Supersymmetry on Curved Spaces and Holography

based on:

1205.1062 CK-Tomasiello-Zaffaroni
1207.2181 Cassani-CK-Martelli-Tomasiello-Zaffaroni

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20/9/12 - Eurostrings Corfu

Outline

Motivations

Basic Strategy

Results

Future Directions

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Future Directions

Exact results in SUSY QFT's

→ imagine talk by D. Jafferis

- Localization reduces the path integral to a matrix model yielding new observables & exact results

Exact results in SUSY QFT's

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- Localization reduces the path integral to a matrix model yielding new observables & exact results
  - Check of Seiberg dualities ← 4d, 3d
  - AGT ← 4d/2d
  - $N^{3/2}$-, N^3- scaling ← 3d, 5/6d
  - Z-minimization & F-theorem ← 3d
  - SCFT index ← 4d
  - …

Exact results in SUSY QFT's

- Localization reduces the path integral to a matrix model
- Typically this requires curved spaces (e.g. S^n)

Question:
Which spaces are allowed?

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Problem

- Curvature breaks SUSY
- Complicated construction of $\mathcal{L}$ and $\delta\Phi$ following the "Noether procedure"

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- Complicated construction of $\mathcal{L}$ and $\delta\Phi$ following the "Noether procedure"

Elegant Solution

[Festuccia-Seiberg '11]

- Couple the QFT to supergravity
- Freeze the gravity fields to (complex) background value, s.t. $\delta\psi_\mu = 0$

$\Rightarrow$ Get $\mathcal{L}$ and $\delta\Phi$ "for free"

- If the QFT is a CFT, naturally: couple to "conformal SUGRA"!

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Recall: in conformal SUGRA the whole superconformal group is gauged
vs. "Poincaré SUGRA" ↔ only usual SUSY group is local

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- Superconformal theories can also be understood via their gravity dual. The curved manifold arises then as the boundary of a bulk geometry.

$\Rightarrow$ Relation between the two!?

The bulk theory: $\mathcal{N} = 2$ gauged SUGRA.

minimal: (g_{mn}, A_m, ψ_m)

The bulk theory: Minimal gauged SUGRA

↓ r^{-1} - expansion

On the boundary: Conformal SUGRA!

The bulk theory: Minimal gauged SUGRA

$\downarrow$ r^{-1} - expansion

On the boundary: Conformal Killing spinor (CKS)

$$\delta\psi_M = 0 \quad \Rightarrow \quad \boxed{\left(\nabla^A_m - \frac{1}{d}\gamma_m D^A\right)\epsilon = 0} \qquad d = 3, 4$$

with $\nabla^A_m = \nabla_m - iA_m, \quad D = \gamma^m \nabla_m$

Manifolds admitting a conformal Killing spinor (CKS), admit SUSY.

$$P_m^A \epsilon \equiv \left(\nabla_m^A - \frac{1}{d} \gamma_m D^A \right) \epsilon = 0$$

Manifolds with a CKS / with SUSY

$$P^A_m \epsilon \equiv \left(\nabla^A_m - \frac{1}{d}\gamma_m D^A\right)\epsilon = 0$$

The $A = 0$ case is studied:

Manifolds with a CKS / with SUSY

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The $A = 0$ case is studied:

- Euclidean Case $\rightarrow$ $\nabla_m \epsilon \sim \gamma_m \epsilon$

Dimension	3	4	5	6
Manifold	S^3	S^4	SE	nearly Kähler
Cone	$\mathbb{R}^4$	$\mathbb{R}^5$	CY	G_2

Manifolds with a CKS / with SUSY

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The $A = 0$ case is studied:

- Euclidean Case → $\nabla_m \epsilon \sim \gamma_m \epsilon$
- Lorentzian Case (in 4d) → either Fefferman or pp-wave metrics

Manifolds with a CKS / with SUSY

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Example (4d Euclidean):

$$\epsilon_+ \otimes \epsilon_+^\dagger \sim e^{-ij}$$
$$\epsilon_+ \otimes \epsilon_+^T \sim \omega$$

$\Rightarrow$ $U(2)$-structure

$$j^2 = \omega \wedge \bar{\omega}$$
$$\omega^2 = 0$$

Manifolds with a CKS / with SUSY

General strategy:

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2. CKS equation $\Rightarrow$ conditions on the forms

↑

$$P_m^A \epsilon = 0$$

Manifolds with a CKS / with SUSY

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4d Euclidean:

[CK-Tomasiello-Zaffaroni
Dumitrescu-Festuccia-Seiberg]

$$\left.\begin{aligned} dj &= W_4 \wedge j \\ d\omega &= W_5 \wedge \omega \end{aligned}\right\} \Rightarrow \boxed{M_4 \text{ is complex!}}$$

Manifolds with a CKS / with SUSY

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+ (roughly): $A \sim W_4 + W_5 \;\Rightarrow$ determines A

Manifolds with a CKS / with SUSY

Comments:

- This A determines $\mathcal{L}$ and $\delta\Phi$
- $*$ is a local statement

4d Euclidean:

$$\left.\begin{aligned} dj &= W_4 \wedge j \\ d\omega &= W_5 \wedge \omega \end{aligned}\right\} \Rightarrow \boxed{M_4 \text{ is complex!}}^{\,*}$$

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Manifolds with a CKS / with SUSY

- Results:

Dimension	G	Forms	Condition
4d (++++)	$U(2)$	j, ω	M_4 complex $(i.e.\ d\omega = W \wedge \omega)$
4d (–+++)	$\mathbb{R}^2$	$z\ (\text{real \& null})$ $\omega = z \wedge w$	z conformal Killing $(i.e.\ \nabla_{(\mu} z_{\nu)} = \lambda g_{\mu\nu})$
3d (+++)	I	e^3 $o = e^1 + ie^2$	$do = W \wedge o$

Comment: The non-conformal case

- (Non-conformal) SUSY Theories with an R-symmetry are automatically included in the analysis as the special case where $\epsilon_+ \neq 0$

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- (Non-conformal) SUSY Theories with an R-symmetry are automatically included in the analysis as the special case where $\epsilon_+ \neq 0$

- Get the same manifolds as before!
  ...with subtleties:
    - Local → global
    - In 4d Lorentz, CKV → KV $(\nabla_{(\mu} z_{\nu)} = 0)$

SUSY Theories with an R-symmetry (the non-conformal case)

SUSY Theories with an R-symmetry

- Naturally: couple to "new minimal SUGRA"

$$\nabla_m \epsilon_+ = -i\left(\frac{1}{2}v^n \gamma_{nm} + (v-a)_m\right)\epsilon_+,$$

constraint: $d * v = 0$

SUSY Theories with an R-symmetry

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constraint: $d * v = 0$

(3d follows by dimensional reduction...)

→ see also:
talk by G. Tartaglino-Mazzucchelli

SUSY Theories with an R-symmetry

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$$\nabla_m \epsilon_+ = -i\Big(\frac{1}{2}v^n\gamma_{nm} + (v-a)_m\Big)\epsilon_+ ,$$

- Fact: This is a special case of CKS equation

$$\nabla^A_m \epsilon_+ = \frac{1}{d}\gamma_m D^A \epsilon_+$$

SUSY Theories with an R-symmetry

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- Fact: This is the special case of CKS equation for $\epsilon_+ \neq 0$,

where $D^A \epsilon_+ \equiv v \cdot \epsilon_+$ and $a \equiv A + \frac{3}{2}v$

SUSY Theories with an R-symmetry

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- The constraint $d * v = 0$ can be imposed by fixing an ambiguity in the definition of v

SUSY Theories with an R-symmetry

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$$\nabla_m \epsilon_+ = -i\Big(\frac{1}{2}v^n\gamma_{nm} + (v-a)_m\Big)\epsilon_+,$$

- Fact: This is the special case of CKS equation for $\epsilon \neq 0$

- Lorentzian case: reality of v ↔ Weyl rescaling

$\Rightarrow \lambda = 0$ (z becomes Killing vector)

SUSY Theories with an R-symmetry

Comment: No surprise!

Recall: Superconformal tensor calculus …

Conformal SUGRA ⟶ New minimal SUGRA

gauge fixing the conformal symmetries (upon a "compensator multiplet" $v = *db$)

Future Directions

- Regularity in bulk ↔ Imprint on boundary (via perturbative construction in $\frac{1}{r}$)

  → see also: A. Passias' talk

- Other dimensions:

  → $5d/6d$ interesting in the context of the M5 and AGT.
  New feature: Non-abelian!

  → see also: N. Lambert's talk
  D. Jafferis' talk
  F. Bonetti's talk

- $\mathcal{N} = 2$ in 4d

Thank You…

…Questions?

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More explicitly: 4d Euclidean

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- Extra Detail: $\|\epsilon\|^2 = e^B$

- Equations for $U(2)$-structure ω, j :

A is complex!

$$w^3 = 0$$

$$iA_{1,0} = -\frac{1}{2}\overline{w^5_{0,1}} + \frac{1}{4}w^4_{1,0} + \frac{1}{2}\partial B$$

$$iA_{0,1} = +\frac{1}{2}w^5_{0,1} - \frac{3}{4}w^4_{0,1} + \frac{1}{2}\bar{\partial} B$$

$$d\omega = w^5 \wedge \omega \qquad dj = w^4 \wedge j$$

More explicitly: 4d Euclidean

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Example: Kähler manifolds $dj = 0 \Rightarrow w^4 = 0$

$$d\omega = 2i\mathrm{Re}A \wedge \omega \qquad \mathrm{Im}A = 0$$

(A is real.)

More explicitly: 4d Euclidean

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Example: Kähler manifolds $dj = 0 \Rightarrow w^4 = 0$

Note: in new minimal set-up: $v_m = 0$

$$(\nabla_m - ia_m)\epsilon = 0 \longleftarrow$$

Characterises Kähler manifolds

More explicitly: 4d Euclidean

$$w^3 = 0$$

$$iA_{1,0} = -\tfrac{1}{2}\overline{w^5_{0,1}} + \tfrac{1}{4}w^4_{1,0} + \tfrac{1}{2}\partial B$$

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Example: complex but non-Kähler?

More explicitly: 4d Euclidean

$$w^3 = 0$$

$$iA_{1,0} = -\tfrac{1}{2}\overline{w^5_{0,1}} + \tfrac{1}{4}w^4_{1,0} + \tfrac{1}{2}\partial B$$

$$iA_{0,1} = +\tfrac{1}{2}w^5_{0,1} - \tfrac{3}{4}w^4_{0,1} + \tfrac{1}{2}\bar{\partial} B$$

Example: $S^3 \times S^1$ $\qquad ds^2 = dt^2 + ds^2_{S^3}$

$$dj = -2dt \wedge j$$
$$d\omega = -2dt \wedge \omega$$

$$\Rightarrow \quad A = -\tfrac{i}{2}dt \qquad dB = 0$$

(A is imaginary!)

More explicitly: 3d Euclidean

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- Get by dimensional reduction: $$v_m \to v_i, v_4$$ $$a_m \to a_i, v_4$$

- Extra Detail: $\|\epsilon\|^2 = e^B$

- Equations for I-structure $e_3, o, \bar{o}$:

$$(o = e_1 + ie_2)$$

$$de_3 = -(dB + 2\,\mathrm{Im}a) \wedge e_3 + 4 * \mathrm{Re}v + i\,\mathrm{Im}v_4\, o \wedge \bar{o}$$

$$do = (2\, v_4 e_3 + 2i\, a - dB) \wedge o$$

More explicitly: 3d Euclidean

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Ex.: (Squashed) 3-sphere $ds^2 = l_1^2 + l_2^2 + \frac{1}{s^2} l_3^2$

↑

$$dl_i = \epsilon_{ijk} l_j \wedge l_k$$

More explicitly: 3d Euclidean

$$de_3 = -(dB + 2\,\mathrm{Im}a) \wedge e_3 + 4 * \mathrm{Re}v + i\,\mathrm{Im}v_4\, o \wedge \bar{o}$$

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Ex.: (Squashed) 3-sphere $ds^2 = l_1^2 + l_2^2 + \frac{1}{s^2} l_3^2$

Choose vielbein: $e_1 = l_1, \quad e_2 = l_2, \quad e_3 = \frac{l_3}{s}$

Determine background fields:

$$v_4 = \frac{i}{s}, \quad a = (1 - \frac{1}{s^2}) l_3, \quad v = B = 0$$

More explicitly: 4d Lorentzian

[Cassani-CK-Martelli-Tomasiello-Zaffaroni]

More explicitly: 4d Lorentzian

- The bilinears define a vielbein...

$$\boxed{ds^2 = ze^- + w\bar{w}} \qquad z^2 = w^2 = (e^-)^2 = 0$$

...up to an ambiguiety:

$$w \to w + \alpha z$$
$$e^- \to e^- - \bar{\alpha}w - \alpha\bar{w} - |\alpha|^2 z$$

("$\mathbb{R}^2$-structure")

More explicitly: 4d Lorentzian

- The bilinears define the vielbein:

$$ds^2 = ze^- + w\bar{w} \qquad z^2 = w^2 = (e^-)^2 = 0$$

- SUSY $\Rightarrow$ z Killing $\Rightarrow$ special coordinates: $z = \frac{\partial}{\partial y}$

Parametrise most general metric:

$$ds^2 = 2H^{-1}(du + \beta)\big(dy + \rho + \mathcal{F}(du + \beta)\big) + H h_{mn} dx^m dx^n$$

for some $H, \mathcal{F}, \rho, \beta$ independent of y

More explicitly: 4d Lorentzian

$$ds^2 = 2H^{-1}(du+\beta)\big(dy+\rho+\mathcal{F}(du+\beta)\big) + H h_{mn} dx^m dx^n$$

We have explicit coordinate expressions for the background fields:

$$\begin{aligned} a^{\perp} &= a - \frac{1}{2}(a\cdot e^{-})z \\ &= \frac{1}{4} *_2 \left[d_2(H^{-1}\bar{w}) -_u \left(H^{-1}\beta\wedge\bar{w}\right)\right] w + \text{c.c.} \end{aligned}$$

etc. ...

More explicitly: 4d Lorentzian

Note: in Lorentzian case, we have a complete classification of the gravity duals!

[Gauntlett-Gutowski '03]

Reduction to boundary reproduces our CKS results from the full bulk solutions

Hard but interesting: Which of our boundary solutions extent to regular ones in the bulk?

Disposal

New Minimal in 3d:

$$\nabla_m \chi = -i\big(v^n \sigma_{nm} + (v-a)_m\big)\chi + \frac{v_4}{2}\sigma_m \chi$$

Disposal

The other vector in 4d Lorentzian:

$$v^{\perp} = v - \frac{1}{2} v \cdot e^{-} z =$$

$$\frac{1}{4} H^{-2} \left[*_2 (\beta \wedge \partial_u \beta - d_2 \beta) \right] e^{-} + \frac{1}{2} H *_2 \left[\partial_u (H^{-1} \beta) - d_2 (H^{-1}) \right]$$