\mu \;.$$

There are, for $d = 4$, therefore $1+4+10+4 = 19$ independent tensor structures.

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In d dimensions, tensor up to rank k, $N(d,k)$ number of independent tensor structures

$$N(d,k) = \binom{d-1+k}{k} + \sum_{p=0}^{k+1} \binom{d-1+p}{p} \;. \tag{1}$$

In the table below we give the results for various ranks and dimensionalities.

k	0	1	2	3	4
d =1	3	4	5	6	7
2	4	8	13	19	26
3	5	13	26	45	71
4	6	19	45	90	161
5	7	26	71	161	322
6	8	34	105	266	588

Values of $N(d,k)$

Two-loop amplitudes

The OPP-"miracle" is that the OPP equation works with only 10(6) different coefficients

$$1 = \sum_{i=1}^{5} D_i(q)(c_i^{(0)} + c_i^{(1)}\epsilon_i(q))$$

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Back to two loops: iGraphs can be denoted by the triplet (n_1, n_2, n_3), $n = n_1 + n_2 + n_3$

$$n_{1,2,3} \leq 4\ (= d)\ ,\quad n_1 + n_2 + n_3 \leq 11\ (= 2d + 3)\ .$$

Linear terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j)$$

$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$	$d=1$
3	39-0	33-0	27-0	21-0	15-0	9-0
4	52-0	44-0	36-0	28-0	20-0	12-2
5	65-1	55-1	45-1	35-1	25-1	15-5
6	78-3	66-3	54-3	42-3	30-3	
7	91-6	77-6	63-6	49-6	35-8	
8	104-10	88-10	72-10	56-10		
9	111-15	99-15	81-15	63-17		
10	130-21	110-21	90-21			
11	143-28	121-28	99-30			
12	156-36	132-36				
13	169-45	143-47				
14	182-55					
15	195-55					
$T(d)$	127	96	69	46	27	10

Quadratic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \le k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k) + \cdots$$

$$T(d) = 4d^3/3 + 10d^2 + 20d/3 - 2 \qquad (2)$$

n	$d = 4$	$d = 3$	$d = 2$
3	135-4	84-3	45-3
4	180-6	128-6	60-6
5	225-18	140-16	75-15
6	270-38	168-32	90-30
7	315-65	196-53	
8	360-98	224-80	
9	405-136	252-108	
10	450-180		
11	495-225		
$T(d)$	270	144	60

Cubic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \cdots + \sum_{j \leq k} g_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_1 \cdot t_l) + \cdots$$

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1$$

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$
5				420/332
6				504/352
7			1155/803	588/360
8			1320/823	672/360
9		2574/1603	1485/831	
10		2860/1623	1650/831	
11	5005/2848	3146/1631		
12	5460/2868	3432/1631		
13	5915/2876			
14	6370/2876			
$T(d)$	2876	1631	831	360

OPP at two loops

$$1 = \sum D_i R_i + \sum D_i D_j R_{ij} + \sum D_i D_j D_k R_{ijk} + \cdots$$

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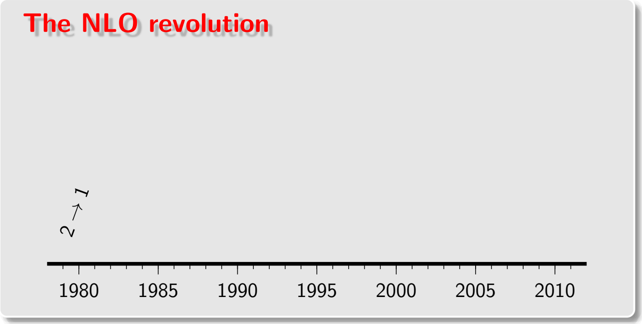
- Rational terms $R_1 + R_2$?

- The NLO revolution



1979: NLO Drell-Yan [Altarelli, Ellis & Martinelli]

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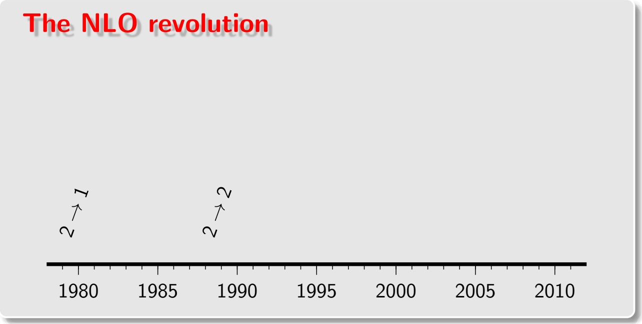
NNLO in the future

- The NLO revolution



1987: NLO high-p_t photoproduction [Aurenche et al]
1988: NLO $b\bar{b}$, $t\bar{t}$ [Nason et al]
1993: dijets, Vj [JETRAD, Giele, Glover & Kosower]

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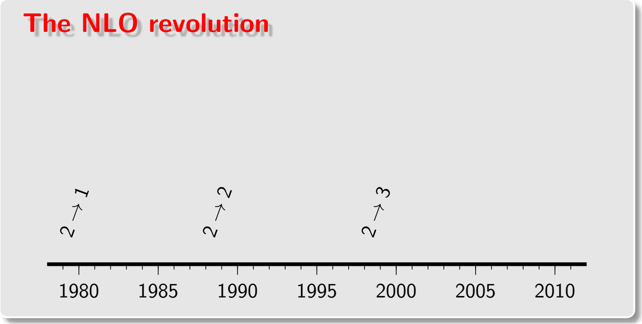

1998: NLO $Wb\bar{b}$ [MCFM: Ellis & Veseli]
2000: NLO $Zb\bar{b}$ [MCFM: Campbell & Ellis]
2001: NLO $3j$ [NLOJet++: Nagy]
. . .
2007: NLO $t\bar{t}j$ [Dittmaier, Uwer & Weinzierl '07]
. . .

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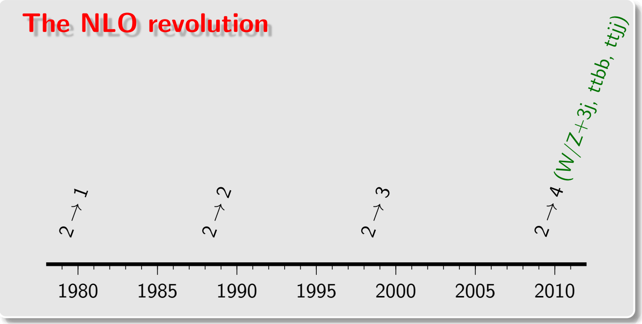

2009: NLO $W{+}3j$ [Rocket: Ellis, Melnikov & Zanderighi] [unitarity]
2009: NLO $W{+}3j$ [BlackHat: Berger et al] [unitarity]
2009: NLO $t\bar{t}b\bar{b}$ [Bredenstein et al] [traditional]
2009: NLO $t\bar{t}b\bar{b}$ [HELAC-NLO: Bevilacqua et al] [unitarity]
2009: NLO $q\bar{q} \to b\bar{b}b\bar{b}$ [Golem: Binoth et al] [traditional]
2010: NLO $t\bar{t}jj$ [HELAC-NLO: Bevilacqua et al] [unitarity]
2010: NLO $Z{+}3j$ [BlackHat: Berger et al] [unitarity]

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NNLO IN THE FUTURE

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I. Bierenbaum, M. Czakon and A. Mitov, Nucl. Phys. B **856** (2012) 228 [arXiv:1107.4384 [hep-ph]].

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    M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].

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Back-up slides

For those who are interested in more details ...

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... have a drink with me.