Some Aspects of $N = (2,2)$ Non-Linear $\sigma$-Models

A short review

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Motivation

• The NSR model $\equiv N = (2, 2)$ supersymmetric non-linear $\sigma$-models in d=2 gives a very explicit description of strings backgrounds with NSNS fluxes (including D-brane configurations interpolating between A & B type (AS, Staessens, Wijns ’08, ’09)).

• Deep connection with generalized Kähler & CY geometry (Gualtieri ’03; ...); Ulf Lindström yesterday

• Doubled formalism $\leftrightarrow$ make contact with results by Hull, Hohm, Zwiebach, ...
The NSR approach

- $N \leq (1, 1)$ very simple and rather uninteresting. For type II string theory one has to study $N = (2, 2)$ non-linear $\sigma$-models in $d = 2$.
- The full off-shell description (in $N = (2, 2)$ superspace) is known $\Rightarrow$ good understanding of local, classical geometry. Elegant and natural generalization of Kähler geometry!
- Quantum calculations simplify considerably.
The NSR approach: geometric data

Geometric data: target manifold (Gates, Hull, Roček, ’84)

- Target manifold $\mathcal{M}$, local coordinates $x^a$, $a \in \{1, \cdots, d\}$.
- Metric $g(X, Y)$.
- Closed 3-form $H$, $dH = 0$, locally $H = db$ and $b \simeq b + dk$.

Geometric data: worldsheet

- Worldsheet lightcone coordinates:
  \[
  \sigma^\pm = \tau + \sigma, \quad \sigma^- = \tau - \sigma.
  \]
- Grassman coordinates: $\theta^+$ and $\theta^-$; derivatives:
  \[
  D_+^2 = -\frac{i}{2} \partial^\pm, \quad D_-^2 = -\frac{i}{2} \partial^-, \quad \{D_+, D_-\} = 0.
  \]
The NSR approach: the action

- The $N = (1, 1)$ action in $N = (1, 1)$ superspace:

\[
S = 4 \int d^2\sigma \; d^2\theta \left( D_+ x^a D_- x^b \left( g_{ab} + b_{ab} \right) \right)
\]

- The $N = (1, 1)$ action in ordinary space:

\[
S = 2 \int d^2\sigma \left( (g_{ab} + b_{ab}) \partial_\mp x^a \partial_\pm x^b + 
2i g_{ab} \psi_+^a \nabla^{(+)\,\pm} \psi_+^b + 2i g_{ab} \psi_-^a \nabla^{(-)\,\mp} \psi_-^b + 
R_{abcd} \psi_-^a \psi_-^b \psi_+^c \psi_+^d + 
2(F^a - i\Gamma_{(-)\,cd}^a \psi_-^c \psi_+^d)g_{ab}(F^b - i\Gamma_{(-)\,ef}^b \psi_-^e \psi_+^f) \right).
\]

- Note:

\[
\Gamma_{\pm bc}^a \equiv \{ a \}_{bc} \pm \frac{1}{2} H^a_{bc}.
\]
Additional supersymmetries

- Only possibility for extra susy transformations:
  \[ \delta x^a = \varepsilon^+ J_{+b}(x) D_+ x^b + \varepsilon^- J_{-b}(x) D_- x^b \]

- Closure of algebra \( \Leftrightarrow J_+ \) and \( J_- \) are complex structures!
  So \( d = 2n \).

- Note: generically no off-shell closure of the algebra!

- Invariance of the action \( \Leftrightarrow \)
  - Metric is hermitian: \( g(J_{\pm X}, J_{\pm Y}) = g(X, Y) \).
  - Introduce \( \omega_{\pm}(X, Y) = -g(X, J_{\pm Y}) \)
    \[ d\omega_{\pm}(X, Y, Z) = \mp H(J_{\pm X}, J_{\pm Y}, J_{\pm Z}) \]
Solving the conditions (locally)

Can one solve:

- $J_+$ and $J_-$ are complex structures!
- $g(J_±X, J_±Y) = g(X, Y)$.
- $d\omega_±(X, Y, Z) = ±H(J_±X, J_±Y, J_±Z)$
Off-shell closure

- All off-shell non-closing terms are proportional to \([J_+, J_-]\)!
  Note \(\ker[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)\).
- \(T_M = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \im[J_+, J_-]g^{-1}\).
  - \(\im[J_+, J_-]g^{-1} \Rightarrow\) semi-chiral \(N = (2, 2)\) superfields.
    (Warning: type-changing!)
  - \(\ker(J_+ + J_-) \Rightarrow\) twisted chiral \(N = (2, 2)\) superfields.
  - \(\ker(J_+ - J_-) \Rightarrow\) chiral \(N = (2, 2)\) superfields.

(Lindström, Roček, von Unge, Zabzine ’05) (AS, Troost ’96)
$N = (2, 2)$ superspace

- Coordinates: $\sigma^+, \sigma^-, \theta^+, \theta^-, \hat{\theta}^+, \hat{\theta}^-$ (and corresponding derivatives).
- Action:
  $$S = 4 \int d^2\sigma \ d^2\theta \ d^2\hat{\theta} \mathcal{V}(X).$$
- $\mathcal{V}$ can only be some function of the scalar superfields $\Rightarrow$ constraints needed!
$N = (2, 2)$ superfields

- Simplest choice:
  \[ \hat{D}_\pm X^a = J_{\pm}^a b(X) D_\pm X^b. \]

- But:
  \[ \hat{D}_+^2 = D_+^2 = -\frac{i}{2} \partial_+, \quad \hat{D}_-^2 = D_-^2 = -\frac{i}{2} \partial_- \]
  and all other (anti-)commutators zero.

- Integrability conditions $\Rightarrow J_+$ and $J_-$ are commuting complex structures!
\textbf{N = (2, 2) superfields}

- ⇒ they can be simultaneously diagonalized with eigenvalues ±i.
- ⇒ two cases:
  1. They have the same eigenvalue ↔ chiral superfields ↔ ker \((J_+ - J_-)\). We call them \(z\) and \(\bar{z}\): \(\hat{D}_+ z = +i D_+ z\), \(\hat{D}_- z = +i D_- z\).
  2. They have the opposite eigenvalue ↔ twisted chiral superfields ↔ ker \((J_+ + J_-)\). We call them \(w\) and \(\bar{w}\): \(\hat{D}_+ w = +i D_+ w\), \(\hat{D}_- w = -i D_- w\). (Gates, Hull, Roček, ’84)
- Twisted chiral and chiral \(N = (2, 2)\) superfields have the same number of components as \(N = (1, 1)\) superfields ⇒ no new auxiliary dof’s are introduced.
$N = (2, 2)$ superfields

- Only other possibility: chiral constraints $\Rightarrow$ these $N = (2, 2)$ superfields have twice as many components compared to $N = (1, 1)$, half auxiliary?
- Auxiliary fields $\Leftrightarrow$ must come in complex pairs.
- These are semi-chiral superfields, $\text{im}[J_+, J_-]g^{-1}$. We call them $l, \bar{l}, r$ and $\bar{r}$. (Buscher, Lindström, Roček, ’88)

$$\hat{D}_+ l = i D_+ l, \quad \hat{D}_- r = i D_- r,$$

$\hat{D}_- l$ and $\hat{D}_+ r$ are auxiliary.
Note: other superfields possible

Besides semi-chiral, twisted chiral and chiral superfields other matter superfields are possible. Complex and real linear and twisted linear superfields exist as well: they are defined by constraints quadratic in the superspace derivatives. They provide a dual description to models in terms of chiral and twisted chiral superfields.

There are also gauge superfields, they are unconstrained real or complex superfields.
Action

• The action is simply

\[ S = 4 \int d^2 \sigma \, d^2 \theta \, d^2 \hat{\theta} \, V(X), \]

where \( V \) is an (arbitrary) real function of the semi-chiral, twisted chiral and chiral superfields.

• Integrating over \( \hat{\theta}^+ \) and \( \hat{\theta}^- \) and eliminating the auxiliary fields yields explicit expressions for \( J_+, J_-, g \) and \( b \). Generically they are non-linear expressions of derivatives of the generalized Kähler potential. Elegant expressions available.
Uniqueness

- $l \rightarrow l'(l, w, z)$, $r \rightarrow r'(r, \bar{w}, z)$, $w \rightarrow w'(w)$, $z \rightarrow z'(z)$.
- The potential is determined modulo a generalized Kähler transformation
  \[ V \rightarrow V + F(l, w, z) + \bar{F}(\bar{l}, \bar{w}, \bar{z}) + G(r, \bar{w}, z) + \bar{G}(\bar{r}, w, \bar{z}). \]
- Semi-chiral $\leftrightarrow$ Legendre transformations:
  \[ \hat{V}(\hat{l}, \bar{\hat{l}}, \hat{r}, \bar{\hat{r}}) = V(l, \bar{l}, r, \bar{r}) - F(l, \bar{l}) - \bar{F}(\bar{l}, \bar{l}) + G(r, \bar{r}) + \bar{G}(\bar{r}, \bar{r}) \]
- The local “mirror” transformation is
  \[ V(l, \bar{l}, r, \bar{r}, w, \bar{w}, z, \bar{z}) \rightarrow -V(l, \bar{l}, \bar{r}, r, z, \bar{z}, w, \bar{w}). \]
UV properties

One-loop $\beta$-function: necessary condition for $N = (2, 2)$ superconformal invariance @ quantum level. Counterterm: (Grisaru, Massar, AS, Troost, ’99)

$$S_{1\text{-loop}} \propto \frac{1}{\varepsilon} \int d^2 \sigma \, d^2 \theta \, d^2 \hat{\theta} \ln \frac{\det(N_+)}{\det(N_-)}$$

with

$$N_+ = \begin{pmatrix} V_{\bar{l}l} & V_{lr} & V_{l\bar{w}} \\ V_{\bar{r}l} & V_{\bar{r}r} & V_{\bar{r}\bar{w}} \\ V_{\bar{w}l} & V_{wr} & V_{w\bar{w}} \end{pmatrix}, \quad N_- = \begin{pmatrix} V_{\bar{l}l} & V_{l\bar{r}} & V_{l\bar{z}} \\ V_{\bar{r}l} & V_{\bar{r}\bar{r}} & V_{\bar{r}\bar{z}} \\ V_{\bar{z}l} & V_{z\bar{r}} & V_{z\bar{z}} \end{pmatrix}.$$ 

and vanishes $\iff$

$$\frac{\det(N_+)}{\det(N_-)} = \pm |f_+(l, w, z)|^2 |f_-(r, \bar{w}, z)|^2$$

Generalized CY $\iff$ $$\frac{\det(N_+)}{\det(N_-)} = \text{constant} \quad \text{E. g. (Hull, Lindström, Roček, von Unge, Zabzine ’12)}$$
Examples

Simple example: $SU(2) \times U(1) = S^3 \times S^1$.

- Parameterization:
  \[
  g = e^{i\rho} \begin{pmatrix}
  \cos \psi e^{i\varphi_1} & \sin \psi e^{i\varphi_2} \\
  -\sin \psi e^{-i\varphi_2} & \cos \psi e^{-i\varphi_1}
  \end{pmatrix},
  \]

  and

  $\varphi_1, \varphi_2, \rho \in \mathbb{R} \mod 2\pi$ and $\psi \in [0, \pi/2]$
**Type (1, 1):** 1 twisted chiral + 1 chiral

Superfields: (Roček, Schoutens, AS ’91)

\[ w = \cos \psi \ e^{-\rho-i\varphi_1}, \quad z = \sin \psi \ e^{-\rho+i\varphi_2} \]

Generalized Kähler potential:

\[ V_{\psi \neq \pi/2} = \int \frac{z\bar{z}}{\bar{w}w} \frac{dq}{q} \ln (1 + q) - \frac{1}{2} (\ln w\bar{w})^2 \]

or

\[ V_{\psi \neq 0} = -\int \frac{w\bar{w}}{z\bar{z}} \frac{dq}{q} \ln (1 + q) + \frac{1}{2} (\ln z\bar{z})^2 \]

and

\[ V_{\psi \neq \pi/2} - V_{\psi \neq 0} = -\ln (z\bar{z}) \ln (w\bar{w}) \]

**Note:** has in fact \( N = (4, 4) \) susy, can be lifted to projective superspace (see U. Lindström yesterday):

\[ V \propto \int_{\mathcal{C}} \frac{d\zeta^+}{\zeta^+} \int_{\mathcal{C}'} \frac{d\zeta^-}{\zeta^-} \ln \gamma \]
Type (0, 0): 1 semi-chiral

Superfields: (Troost, AS ’96)

\[ l = w, \quad r = \frac{\bar{w}}{z} \]

Generalized Kähler potential:

\[ V_{\psi \neq 0} = \ln \frac{l}{r} \ln \frac{\bar{l}}{r} - \int^{r\bar{r}} \frac{dq}{q} \ln (1 + q) \]

or

\[ V_{\psi \neq \frac{\pi}{2}} = -\ln \frac{l}{r} \ln \frac{\bar{l}}{r} + \int^{r\bar{r}} \frac{dq}{q} \ln (1 + q) \]

and

\[ V_{\psi \neq \frac{\pi}{2}} (l', \bar{l}', r', \bar{r}') = \frac{1}{2} (\ln r')^2 - \frac{1}{2} (\ln \bar{r}')^2 = V_{\psi \neq 0} (l, \bar{l}, r = r'^{-1}, \bar{r} = \bar{r}'^{-1}) - \ln l \ln l' - \ln \bar{l} \ln \bar{l}' \]

with

\[ l' = \frac{\bar{l}}{r}, \quad r' = \frac{1}{r} \]
• Both descriptions are T-dual, dualize along the $S^1$ of $S^1 \times S^3$.

• Note: type-changing occurs: at $\psi = \frac{\pi}{2}$: $J_- = -J_+$; at $\psi = 0$: $J_- = +J_+$.
Outlook

T-duality and doubled formalism

Intimate relation relation between \( N = (2, 2) \) \( \sigma \)-models and generalized CY-geometry. Hohm-Hull-Zwiebach: doubled formalism suggests an intricate generalized geometrical structure.

T-duality in \( N = (2, 2) \) superspace:

- Chiral \( \leftrightarrow \) twisted chiral (Gates, Hull, Roček, ’84)
  \[ V(w + \bar{w}, \cdots) \leftrightarrow \hat{V}(z + \bar{z}, \cdots) \]

- Chiral + twisted chiral \( \leftrightarrow \) semi-chiral (Grisaru, Massar, AS, Troost, ’98)
  \[ V(z + \bar{z}, w + \bar{w}, i(z - \bar{z} - w + \bar{w}), \cdots) \leftrightarrow \hat{V}(l + \bar{l}, r + \bar{r}, i(l - \bar{l} - r + \bar{r}), \cdots) \]

Doubled formalism: include both original and dual fields + “chirality constraint” in doubled space.
Outlook

Simple example

- The potentials,
  \[ V = \frac{1}{2} (z + \bar{z})^2 \]
  and
  \[ \hat{V} = -\frac{1}{2} (w + \bar{w})^2 \]
  are T-dual.
- In the doubled space with coordinates \( z, \bar{z}, w \) and \( \bar{w} \), the constraint,
  \[ w + \bar{w} = z + \bar{z} \]
  holds, \emph{i.e.} a kind of coisotropic brane is singled out, eliminating the “overdoubled” coordinates.
Outlook

• The Hull constraints \((dX \propto \ast d\tilde{X})\) follow:
  \[
  \hat{D}_\pm (w + \bar{w}) = \hat{D}_\pm (z + \bar{z}) \Rightarrow D_\pm (w - \bar{w}) = \pm D_\pm (z - \bar{z}).
  \]

Notation

• Introduce \(\mathbb{D}_\pm = \hat{D}_\pm + i D_\pm, \bar{\mathbb{D}}_\pm = \hat{D}_\pm - i D_\pm\).
  \(\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = \partial_\pm, \quad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = \partial_\mp\),
  all other (anti-)commutators vanish.

• Chiral field \(z\): \(\bar{\mathbb{D}}_+ z = \mathbb{D}_- z = 0\) (also \(\mathbb{D}_+ \bar{z} = \bar{\mathbb{D}}_- \bar{z} = 0\)).

• Twisted chiral field \(w\): \(\bar{\mathbb{D}}_+ w = \mathbb{D}_- w = 0\) (also \(\mathbb{D}_+ \bar{w} = \bar{\mathbb{D}}_- \bar{w} = 0\)).
Outlook

- Implies a theory of chiral bosons
  \[ D_{\pm}(w + \bar{w}) = D_{\pm}(z + \bar{z}) \Rightarrow D_+(z - w) = D_-(z - \bar{w}) = 0. \]
  But also,
  \[ \bar{D}_+(z - w) = \bar{D}_-(z - \bar{w}) = 0, \]
  implying,
  \[ \partial_{\pm}(z - w) = \partial_{\mp}(z - \bar{w}) = 0. \]
  So a kind of “chiral” semi-chiral multiplet.

- Extend PST to \( N = (2, 2) \) superspace: subtle but feasible.
- Study the doubled formulation (classical & quantum).

To be continued...