Calabi-Yau Manifolds, Hermitian Yang-Mills Instantons and Mirror Symmetry

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Based on joint works with Jungjai Lee, John J. Oh, Chanyong Park and Sangheon Yun

1. Yang-Mills Instantons from Gravitational Instantons

   J. J. Oh, C. Park & HSY, JHEP 1104 (2011) 087, arXiv:1101.1357

2. Einstein Manifolds As Yang-Mills Instantons

   J. J. Oh & HSY, arXiv:1101.5158

3. Calabi-Yau Manifolds, Hermitian Yang-Mills Instantons and Mirror Symmetry

   HSY & S. Yun, arXiv:1107. 2095

4.5. An Efficient Representation of Euclidean Gravity I & II
   J. Lee, J. J. Oh & HSY, to appear

Euclidean Gravity As Gauge Theory

On a Riemannian manifold M of dimension d, the spin connection ω is an $SO(d)$-valued one-form and can be identified with an $SO(d)$ gauge field:

$$\omega \to \omega' = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \quad \text{where} \quad \Lambda \in SO(d).$$

First introduce the d-dimensional Clifford algebra

$$\{\gamma^A, \gamma^B\} = 2\delta^{AB}, \qquad A, B = 1, \cdots, d.$$

Then the $SO(d)$ Lorentz generators are given by $J^{AB} = \frac{1}{4}[\gamma^A, \gamma^B]$ satisfying Lorentz algebra

$$[J^{AB}, J^{CD}] = -(\delta^{AC}J^{BD} - \delta^{AD}J^{BC} - \delta^{BC}J^{AD} + \delta^{BD}J^{AC}).$$

The $SO(d)$-valued spin connection is defined by $\omega = \frac{1}{2}\omega_{AB}\, J^{AB}$ where $\omega_{AB} = \omega_{MAB}\, dx^M$ are one-forms on M.

Then the $SO(d)$-valued Riemann curvature tensor is defined by

$$R = d\omega + \omega \wedge \omega \qquad (1)$$

$$= \frac{1}{2} R_{AB} J^{AB} = \frac{1}{2}(d\omega_{AB} + \omega_{AC} \wedge \omega_{CB})\, J^{AB}$$

$$= \frac{1}{4}(R_{MNAB} J^{AB})\, dx^M \wedge dx^N$$

$$= \frac{1}{4}[(\partial_M \omega_{NAB} - \partial_N\, \omega_{MAB} + \omega_{MAC}\, \omega_{NCB} - \omega_{NAC}\, \omega_{MCB})\, J^{AB}] dx^M \wedge dx^N$$

Now we introduce an $SO(d)$-valued gauge field defined by $A = A^a T_a$ where $A^a = A^a_M \, dx^M$ $\left(a = 1, \cdots, \frac{d(d-1)}{2}\right)$ are connection one-forms on M and T_a are Lie algebra generators of $SO(d)$ satisfying

$$[T_a, T_b] = -f_{ab}{}^{c} \; \mathrm{T_c}.$$

The identification we want to make is then given by

$$\omega = \frac{1}{2} \, \omega_{AB} J^{AB} \equiv A = A^a \, T_a. \qquad (2)$$

In terms of gauge theory variables, the curvature tensor is given by

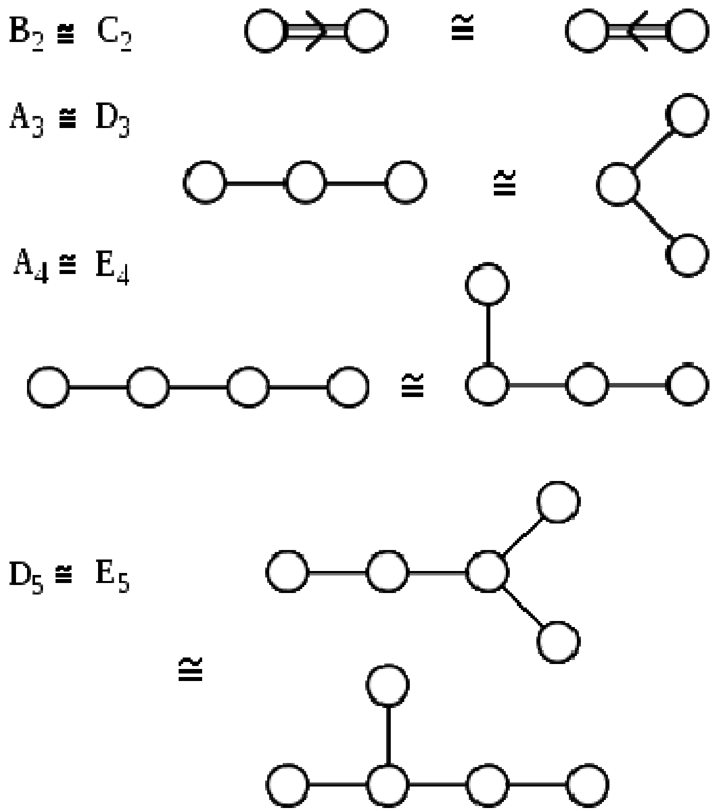
$$F = dA + A \wedge A \qquad (3)$$

$$= F^a\, T_a = \left(dA^a - \frac{1}{2}\, f_{bc}{}^{a}\, A^b \wedge A^c \right) T^a$$

$$= \frac{1}{2} (F^a_{MN} T^a)\, dx^M \wedge dx^N$$

$$= \frac{1}{2} \left[\left(\partial_M A^a_N - \partial_N A^a_M - f_{bc}{}^{a}\, A^b_M\, A^c_N \right) T^a \right] dx^M \wedge dx^N .$$

Lie group homomorphisms



$$SO(3) \cong SU(2)$$

$$SO(4) \cong SU(2)_L \times SU(2)_R$$

$$SO(5) \cong Sp(2)$$

$$SO(6) \cong SU(4)$$

Clifford and Exterior Algebras

- Clifford algebra $\mathbb{C}l(d)$ is isomorphic to the exterior algebra

$$\Lambda^{*}M = \bigoplus_{k=0}^{d} \Lambda^{k} T^{*}M$$

$$\mathbb{C}l(d) \cong \Lambda^{*}M = \bigoplus_{k=0}^{d} \Lambda^{k} T^{*}M$$

where the chirality operator Γ^{d+1} corresponds to the Hodge operator $*: \Lambda^{k} T^{*}M \to \Lambda^{d-k} T^{*}M$ and $\mathbb{C}l(d)$ is generated by the Dirac algebra

$$\{\Gamma^{a}, \Gamma^{b}\} = 2\,\delta^{ab} 1_{n}$$

Where $n = 2^{\left\lfloor \frac{d}{2} \right\rfloor}$. More precisely, the Clifford algebra may be thought of as a quantization of the exterior algebra in the same sense that the Weyl algebra Is a quantization of the symmetric algebra.

4-dimensional Einstein Gravity

The Hodge $*$-operator acts on a vector space $\Lambda^p T^*M$ and defines an automorphism of $\Lambda^2 T^*M$ with eigenvalues ± 1. Therefore, we have the decomposition

$$\Lambda^2 T^*M = \Lambda_3^+ \oplus \Lambda_3^-$$

Where $\Lambda_3^\pm \equiv P_\pm \Lambda^2\, T^*M$ and $P_\pm = \frac{1}{2}\,(1\ \pm *\,)$.

The above Hodge decomposition can harmoniously be incorporated with the group isomorphism $Spin(4) = SU(2)_L \times SU(2)_R$.

This feature is a mystique of 4 dimensions:

$SO(4)$ is the only non-simple Lorentz group and one can define a self-dual 2-form !

The ‘t Hooft symbols η^{a}_{AB} and $\bar{\eta}^{\dot{a}}_{AB}$ take a superb mission consolidating the Hodge decomposition and the Lie algebra isomorphism $so(4) \cong su(2)_L \times su(2)_R$, which intertwines the group structure of the index a and $\dot{a}$ with the spacetime structure of the indices A, B.

The 't Hooft matrices are two independent spin $s = \frac{3}{2}$ representations of $SU(2)$ Lie algebra.

A deep geometrical meaning of the 't Hooft symbols is to specify the triple (I, J, K) of complex structures of $\mathbb{R}^4 \cong \mathbb{C}^2$ as the simplest hyper-Kähler manifold for a given orientation.

The triple complex structures (I, J, K) form a quaternion which can be identified with the $SU(2)$ generators T^{a}_{+} and $T^{\dot{a}}_{-}$.

Since the group $SO(4)$ is a direct product of normal subgroups $SU(2)_L$ and $SU(2)_R$, i.e. $SO(4) = SU(2)_L \times SU(2)_R$, the four dimensional Euclidean gravity, when formulated as the $SO(4)$ gauge theory, will basically be two copies of $SU(2)$ gauge theories.

This structure can explicitly be realized by considering the following decomposition for spin connections

$$\omega_{MAB} \equiv A_M^{(+)a}\, \eta_{AB}^a + A_M^{(-)\dot{a}}\, \bar{\eta}_{AB}^{\dot{a}} \qquad (4)$$

and Riemann curvature tensor

$$R_{MNAB} \equiv F_{MN}^{(+)a}\, \eta_{AB}^a + F_{MN}^{(-)\dot{a}}\, \bar{\eta}_{AB}^{\dot{a}}, \qquad (5)$$

where

$$F_{MN}^{(\pm)} = \partial_M A_N^{(\pm)} - \partial_N A_M^{\pm} + \left[A_M^{(\pm)},\ A_N^{\pm}\right].$$

It turns out that $A_M^{(+)a}$ and $A_M^{(-)\dot{a}}$ are $\mathrm{SU}(2)_\mathrm{L}$ and $SU(2)_R$ gauge fields and $F_{MN}^{(+)a}$ and $F_{MN}^{(-)\dot{a}}$ are their field strengths.

Question: What is the Einstein equation from the gauge theory formulation ?

$$R_{AB} - \frac{1}{2}\delta_{AB} R + \Lambda\delta_{AB} = 0 \Leftrightarrow \ ?$$

We list some useful identities of the 't Hooft tensors

$$\eta^{(\pm)a}_{AB} = \pm\frac{1}{2}\varepsilon_{AB}{}^{CD}\eta^{(\pm)a}_{CD}, \tag{1.8}$$

$$\eta^{(\pm)a}_{AB}\eta^{(\pm)a}_{CD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} \pm \varepsilon_{ABCD}, \tag{1.9}$$

$$\varepsilon_{ABCD}\eta^{(\pm)a}_{DE} = \mp(\delta_{EC}\eta^{(\pm)a}_{AB} + \delta_{EA}\eta^{(\pm)a}_{BC} - \delta_{EB}\eta^{(\pm)a}_{AC}), \tag{1.10}$$

$$\eta^{(\pm)a}_{AB}\eta^{(\mp)b}_{AB} = 0, \tag{1.11}$$

$$\eta^{(\pm)a}_{AC}\eta^{(\pm)b}_{BC} = \delta^{ab}\delta_{AB} + \varepsilon^{abc}\eta^{(\pm)c}_{AB}, \tag{1.12}$$

$$\eta^{(\pm)a}_{AC}\eta^{(\mp)b}_{BC} = \eta^{(\mp)b}_{AC}\eta^{(\pm)a}_{BC}, \tag{1.13}$$

$$\varepsilon^{abc}\eta^{(\pm)b}_{AB}\eta^{(\pm)c}_{CD} = \delta_{AC}\eta^{(\pm)a}_{BD} - \delta_{AD}\eta^{(\pm)a}_{BC} - \delta_{BC}\eta^{(\pm)a}_{AD} + \delta_{BD}\eta^{(\pm)a}_{AC} \tag{1.14}$$

where $\eta^{(+)a}_{AB} \equiv \eta^a_{AB}$ and $\eta^{(-)a}_{AB} \equiv \bar{\eta}^a_{AB}$.

Also let us introduce two families of 4×4 matrices defined by

$$[T^a_+]_{AB} \equiv \eta^a_{AB}, \qquad [T^a_-]_{AB} \equiv \bar{\eta}^a_{AB}. \tag{1.15}$$

According to the above definition, the matrix representation of the generators in (1.15) is given by

$$T^1_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T^2_+ = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T^3_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.16}$$

$$T^1_- = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T^2_- = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad T^3_- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1.17}$$

Then Eqs. (1.12) and (1.13) immediately show that $T^a_\pm$ satisfy $SU(2)$ Lie algebras, i.e.,

$$[T^a_\pm, T^b_\pm] = -2\varepsilon^{abc}T^c_\pm, \qquad [T^a_\pm, T^b_\mp] = 0. \tag{1.18}$$

Lemma If M is an oriented 4-manifold, the Hodge $*$-operation is an involution of $\Lambda^2 T^*M$ which decomposes the two forms into self-dual and anti-self dual parts, $\Lambda^2\, T^*M = \Lambda_3^+ \oplus \Lambda_3^-$. The Riemann curvature 2-form can then be written as Eq.(5). With the decomposition (5), the Einstein equation

$$R_{AB} - \frac{1}{2}\,\delta_{AB} R + \Lambda\,\delta_{AB} = 0$$

for the 4-manifold M is equivalent to the self-duality equation of Yang-Mills instantons

$$F_{AB}^{(\pm)} = \pm\frac{1}{2}\varepsilon_{AB}{}^{CD}\; F_{CD}^{(\pm)}$$

where $F_{AB}^{(+)a}\eta_{AB}^{a} = F_{AB}^{(-)\dot{a}}\bar{\eta}_{AB}^{\dot{a}} = 2\Lambda.$

Proof: Let us further decompose the $SU(2)$ field strengths $F_{AB}^{(\pm)} \equiv E_A^M E_B^N F_{MN}^{(\pm)}$ in Eq.(5) according to the group structure of $SO(4) = SU(2)_L \times SU(2)_R$:

$$F_{AB}^{(+)a} \equiv f_{(++)}^{ab}\, \eta_{AB}^{b} + f_{(+-)}^{a\dot{a}}\, \bar{\eta}_{AB}^{\dot{a}},$$
$$F_{AB}^{(-)\dot{a}} \equiv f_{(-+)}^{\dot{a}a}\, \eta_{AB}^{a} + f_{(--)}^{\dot{a}\dot{b}} \bar{\eta}_{AB}^{\dot{b}}.$$

Then we get the following decomposition of the Riemann curvature tensor in Eq.(5)

$$R_{ABCD} = f_{(++)}^{ab}\, \eta_{AB}^{a}\eta_{CD}^{b} + f_{(+-)}^{a\dot{a}}\, \eta_{AB}^{a}\bar{\eta}_{CD}^{\dot{a}} + f_{(-+)}^{\dot{a}a}\, \bar{\eta}_{AB}^{\dot{a}}\eta_{CD}^{a} + f_{(--)}^{\dot{a}\dot{b}} \bar{\eta}_{AB}^{\dot{a}}\bar{\eta}_{CD}^{\dot{b}}.$$

The symmetry property of Riemann curvature tensor, $R_{ABCD} = R_{CDAB}$, leads to the following relation between coefficients

$$f_{(++)}^{ab} = f_{(++)}^{ba}, \qquad f_{(--)}^{\dot{a}\dot{b}} = f_{(--)}^{\dot{b}\dot{a}}, \qquad f_{(+-)}^{a\dot{a}} = f_{(-+)}^{\dot{a}a}\,.$$

The first Bianchi identity $\varepsilon^{ACDE} R_{BCDE} = 0$ further constrains the cofficients

$$f^{ab}_{(++)} \delta^{ab} = f^{\dot{a}\dot{b}}_{(--)} \, \delta^{\dot{a}\dot{b}}.$$

The above result can be applied to the Ricci tensor $R_{AB} \equiv R_{ACBC}$ and the Ricci scalar $R \equiv R_{AA}$ to yield

$$R_{AB} = (f^{ab}_{(++)} \, \delta^{ab} + f^{\dot{a}\dot{b}}_{(--)} \, \delta^{\dot{a}\dot{b}}) \, \delta_{AB} + 2 \, f^{a\dot{a}}_{(+-)} \, \eta^{a}_{AC} \bar{\eta}^{\dot{a}}_{BC},$$
$$R = 4 \, (f^{ab}_{(++)} \delta^{ab} + f^{\dot{a}\dot{b}}_{(--)} \, \delta^{\dot{a}\dot{b}}).$$

Hence the Einstein tensor $G_{AB} \equiv R_{AB} - \frac{1}{2} \, \delta_{AB} \, R$ has 10 independent components given by

$$G_{AB} = 2 \, f^{a\dot{a}}_{(+-)} \, \eta^{a}_{AC} \bar{\eta}^{\dot{a}}_{BC} - 2 f^{ab}_{(++)} \delta^{ab} \, \delta_{AB}.$$

The condition for the Einstein manifold satisfying $R_{AB} = \Lambda\,\delta_{AB}$ is given by

$$f^{ab}_{(++)}\delta^{ab} = f^{\dot a\dot b}_{(--)}\,\delta^{\dot a\dot b} = \frac{\Lambda}{2}, \qquad f^{a\dot a}_{(+-)} = 0. \tag{6}$$

Therefore the curvature tensor for an Einstein manifold reduces to

$$\begin{aligned} R_{ABCD} &= F^{(+)a}_{AB}\,\eta^{a}_{CD} + F^{(-)\dot a}_{AB}\,\bar\eta^{\dot a}_{CD} \\ &= f^{ab}_{(++)}\,\eta^{a}_{AB}\eta^{b}_{CD} + f^{\dot a\dot b}_{(--)}\bar\eta^{\dot a}_{AB}\bar\eta^{\dot b}_{CD} \end{aligned} \tag{7}$$

with coefficients satisfying (6).

It is obvious that the $SU(2)$ field strengths in Eq. (7) satisfy the self-duality equation

$$F^{(\pm)}_{AB} = \pm\frac{1}{2}\,\varepsilon_{AB}{}^{CD}\,F^{(\pm)}_{CD}\,. \qquad \text{QED.}$$

It is also easy to show the Yang–Mills equations

$$D_A^{(\pm)} F_{AB}^{(\pm)} = 0.$$

As a consequence, we arrive at an interesting result that any Einstein manifold with or without a cosmological constant always arises as the sum of $SU(2)_L$ instantons and $SU(2)_R$ anti–instantons. It explains why an Einstein manifold is stable because two kinds of instantons belong to different gauge groups, one in $SU(2)_L$ and the other in $SU(2)_R$, and so they cannot decay into a vacuum.

20 (Riemann) = 10 (Weyl) + 10 (Ricci).

Among the 20 components of Riemann tensors, only 10 components represented by the Weyl tensor are gravitational degrees of freedom which are not determined by matter distributions and the remaining 10 components given by the Ricci tensor are matter degrees of freedom which are determined by matter distribution through the Einstein equations.

So it will be interesting to see how the energy-momentum tensor of matter fields in the Einstein equation

$$R_{AB} - \frac{1}{2}\delta_{AB} R + \Lambda\delta_{AB} = 8\,\pi\, G\, T_{AB} \qquad (8)$$

deforms the structure of an Einstein manifold described by (7).
To be specific, consider the Einstein-Yang-Mills theory where the energy-momentum tensor of Yang-Mills gauge fields is given by

$$T_{AB} = \frac{2}{g_{YM}^2} Tr\left(F_{AC}F_{BC} - \frac{1}{4}\,\delta_{AB}\,F_{CD}F^{CD}\right).$$

The Yang-Mills field strength F_{AB} in the adjoint representation of gauge group G can similarly be decomposed according to the Lorentz group $SO(4)$

$$F_{AB} = f_{(+)}^{a}\,\eta_{AB}^{a} + f_{(-)}^{\dot{a}}\,\bar{\eta}_{AB}^{\dot{a}}.$$

Then the energy-momentum tensor is given by

$$T_{AB} = \frac{4}{g_{YM}^2} Tr\left(f_{(+)}^{a} f_{(-)}^{\dot{a}} \right) \eta_{AC}^{a} \bar{\eta}_{BC}^{\dot{a}}.$$

Finally the Einstein equation (8) can be written as the form

$$f_{(++)}^{ab}\delta^{ab} = f_{(--)}^{\dot{a}\dot{b}}\,\delta^{\dot{a}\dot{b}} = \frac{\Lambda}{2}, \qquad f_{(+-)}^{a\dot{a}} = \frac{16\pi G}{g_{YM}^2} Tr\left(f_{(+)}^{a} f_{(-)}^{\dot{a}} \right). \qquad (9)$$

The Einstein equations written in the form (9) show us a crystal-clear picture how (non-)Abelian gauge fields deform the structure of the Einstein manifold. They introduce a mixing of $SU(2)_L$ and $SU(2)_R$ sectors without disturbing the conformal structure and the instanton structure described by Eq. (7). This will not be the case for other fields such as scalar and Dirac fields.

Since Einstein manifolds carry a topological information in the form of Yang-Mills instantons as was shown using the gauge theory formulation of Euclidean gravity, it will be interesting to see how the topology of spacetime fabric is encoded into the local structure of gauge fields. In particular, the representation (7) provides us a powerful way to prove some inequalities about topological invariants for a compact Einstein manifold without boundary. The Euler characteristic $\chi(M)$ and the Hirzebruch signature $\tau(M)$ for a compact manifold *M* are, respectively, given by

$$\chi(M) = \frac{1}{32\pi^2} \int_M \varepsilon^{ABCD} R_{AB} \wedge R_{CD}$$
$$= \frac{1}{2\pi^2} \int_M d^4x \sqrt{g} \left[\left(f^{ab}_{(++)}\right)^2 + \left(f^{\dot{a}\dot{b}}_{(--)}\right)^2 \right] \geq 0.$$

$$\tau(M) = \frac{1}{24\pi^2} \int_M R_{AB} \wedge R_{AB}$$
$$= \frac{1}{3\pi^2} \int_M d^4x \sqrt{g} \left[\left(f^{ab}_{(++)}\right)^2 - \left(f^{\dot{a}\dot{b}}_{(--)}\right)^2 \right].$$

Hitchin–Thorpe inequality

$$\chi(M) - \frac{3}{2}\,\tau(M) = \frac{1}{\pi^2}\int_M d^4x\,\sqrt{\mathrm{g}}\left(f^{\dot{a}\dot{b}}_{(--)}\right)^2 \geq 0$$

where the equality hold iff $f^{\dot{a}\dot{b}}_{(--)} = 0$, i.e., M is half-flat (a gravitational instanton).

We can deduce the following relation for the topological numbers:

$$\chi(M) = \chi_+(M) + \chi_-(M) \equiv m \in \mathbb{Z}$$
$$\tau(M) = \tfrac{2}{3}\,(\chi_+(M) - \chi_-(M)) \equiv n \in \mathbb{Z},$$

where

$$\chi_+(M) \equiv \frac{1}{2\pi^2}\int_M d^4x\,\sqrt{g}\,\left(f^{ab}_{(++)}\right)^2 \geq 0,$$
$$\chi_-(M) \equiv \tfrac{1}{2\pi^2}\int_M d^4x\,\sqrt{g}\left(f^{\dot{a}\dot{b}}_{(--)}\right)^2 \geq 0.$$

This inequality means that

$$\chi_+(M) = \frac{2m + 3n}{4} \geq 0, \quad \chi_-(M) = \frac{2\mathrm{m} - 3\mathrm{n}}{4} \geq 0. \tag{9}$$

Furthermore the Poincaré duality implies that

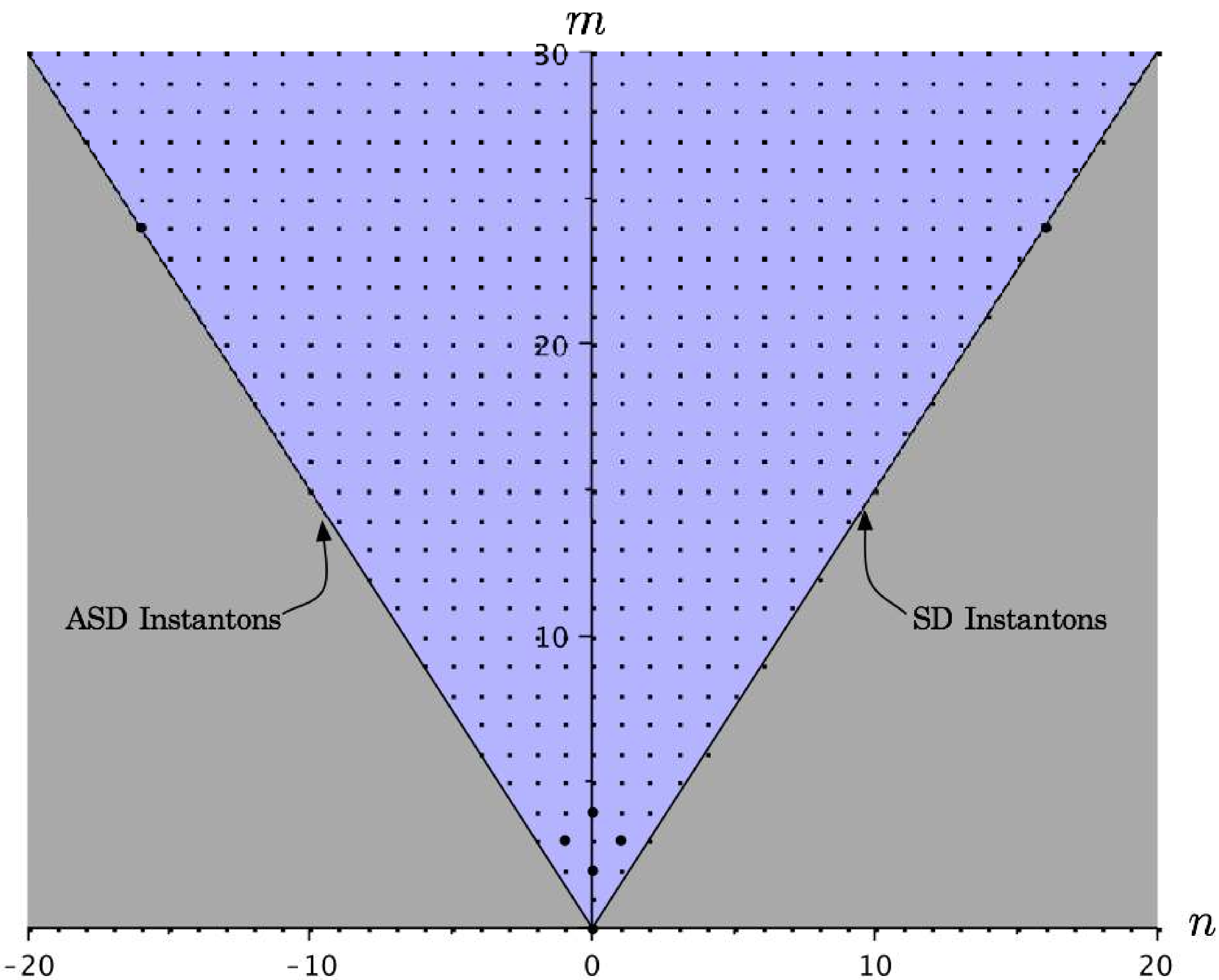
$$\chi(M) \equiv \tau(M) \quad (\mathrm{mod}\ 2),$$

that is, $m \equiv n$ (mod 2). Therefore the topological numbers (m, n) should be placed on an even integer lattice, i.e.,

$(m, n) =$(even, even) or (odd, odd)

satisfying the inequality (9).

$M_4(m,n)$
$K3(24,16)$
$\mathrm{Page}(4,0)$
$S^2 \times S^2(4,0)$
$T^2 \times S^2(2,0)$
$S^4(2,0)$
$CP^2(3,1)$
$T^4(0,0)$
$S^1 \times S^3(0,0)$



Generalizations to Higher Dimensions

In 4 dimensions,

gravitational instantons = $SU(2)$ Yang-Mills instantons.

$SO(4)$ = holonomy group of orientable Riemannian manifolds
$SU(2)_L \times SU(2)_R$ = gauge group of Yang-Mills theory
→ $SU(2)$ = holonomy group of Calabi-Yau 2-folds
= gauge group of Yang-Mills instantons.

In 6 dimensions,

Calabi-Yau 3-folds = $SU(3)$ Hermitian Yang-Mills instantons

$SO(6)$ = holonomy group of orientable Riemannian manifolds
$SU(4)$ = gauge group of Yang-Mills theory
→ $SU(3)$ = holonomy group of Calabi-Yau 3-folds
= gauge group of Hermitian Yang-Mills instantons.

In 6 dimensions, the 2-forms $\Lambda^2 T^* M$ can be classified by the ω -Hodge operator

$$*_\omega \equiv * (\bullet \wedge \omega) : \Lambda^2 T^* M \to \Lambda^4 T^* M \to \Lambda^2 T^* M.$$

$$\Lambda^2 T^* M = \Lambda^2_1 \oplus \Lambda^2_6 \oplus \Lambda^2_8 \,,$$

where ω is the Kähler 2-form of the Calabi-Yau 3-fold.
Then Λ^2_8 is an eigenvector space of $*_\omega$ with eigenvalue -1, i.e.,

$$* F_A = -F_A \wedge \omega,$$

which is called the Hermitian Yang-Mills (HYM) equation.

The vector space Λ^2_8 will be isomorphic to the $su(3)$ Lie algebra which supports the claim that
Calabi-Yau 3-folds = $SU(3)$ Hermitian Yang-Mills instantons satisfying the HYM (or Donaldson-Uhlenbeck-Yau) equation.

In 7 dimensions, I expect that a similar story will appear.
In this case the 2-forms $\Lambda^2 T^*M$ can be classified
by the ω -Hodge operator

$$*_\omega \equiv *(\bullet \wedge \omega): \Lambda^2 T^*M \to \Lambda^5 T^*M \to \Lambda^2 T^*M.$$

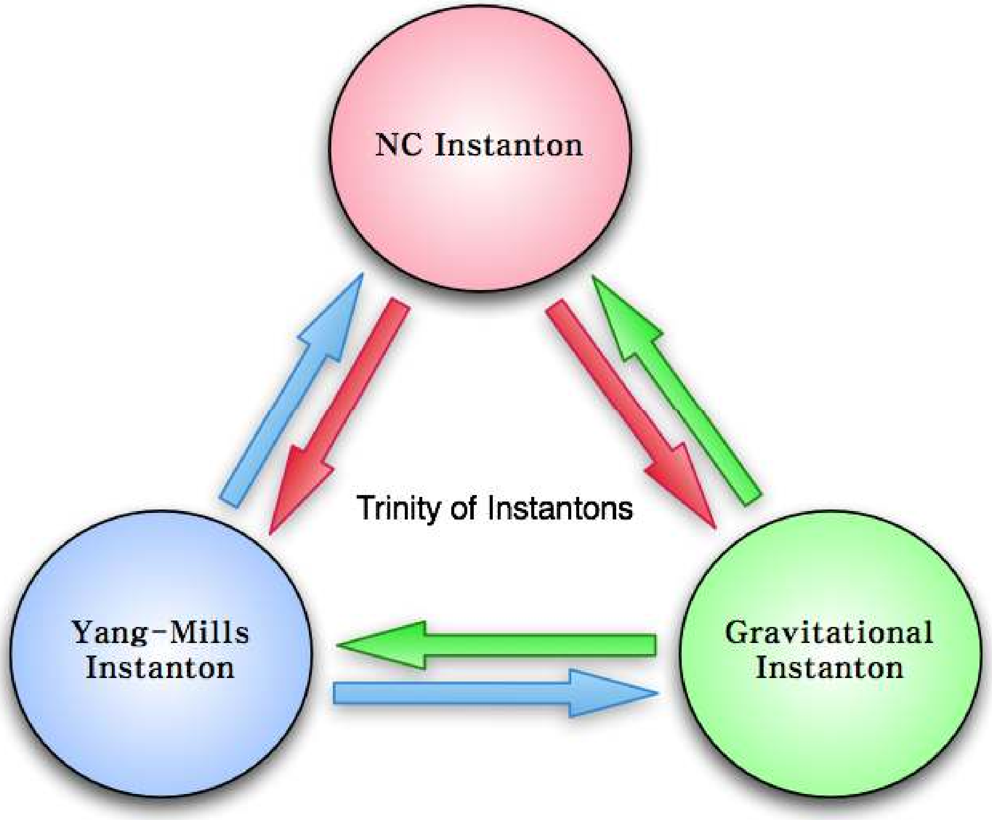
$$\Lambda^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{14},$$

where ω is the nondegenerate positive 3-form of the G_2-manifold.
Then Λ^2_{14} is an eigenvector space of $*_\omega$ with eigenvalue -1, i.e.,

$$*F_A = -F_A \wedge \omega,$$

which is called the Donaldson-Thomas equation.
The vector space Λ^2_{14} will be isomorphic to the Lie algebra of G_2.

Thus maybe G_2-manifolds = G_2 Yang-Mills instantons satisfying the Donaldson-Thomas equation.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Trinity of Instantons
Yang-Mills
Instanton
Gravitational
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