A Short Introduction to String theory

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Outline of the lecture

1. Lecture I: String theory: motivation and overview
2. Lecture II: Quantization and Spectrum
3. Lecture III: Interactions and Effective action
4. Lecture IV: Duality symmetries

- A short list of **References** for this course

There is no intellectual exercise that is not ultimately pointless
Jorge Luis Borges

Part I

String theory: motivation and overview

String theory postulates that the elementary objects are open or closed *fluctuating fundamental string*.

An extended object of extension p swaps a $p+1$ world-sheet $\Sigma^{(p+1)}$ embedded in a space-time $\mathcal{M}^{1,D-1}$ with metric signature $(-+\cdots+)$.

$$\Sigma^{(p+1)} \to \mathbb{R}^{1,D-1}$$

$$\xi^i \mapsto x^\mu(\xi)$$

The world-sheet coordinates are ξ^i with $i=0,\dots,p$ equipped with a metric of Lorentz signature $(-+\cdots+)$ and the embedding coordinates $x^\mu(\xi)$ with $\mu=0,\dots,D-1$

- The reparametrisation invariant world-volume action is given by the volume of the induced metric on the word-sheet $\hat{g}_{ab}$ pull-back of the spacetime metric $g_{\mu\nu}$

$$S_p^{NG} = -T_p \int d^{p+1}\xi \, \sqrt{-\hat{g}}$$

where $\hat{g} := \det \hat{g}_{ab}$

$$ds^2 = \hat{g}_{ab}(x(\xi))\, d\xi^a d\xi^b;$$

$$\hat{g}_{ab}(x(\xi)) = \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} g_{\mu\nu}$$

from now we use $\partial_a x^\mu := \frac{\partial x^\mu}{\partial \xi^a}$

- T_p is the tension. In a static gauge where the world-sheet time is proportional to the space-time time $\xi^0 \propto t$

$$S_p^{NG} = -T_p \int dt \, (\text{volume of extended object})$$

therefore the tension is energy per unit of p-dimensional volume

$$T_p = \frac{\text{Energy}}{\text{vol}_p} \sim (\text{Length})^{-1-p}$$

in units where $\hbar = c = 1$

- Introducing the auxiliary metric on the world-sheet $h_{ab}(\xi)$, it's inverse is $h^{ab}(\xi)$ and its determinant is $h := \det h_{ab}$

$$S_p = -\frac{T_p}{2} \int d^{p+1}\xi \, \sqrt{-h} \, \left(h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \Lambda_{(p)} \right)$$

- $\Lambda_{(p)} = p - 1$ is the world-sheet cosmological constant
- Varying the action with respect to to h^{ab} gives the stress-energy tensor

$$T_{ab} := \frac{2}{T_p} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}} = 0$$

$$T_{ab} = \hat{g}_{ab} - \frac{1}{2} h_{ab} \left(h^{cd} \, \hat{g}_{cd} + \Lambda_{(p)} \right)$$

- Solving the equation $T_{ab} = 0$ allows to express h_{ab} in terms of $\hat{g}$ and we get the Nambo-Goto action $\int d^{p+1}\xi \, \sqrt{-\hat{g}}$

What makes fundamental strings special? I

Specializing to $p = 1$ gives a 1-dimensional object swapping a 2-dimensional world-volume

String like objects can be used to describe many physical situation

- magnetic flux tubes in superconductor
- QCD color flux tubes
- cosmic strings

They are not all *fundamental* strings because they are in general effective objects.

The world-volume action gets higher derivative correction extrinsic curvature corrections, and have a *width*

$$S_1^{QCD-string} = -\int d^2\xi\left(T_1(\partial X)^2 + \frac{1}{\mu}(\partial^2 X)^2 + \cdots\right)$$

What makes fundamental strings special? II

curvature and instrinsic curvature corrections that are subleading compare to the Nambu-Goto action with higher derivative corrections from quantum fluctuations

$$S_p = -T_p \int d^2\xi\, \sqrt{-\hat{g}_{ab}}\, \left(1 + \alpha_1 \mathcal{R} + \alpha_2 R^2 + \cdots\right)$$

String theory posits the fundamental strings to have *zero width*

What makes fundamental strings special? III

- Let's consider a string fluctuation around a classical solution $X^\mu = x_0^\mu + \sqrt{T_F}^{-1}\, Y^\mu$

$$S = T_F \int d^2\xi\, g_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu$$
$$= \int d^2\,\xi\,(g_{\mu\nu}(x_0) + \sqrt{T}^{-1}\nabla_\rho g_{\mu\nu}\, Y^\rho + \cdots)\partial_a Y^\mu \partial^a Y^\mu$$

- This expansion makes sense if the curvature of space-time is small enough

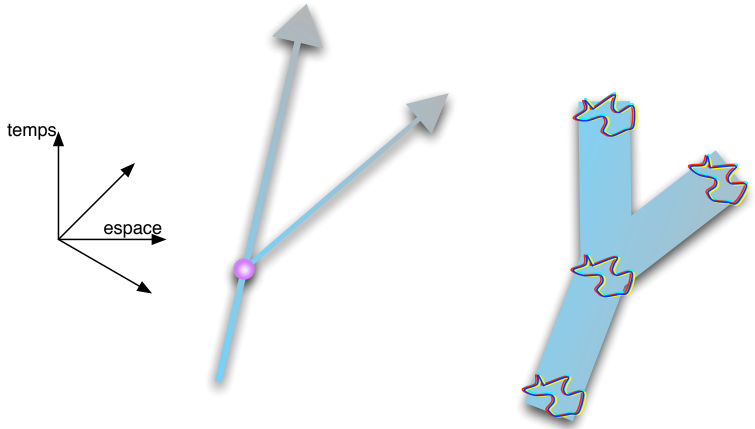
$$\nabla_\rho g_{\mu\nu} \sim \frac{1}{r} \ll \sqrt{T_F}$$

- with

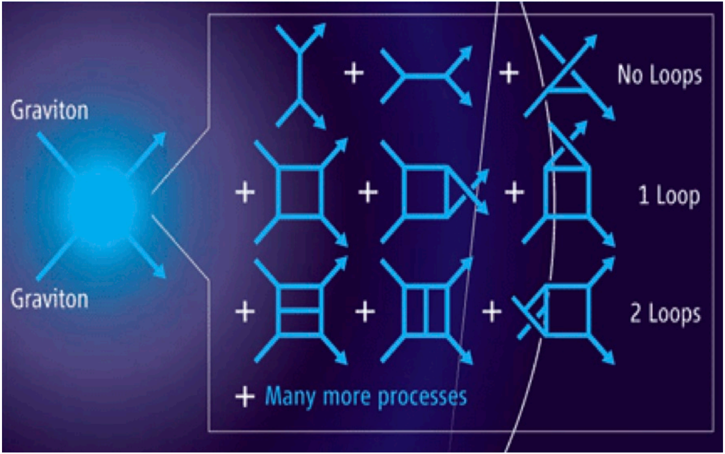
$$T_F \leqslant M_{pl}^2 = (10^{18}\mathrm{GeV})^2$$

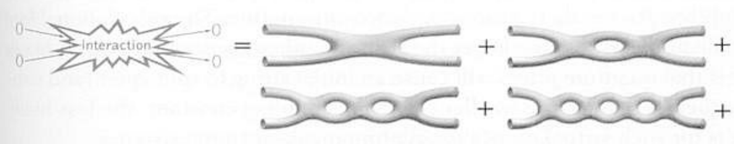
- To understand what value this should take we need to make contact with Gravity as will be done in a later lecture

WHAT MAKES FUNDAMENTAL STRINGS SPECIAL? IV



WHAT MAKES FUNDAMENTAL STRINGS SPECIAL? V





Quantization I

Quantization of this model will provide constraints on the value of the tension T_F, the dimension of the embedding space-time, the charge at the end of the strings

There are two approaches to string quantization

1rst quantized approach

- One chooses a specific classical background $\mathcal{M}^{1,D-1}$ on which one quantizes the world-sheet string action

$$
\begin{aligned}
\delta S \sim T_F & \int_\Sigma d^2\xi \sqrt{-h}\, h^{ab} \mathcal{G}_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu \\
& + q \oint_{\partial\Sigma} \mathcal{A}_\mu(X) \partial_a X^\mu dn^a \\
& + \int_\Sigma \Phi \mathcal{R}^{(2)}(h)
\end{aligned}
$$

QUANTIZATION II

- Interactions are derived by weak field perturbations of the background $g_{\mu\nu} = g^{o}_{\mu\nu} + \tilde{g}_{\mu\nu}$, $\Phi = \varphi + \Phi'$, $A_\mu = A^0_\mu + a_\mu$

$$e^{-\varphi\,\chi(\Sigma)} \int \mathcal{D}h\mathcal{D}X,\, e^{-S_1} \left(\int \tilde{g}_{\mu\nu}\partial X^\mu \cdot \partial X^\nu\right)\cdots\left(q\oint_{\partial\Sigma} dn\cdot a\right)$$

- The fundamental string feels the space-time background thru various corrections allowed by Weyl invariance. The conformal invariance of the string imposes that the background field satisfy their equations of motions and one can only compute *on-shell* S-matrix elements
- The contact with low-energy field theory is easy $T_F \to \infty$
- This approach is not background independent and does not capture the non-perturbative physics: the dualities of string theory allow to overcome this problem

Quantization III

2nd quantized approach: Another approach is to use interacting nonlinear action for string fields $\Psi[X^\mu(\sigma)]$ and compute amplitudes [Witten]

$$S[\Psi(X(\sigma))] = \langle \Phi, Q\Psi\rangle + \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle$$
$$Z = \int \mathcal{D}\Psi\, e^{\frac{i}{\hbar} S[\Psi(X(\sigma))]}$$

- allows to address non-perturbative questions: tachyon condensation [Sen]
- Can compute *off-shell* amplitudes
- This approach is background independent
- Very difficult to use in practice

String theory provides has a consistent unitarity and space-time local interactions. It has a well defined S-matrix approach (guaranteed by its high-energy behaviour) $s=-(k_1+k_2)^2$, $t=-(k_1+k_4)^2$, $u=-(k_1+k_3)^2=-s-t$

$$\lim_{s\gg 1} A_{4pt}^{\text{tree}} \sim \exp(-T_F^{-1}(s\log(T_F^{-1}s)+t\log(T_F^{-1}t)+u\log(T_F^{-1}u))$$

- String provides a consistent description of gauge interactions (non-abelian massless spin 1) and gravity (massless spin 2). The UV cutoff of effective field theory is $\Lambda^2 = T_F \sim \alpha'^{-1}$

$$\mathcal{L}^{\text{YM}} = \int d^4x \left(\frac{1}{g^2_{\text{YM}}}\text{tr}(F^2) + f(\alpha')\,\text{tr}(F^4) + g(\alpha')\text{tr}(F^2)^2 + \cdots\right)$$

$$\mathcal{L}^{\text{Grav}} = \int d^4x \left(\frac{1}{2\kappa^2}\,\mathcal{R} + c(\alpha')\,R^2 + d(\alpha')R^4 + \cdots\right)$$

- We can compute on-shell amplitudes and extract important informations on the structure of the low-energy effective action. This provides crucial information on
  - string degree of freedom
  - its perturbative and non-perturbative duality symmetry

Part II

QUANTIZATION AND SPECTRUM

The bosonic string I

- The coordinates of the string in flat space-time are given by $X^\mu(\tau,\sigma)$ from $\Sigma \to \mathcal{M}^{1,D-1}$ where the world-sheet $\xi = (\tau,\sigma)$ are the world-sheet coordinates with $0 \leqslant \sigma \leqslant L$ where L is the length of the string.
- We set $a, b = 0, 1$ where 0 is for the time direction τ and 1 for the σ direction. The world-sheet metric has signature $(-+)$

$$S = -\frac{T_1}{2}\int d\tau \int_0^L d\sigma\, \sqrt{-h}\, h^{ab}\, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

- From now we rescale the coordinate so that $\sigma \in [0,\pi]$ and introduce a new parameter α' such that

$$T_F = T_1 L = \frac{1}{2\pi\alpha'}$$

The bosonic string II

- For closed strings we ask for the periodicity condition $X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$

- The variation of the action with respect to to X^μ with

$$\delta X^\mu(\tau_{init}, \sigma) = 0 = \delta X^\mu(\tau_{final}, \sigma) \qquad \forall\, \sigma$$

$$\delta S = T_F \int d^2\xi\, \delta X^\mu \partial^2 X_\mu - T_F \int d^2\xi\, \partial^a(\delta X^\mu \partial_a X_\mu)$$

where $\partial^a \bullet = \partial_b(\sqrt{-h}\, h^{ab} \bullet)$ and $\partial^2 = \partial^a \partial_a$

- For the closed string the total derivative drops and

$$\frac{\delta S}{\delta X^\mu} = 0 \Longrightarrow \partial_a\left(\sqrt{-h}\, h^{ab} \partial_b X_\mu\right) = 0$$

THE BOSONIC STRING III

In the conformal gauge, where $h_{ab} = e^{\sigma(\xi)}\eta_{ab}$ and the (light-cone) coordinate system $\xi^{\pm} := \tau \pm \sigma$ with $\partial_{\pm} := \partial/\partial_{\xi^{\pm}} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ the eom take the form

$$\partial^2 X^\mu = 0; \qquad \partial_- \partial_+ X^\mu = 0$$

- For the open strings there are no periodicity conditions . For open strings with $0 \leqslant \sigma \leqslant \pi$ we have in addition to the eom we have the possible boundary conditions

- The free end Neumann boundary conditions

$$\partial_\sigma X^\mu(\tau, \sigma) = 0 \text{ at } \sigma = 0 \text{ or } \pi$$

The bosonic string IV

- The fixed end Dirichlet boundary conditions break Lorentz invariance

$$\delta X^\mu(\tau,\sigma) = 0 \text{ at } \sigma = 0 \text{ or } \pi$$

ie

$$X^\mu(\tau,\sigma) = y^\mu(\tau) \text{ at } \sigma = 0 \text{ or } \pi$$

- The end point of the string localized on an hyperplane. We can see the space-time as the end of the hyperplane where the end of the open string are fixed
- We can have mixed conditions NN, ND, DD

The conservation laws I

If the action is invariant under a transformation δx^μ we get the conserved current

$$j^a := \frac{\delta L}{\delta \partial_a x^\mu} \delta x^\mu$$

$$\partial_a j^a := 0$$

and conserved charges

$$J(\tau) := \int d\sigma\, j^0$$

$$\frac{dJ}{d\tau} := 0$$

The conservation laws II

- The stress-energy tensor is conserved charged for local diffeomorphism

$$
\begin{aligned}
\delta X^{\mu} &= \zeta^{a}\partial_{a}X^{\mu}\\
\delta h^{ab} &= \zeta^{c}\partial_{c}h^{ab}-\partial_{c}\zeta^{a}h^{cd}-\partial_{c}\zeta^{b}h^{ac}\\
T_{ab} &:= \frac{2}{T_p}\,\frac{1}{\sqrt{-h}}\,\frac{\delta S}{\delta h^{ab}}=0\\
T_{ab} &= \hat{g}_{ab}-\frac{1}{2}\,h_{ab}\,(h^{cd}\,\hat{g}_{cd}+\Lambda_{(p)})
\end{aligned}
$$

- We use diffeomorphism conformal invariance to set $h_{ab}=e^{\sigma(\xi)}\eta_{ab}$ where η_{ab} is the flat metric

THE CONSERVATION LAWS III

- The conservation $\partial^a T_{ab} = 0$ and traceless $T_a{}^a = 0$ of the stress-energy tensor in the light-cone gauge this gives

$$T_{++} = \frac{1}{2}\,\partial_+ X \cdot \partial_+ X = 0;$$
$$T_{--} = \frac{1}{2}\,\partial_- X \cdot \partial_- X = 0$$
$$T_{+-} = T_{-+} = 0$$

- The 2nd set of equation is trivialy satisfied because of the traceleness of the stress-energy tensor implied by Weyl invariance.
- The first sector constraints are the Viraroso constraints
- Conservation of the stress-energy tensor $\partial_\pm T_{\mp\mp} = 0$ is implied by the equation of motion $\partial_-\partial_+ X^\mu = 0$

THE CONSERVATION LAWS IV

- The global Poincaré invariance and

$$\delta x^\mu = \Lambda^\mu{}_\nu\, x^\nu + t^\mu; \qquad \Lambda^\mu{}_\nu \in SO(1, D-1)$$
$$\delta h_{ab} = 0$$

- the Lorentz rotations $\delta X^\mu = \Lambda^\mu{}_\nu X^\nu$ with $\Lambda^\mu{}_\nu \in SO(1, D-1)$ imply the conservation of the angular momentum $J^{\mu\nu}$

$$j_a^{\mu\nu} := T_F\,(x^\mu \partial_a x^\nu - x^\nu \partial_a x^\mu)$$
$$J^{\mu\nu} := \int_0^\pi d\sigma\, J^{0\mu\nu}$$

The conservation laws V

- the space-time translations $\delta X^\mu = \zeta^\mu$ lead to the momentum P^μ

$$p^{a\mu} := -T_F \partial^a x^\mu; \qquad P^\mu := \int_0^\pi d\sigma\, p^{0\mu} = T_F \frac{\partial x^\mu}{\partial_\tau}$$

- If we split the center of mass from the oscillatory part $X^\mu = x^\mu(\tau) + Y^\mu$ where

$$x_0^\mu(\tau) := \int_0^\pi d\sigma\, X^\mu(\tau, \sigma); \qquad \int_0^L d\sigma\, Y^\mu(\tau, \sigma) = 0$$

- no momentum is flowing thru the boundary (using $2\alpha' = (2\pi T_F)^{-1}$)

$$x^\mu(\tau) = x_0^\mu + 2\alpha' P^\mu\, \tau$$

The conservation laws VI

- The satisfy the classical Poincaré algebra

$$\begin{aligned}
[p^\mu, p^\nu] &= 0\\
[p^\mu, J^{\nu\rho}] &= i(\eta^{\mu\rho} p^\nu - \eta^{\mu\nu} p^\rho)\\
[J^{\mu\nu}, J^{\rho\lambda}] &= i(\eta^{\mu\rho} J^{\nu\lambda} - \eta^{\nu\rho} J^{\mu\lambda} - \eta^{\mu\lambda} J^{\nu\rho} + \eta^{\nu\lambda} J^{\mu\rho})
\end{aligned}$$

Canonical quantization I

- The canonical conjugate to X^μ is the momentum

$$P_\mu := \frac{\delta S}{\delta(\partial_\tau X^\mu)} = T_F\,\partial_\tau X_\mu(\tau,\sigma)$$

- impose the equal-time canonical commutators

$$[P_\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\delta(\sigma-\sigma')\,\delta^\mu{}_\nu$$
$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = [P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = 0$$

Mode expansion for closed strings I

Using the left and right moving coordinates $\xi^{\pm} = \tau \pm \sigma$ the eom $\partial_+ \partial_- X^\mu = 0$ is solved by $X^\mu(\tau, \sigma) = X_L^\mu(\xi^+) + X_R^\mu(\xi^-)$

We can consider the mode expansion such that

$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$

$$X_L^\mu(\xi^+) = \frac{1}{2}\, x_0^\mu + \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu\, \xi^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n}\, e^{-2in(\tau+\sigma)}$$

$$X_R^\mu(\xi^-) = \frac{1}{2}\, x_0^\mu + \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}_0^\mu\, \xi^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n}\, e^{-2in(\tau-\sigma)}$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, P^\mu$

Mode expansion for closed strings II

- The commutation relations betwen X and P lead to

$$
\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= m\delta_{m+n,0}\,\eta^{\mu\nu}\\
[x^\mu_0, P^\nu] &= i\eta^{\mu\nu}\\
[\alpha^\mu_m, x^\nu_0] &= [\alpha^\mu_n, P^\nu] = 0\\
[\alpha^\mu_n \tilde\alpha^\nu_n] &= 0
\end{aligned}
$$

- The mode expansion for the stress energy tensor

Mode expansion for closed strings III

$$L_n = \frac{T_F}{2}\int_0^\pi d\sigma\, e^{2in\sigma}\, T_{--}(0,\sigma) = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m$$

$$\tilde{L}_n = \frac{T_F}{2}\int_0^\pi d\sigma\, e^{-2in\sigma}\, T_{++}(0,\sigma) = \frac{1}{2}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m$$

These modes satisfy the Virasoro commutation relations

$$[L_n, L_m] = i(n-m)L_{n+m} + A(m)\,\delta_{m+n,0}$$

$$[\tilde{L}_n, \tilde{L}_m] = i(n-m)\tilde{L}_{n+m} + \tilde{A}(m)\,\delta_{m+n,0}$$

$$[L_n, \tilde{L}_m] = 0$$

Mode expansion for closed strings IV

- The Jacobi Identity $[L_n, [L_m, L_k]] + perms = 0$ implies that $A(m) = am^3 + bm$

- The Hamiltonian is given by

$$
\begin{aligned}
H_{closed} := L_0 + \tilde{L}_0 &= \frac{1}{2} \sum_{m\in\mathbb{Z}} \eta_{\mu\nu} \left(\alpha^{\mu}_{-m}\alpha^{\nu}_{m} + \tilde{\alpha}^{\mu}_{-m}\tilde{\alpha}^{\nu}_{m}\right) \\
&= \frac{\alpha'}{4} p^2 + \frac{1}{2} \sum_{m\in\mathbb{Z}\setminus\{0\}} \left(\alpha_{-m}\cdot\alpha_m + \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\right)
\end{aligned}
$$

Mode expansion of the open strings I

- with NN boundary conditions $\partial_\sigma X = 0$ at $\sigma = 0, \pi$ for all τ

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha^\mu_{0\,open}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\,e^{-in\tau}\cos(n\sigma)$$

where $\alpha^\mu_{0\,open} = \sqrt{2\alpha'}\,P^\mu$ the mode satisfy the reality condition $(\alpha_n^\mu)^* = \alpha^\mu_{-n}$ we have $\bar{L}_n = L_n$

- For DD boundary conditions $X = y_{0,\pi}$ at $\sigma = 0, \pi$

$$X^\mu(\tau, \sigma) = y_0^\mu\left(1 - \frac{\sigma}{\pi}\right) + y_\pi^\mu\sigma - \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\,e^{-in\tau}\sin(n\sigma)$$

where $y^\mu_{0,\pi}$ are the location of the D-brane where the end of open string are stuck

Mode expansion of the open strings II

- For ND and DN boundary conditions one has $1/2$ integer mode expansion.

$$X^\mu(\tau,\sigma) = i\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{\alpha_r^\mu}{r}\, e^{-ir\tau} \begin{cases} \sin(r\sigma) & \textit{for DN} \\ \cos(r\sigma) & \textit{for ND} \end{cases}$$

- The Hamiltonian is given by

$$H_{open} := L_0 = \frac{1}{2} \sum_{m\in\mathbb{Z}+r\backslash\{0\}} \alpha_{-m}\cdot\alpha_m + \begin{cases} \alpha' p^2 \& r = 0 & NN \\ (y_\pi - y_0)^2 \& r = 0 & DD \\ 0 \& r = \frac{1}{2} & ND \text{ or } DN \end{cases}$$

Physical states

- α_n^μ are called creation operators for $n < 0$ and annihilation operators for $n > 0$
- We require that the ground state satisfies
$$\alpha_n^i|\Omega\rangle = 0, \quad \forall n > 0$$
- The ground states $|p\rangle$ is defined as
$$|p\rangle = e^{ip\cdot X}|\Omega\rangle$$
- Clearly this state satisfies
$$P^\mu|p\rangle = p^\mu\,|p\rangle$$
- this is the ground state of the string with momentum p^μ

Hilbert space of physical states

Physical states are excited states obtained by the action of α^{μ}_{-n} with $n > 0$ on the ground state.
We want to define the Hilbert space of physical states $|\text{phys}\rangle$
We need to impose the Virasoro constraints $T_{\pm\pm} = 0$ on the states

$$T_{\pm\pm}|\text{phys}\rangle = 0$$

We need to impose the constraints
One can covariantly quantize then impose the constraints that eliminates the negative norm states from the physical Hilbert space

$$(L_n - a\delta_{n,0})|\text{phys}\rangle = 0, \qquad n \geqslant 0$$

where a is a quantum normal ordering ambiguity discussed later.
Covariantly this includes square roots which can be avoided by going to the space-time light-cone gauge. One then breaks space-time lorentz invariance. Convienient for spectrum generating questions.

The light-cone gauge I

This gauge is very practical determining the physical degrees of freedom and the spectrum of the theory
We define the light-cone coordinates

$$X^{\pm} := \frac{X^0 \pm X^{D-1}}{\sqrt{2}}$$

the rest of the coordinates are denoted X^i with $i = 1, \ldots, D-2$
The Virasoro constraints becomes

$$\partial_{\pm} X^+ \partial_{\pm} X^- = \frac{1}{2} \sum_{i=2}^{D-2} \partial_{\pm} X^i \partial_{\pm} X^i$$

We can use reparametrization on the world-sheet to set all the oscillator modes of X^+ to zero (recall $\xi^{\pm} := \tau \pm \sigma$)

$$X^+(\tau, \sigma) = x^+ + 2\alpha' p^+ \tau$$

The light-cone gauge II

we then get a constraint equation for X^-

$$\partial_- X^- = \frac{1}{2\alpha' p^+}\, \partial_- X^i \partial_- X^i$$

This determines all the modes and constant part of X^- in terms of the modes α^i_n for the transverse coordinates.

$$p^- = \frac{1}{2\alpha' p^+} \sum_{n\geqslant 1} \alpha^i_{-n}\alpha^i_n + \frac{1}{2p^+}\, p^i p^i$$

$$\alpha^-_n = \frac{1}{\sqrt{8\alpha'} p^+} \Big(\sum_{m\in\mathbb{Z}} :\alpha^i_{n-m}\alpha^i_m: - a\delta_n \Big)$$

One canonically quantizes the transverse modes, but one needs to pay attention to normal ordering ambiguities from $\{\alpha^i_{-m}, \alpha^j_m\} = m\, \delta^{ij}$.

The light-cone gauge III

$$[\alpha^i_m, \alpha^j_n] = m\delta_{m+n,0}\delta^{ij}; \qquad [x^i, p^j] = i\delta^{ij}; [\alpha^i_n, x^j] = [\alpha^i_n, p^j] = 0$$

We need to check how the light-cone gauge condition $X^+ = x^+\alpha' p^+\tau$ is compactible with the full $SO(1, D-1)$ Lorentz invariance. This leads to non-linear constraints which can develop anomalies at the quantum level. The potential source of anonmaly is in the commutator which should vanish

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2} \sum_{m\geqslant 1} \Delta_m \left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right)$$

where

$$\Delta_m = m\,\frac{26-D}{12} + \frac{1}{m}\left(\frac{D-26}{12} + 2(1-a)\right)$$

which implies that $D = 26$ and $a = 1$ and we have full full $SO(1, 25)$ Lorentz invariance

Spectrum I

In the light-cone gauge the spectrum obtained by considering the excited states of the ground state

$$A^{i_1\cdots i_r}_{n_1\cdots n_r} := \alpha^{i_1}_{-n_1}\cdots\alpha^{i_r}_{-n_r}|p\rangle$$

The mass of the states is obtained from $L_0|\text{state}\rangle = 0$, but we must pay attention to normal ordering issues from the commutation relation $\alpha^i_{-m}\alpha^j_m = \alpha^j_m\alpha^i_{-m} - m\,\delta^{ij}$ we have

$$L_0 - a = \alpha' p^2 + \sum_{n\geqslant 1}\alpha^i_{-n}\alpha^i_n - 1$$

Zeta function regularisation gives

$$a = -\frac{D-2}{2}\sum_{m\geqslant 1} m = -\frac{D-2}{2}\zeta(-1) = \frac{D-2}{24}$$

Spectrum II

Setting the number operators that counts the number of excitations of mode number n

$$N := \sum_{n\geqslant 1} n\, N_n; \qquad N_n := \frac{\alpha^i_{-n}\alpha^i_n}{n}$$

The exited states have a mass

$$\alpha' M^2 = -\alpha' p^2 = \sum_{n\geqslant 1} n\, N_n - \frac{D-2}{24}$$

The normal ordering constant determines the mass of the ground state

$$\alpha' M^2 = -\frac{D-2}{24}$$

Spectrum III

- This ground state has negative mass for $D > 2$, it is tachyonic and the theory is unstable.
- At level one we can construct the massless state $|\zeta, p\rangle = \zeta_i\, \alpha^i_{-1}|p\rangle$ which has the mass

The action of $L_0|\mathrm{phys}\rangle = 0$ implies that

$$\alpha' M^2 = \frac{26 - D}{24} = 0$$

a massless state in $D = 26$

Covariant quantization I

Consider local operator $A(\tau, 0)$ creating a state at one end of the string say $\sigma = 0$
Under change of variables $\tau \to \tau' = \tau + \epsilon$ this operator changes by $A(\tau)(d\tau)^J = A(\tau')(d\tau')^J$

$$\delta A(\tau, 0) = -\epsilon \frac{dA}{d\tau} + JA\frac{d\epsilon}{d\tau}$$

where J is conformal dimension. Since the L_m generate the diffeomorphism on the world-sheet we have

$$[L_m, A] = e^{im\tau}\left(-i\frac{d}{d\tau} + Jm\right)A(\tau, 0)$$

If one decomposes A in modes

$$A(\tau, 0) = \sum_{n\in\mathbb{Z}} A_n\, e^{-in\tau}$$

Covariant quantization II

then

$$[L_m, A_n] = (m(J-1) - n)\, A_{m+n}$$

When $J = 1$, the zero mode A_0 commutes with the L_n and will preserve physical state conditions when acting on another physical state

In particular we can use A_0 on the ground state $|p\rangle$

If one considers the vertex operator

$$V(\zeta, k) :=: \zeta_\mu \partial_\tau X^\mu\, e^{ik\cdot X} :$$

the condition that this state has conformal dimension one implies

$$k^2 = 0; \qquad k\cdot\zeta = 0$$

We see that this state is the covariant version of our level 1 light-cone state by noting that

Covariant quantization III

$$V_0 = \int_0^{2\pi} d\tau \, V(\zeta, k) = \zeta_\mu \alpha^\mu_{-1} + \cdots$$

so $V_0|\Omega\rangle = \zeta_\mu \alpha^\mu_{-1} e^{ik\cdot X}|\Omega\rangle$
Under the transformation $\zeta_\mu \to \zeta_\mu - ik_\mu \Lambda$

$$\delta_\Lambda V(\zeta, k) = -i\Lambda \frac{d}{d\tau} e^{ik\cdot X}$$

We remark that this total derivative total derivative state decouples from amplitudes as should all longtidual states
This corresponds to the Fock space state $k \cdot \alpha_{-1}|k\rangle = L_{-1}|k\rangle$ which is a null state since $k^2 = 0$ and is orthogonal to any physical states $\langle \mathrm{phys}|L_{-1}|k\rangle = 0$

Covariant quantization IV

Now we can interpret the state or the vertex operators as describing a massless spin 1 field for which

$$p_\mu p^\mu \zeta_\nu = 0; \qquad p^\mu \zeta_\mu = 0$$

are the linearized equations of motion and bianchi identities

$$\partial^\mu F_{\mu\nu} = 0; \qquad \partial_{[\mu} F_{\nu\rho]} = 0$$

The vertex operator corresponds to the creation of a massless spin 1 state at the end $\sigma = 0$ of the open string
We can couple the end of the string to an external gauge field $A_\mu = A^0_\mu + a_\mu$ to get

Covariant quantization V

$$\exp(q\int_{\partial\Sigma} d\tau A_\mu(X)\partial_\tau X^\mu)$$
$$= \exp(q\int_{\partial\Sigma} d\tau A^0_\mu(X) d_\tau X^\mu)\,(1 + q\int_{\partial\Sigma} d\tau\, a_\mu(X)\partial_\tau X^\mu + \cdots)$$

for the plane-wave $a_\mu = \zeta_\mu e^{ik\cdot X}$ we have the vertex operator

Gauge groups

- The end of open string can be in some representation R of some group

$$|\Omega; a\rangle = \sum_{i,j} |\Omega; ij\rangle \, \lambda^a_{ij}$$

- Each excited states will carry group indices

$$A^a_\mu = \sum_{ij} A^{ij}_\mu \, \lambda^a_{ij} \sim \alpha^\mu_{-1} e^{ip\cdot X} |\Omega; a\rangle$$

- Where λ^a are some matrices that will be in some representation of a gauge group G
- We will need to consider amplitudes and unitarity for restricting G and anomaly cancellation will restrict it further

The closed string spectrum I

- The spectrum is the tensor product of the open string spectrum $|\Phi\rangle_{closed} = |\Phi\rangle_{open} \otimes |\tilde{\Phi}\rangle_{open}$ and the physical conditions are

$$L_m|\text{phys}\rangle = \bar{L}_m|\text{phys}\rangle = 0; \qquad m > 0$$
$$(L_0 - \frac{D-2}{24})|\text{phys}\rangle = (\tilde{L}_0 - \frac{D-2}{24})|\text{phys}\rangle = 0$$

$$(L_0 + \tilde{L}_0 - \frac{D-2}{12})|\text{phys}\rangle = 0; \qquad \text{Mass-shell condition}$$
$$(L_0 - \tilde{L}_0)|\text{phys}\rangle = 0; \qquad \text{Level matching condition}$$

- The zero mode for the momentum is commmon to left and right movers

The closed string spectrum II

- The level matching condition tells that the string does not have angular momentum

- An exited state

$$\alpha^{i_1}_{-n_1} \cdots \alpha^{i_r}_{-n_r} \tilde\alpha^{j_1}_{-m_1} \cdots \tilde\alpha^{j_s}_{-m_s} |p\rangle$$

- has mass and level matching condition

$$\alpha' M^2 = 2 \left(\sum_{n\geqslant 1} \alpha^i_{-n}\alpha^i_n + \tilde\alpha^i_{-n}\tilde\alpha^i_n - \frac{D-2}{12} \right)$$

$$(\text{zero-modes} =) = 0 = \sum_{n\geqslant 1} \alpha^i_{-n}\alpha^i_n - \sum_{n\geqslant 1} \tilde\alpha^i_{-n}\tilde\alpha^i_n$$

The closed string spectrum III

The absence of anomaly in the Lorentz algebra implies that $D = 26$ the spectrum is

- fundamental is tachyonic $|p\rangle$ of mass $\alpha' M^2 = -4$
- The first exicted states $\zeta_{ij}\alpha^i_{-1}\tilde{\alpha}^j_{-1}|p\rangle$ is massless

Decomposition of irrep for $SO(24)$ one gets from the symmetric traceless part the spin 2 graviton g_{ij}, the antisymmetric part the B_{ij} field and the trace is the dilaton Φ

Again one can check that the physical conditions on the state

$$\zeta_{\mu\nu}(k)\,\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|k\rangle$$

imply the *linearized* equation of motion and gauge invariance for the graviton the B-field and dilaton

$$S_{grav} \propto \int d^{26}x\,\sqrt{-G}\left(\mathcal{R} - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}(dB)^2\right)$$

The Superstring I

The bosonic string has a tachyon in its spectrum and does not contain fermions. The correct spectrum can be obtained by considering supersymmetric extension We will be working with the covariant RNS [Ranmond-Neveu-Schwarz] model is given by the action written by [Deser-Zumino-Brink-di Vecchia-Howe]

In flat space $g_{\mu\nu} = \eta_{\mu\nu}$ the action is given by

$$S = -\frac{T_F}{2}\int d^2\sigma\,\sqrt{-h}\,\big(h^{ab}\partial_a X^\mu \partial_b X^\nu\,\eta_{\mu\nu} - i\bar\psi^\mu\,\rho^a\partial_a\psi^\mu$$
$$+ 2\bar\chi_a\rho^b\rho^a\partial_b X_\mu + \frac{1}{2}\bar\psi^\mu\psi_\mu\bar\chi_a\rho^b\rho^a\chi_b\big)$$

- X^μ and ψ^μ are space-time vector and ψ^μ is a world-sheet fermion
- (h^{ab}, χ_a^α) are the two-dimensional supergravity fields
- where ρ^a are 2×2 gamma-matrices

The Superstring II

In addition to the diffeomorphism and Weyl invariance this action has global susy invariance

$$\delta_\epsilon X^\mu = \bar{\epsilon}\psi^\mu; \qquad \delta_\epsilon \psi^\mu = -i\rho^a \partial_a X^\mu$$

The conserved supercurrent implies a new constaints on physicals states

$$J_{\alpha\,a} := \frac{1}{2}\,\rho^b \rho_a \psi^\mu_\alpha \partial_b X_\mu; \qquad \partial^a J_{a\alpha} = 0$$

The super-conformal gauge $h_{ab} = e^{\sigma(\xi)}\eta_{ab}$ and $\chi_{\alpha\,a} = 0$ with $\psi^\mu := \psi^\mu_L + \psi^\mu_R$

$$S = -\frac{T_F}{2}\int d^2\xi\,(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu_L\,\partial_- \psi^\mu_L - i\bar{\psi}^\mu_R\,\partial_- \psi^\mu_R)$$

The equation of motion are in the world-sheet light-cone gauge

$$\partial_-\partial_+ X^\mu = 0; \qquad \partial_- \psi^\mu_L = 0; \qquad \partial_+ \psi^\mu_R = 0$$

The constraints

$$
\begin{aligned}
T_{++} &= \frac{1}{2}\,\partial_+ X\cdot\partial_+ X + \frac{i}{2}\,\Psi_L\cdot\partial_+\Psi_L\\
T_{--} &= \frac{1}{2}\,\partial_- X\cdot\partial_- X + \frac{i}{2}\,\Psi_R\cdot\partial_-\Psi_R\\
T_{+-} &= T_{-+} = 0\\
J_{L+} &= \frac{1}{2}\,\Psi_L\cdot\partial_+ X\\
J_{R-} &= \frac{1}{2}\,\Psi_R\partial_- X
\end{aligned}
$$

Closed string mode expansion I

periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions for the fermions

$$\begin{aligned} R: &\qquad \psi^\mu(\tau, \sigma+\pi) = \psi^\mu(\tau,\sigma) \\ NS: &\qquad \psi^\mu(\tau, \sigma+\pi) = -\psi^\mu(\tau,\sigma) \end{aligned}$$

- The mode expansion for the right movers

$$\begin{aligned} R: &\qquad \psi^\mu_R(\tau,\sigma) = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} d^\mu_n \, e^{-2in(\tau-\sigma)} \\ NS: &\qquad \psi^\mu_R(\tau,\sigma) = \sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r \, e^{-2ir(\tau-\sigma)} \end{aligned}$$

Closed string mode expansion II

- We define the modes of the supercurrent as

$$G_m = \frac{T_F}{2}\int_0^\pi d\sigma\, J_{L-}(0,\sigma)\, e^{2im\sigma}$$

The NS and R boundary conditions can be chosen independently for the left and right movers giving rise to four sectors $R-R$, $R-NS$, $NS-R$ and $NS-NS$ we will see in a minute that one must restrict this by the GSO projection to get the correct spectrum

Closed string mode expansion III

The bosonic and fermionic constraints now read

- in the R sector

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{n\in\mathbb{Z}}(n+\frac{m}{2})\, d_{-n}\cdot d_{n+m}$$

$$G_m = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{n+m}$$

- in the NS sector

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_{n+m} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}(r+\frac{m}{2})\, b_{-r}\cdot b_{m+r}$$

$$G_s = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{m+s}$$

Open superstring

In the same way than the variation of the bosonic string lead to various boundary condtions at the end of the string for the bosonic coordinate, we have for the fermionic coordinate

$$\delta S_\psi = 0 \Longrightarrow \int d\tau \left[\delta\psi_L \cdot \psi_L - \delta\psi_R \cdot \psi_R\right]_0^\pi = 0$$

Solved by

$$\psi_R^\mu(\tau, 0) = \eta_1 \psi_L^\mu(\tau, 0); \qquad \psi_R^\mu(\tau, \pi) = \eta_2 \psi_L^\mu(\tau, \pi)$$

where $\eta_{1,2} = \pm 1$. If $\eta_1 = \eta_2$ the Ramond boundary conditions, $\eta_1 = -\eta_2 = 1$ the Neveu-Schwarz boundary condition.
We can combine these boundary conditions with the NN, DD, ND DN boundary conditions for the bosonic coordinates

Quantizing the superstring I

We perform the light-cone quantization

$$S = T_F \int d^2\xi \,(\partial_+ X \cdot \partial_- X + i\psi_L \partial_- \psi_L + \psi_R \partial_+ \psi_R)$$

with the constraints

$$\begin{aligned}
\partial_- X \cdot \partial_- X + \frac{i}{2}\psi_R \partial_- \psi_R &= 0\\
\partial_+ X \cdot \partial_+ X + \frac{i}{2}\psi_L \partial_+ \psi_L &= 0\\
\psi_L \cdot \partial_- X &= 0\\
\psi_R \cdot \partial_+ X &= 0
\end{aligned}$$

We make the gauge choice $X^+ = x^+ + 2\alpha' p^+ \tau$ and supersymmetry allows us to set $\psi^+_{L,R} = 0$

Quantizing the superstring II

Now the constraint equation read

$$\partial_- X^- = \frac{1}{2\alpha' p^+} \left(\partial_- X^i \partial_- X^i + \frac{i}{2}\psi_R^i \partial_- \psi_R^i\right)$$
$$\psi_R^- = \frac{1}{2\alpha' p^+} \psi_R^i \partial_- X^i$$

A state is obtained by acting with bosonic creation oscillators and the fermionic ones

$$\alpha^{i_1}_{-n_1} \cdots \alpha^{i_r}_{-n_r} b^{j_1}_{-m_1+\delta} \cdots b^{j_s}_{-m_s+\delta} |p\rangle$$

where $\delta = 0$ for the Ramond boundary conditions and $\delta = 1/2$ for the Neveu-Schwarz boundary conditions

Quantizing the superstring III

As before we need to consider normal ordering

$$L_0^{\rm NS} - a_{\rm NS} = \alpha' p^2 + \sum_{n\geqslant 1} \alpha^i_{-n}\alpha^i_n + \sum_{r\geqslant \frac12} r b^i_{-r} b^i_r + \frac{D-2}{2}\,(1+\frac{1}{2}) \sum_{m\geqslant 1} m$$

$$L_0^{\rm R} - a_{\rm R} = \alpha' p^2 + \sum_{n\geqslant 1} \alpha^i_{-n}\alpha^i_n + \sum_{n\geqslant 1} n d^i_{-n} d^i_n + \frac{D-2}{2}\,(1-1) \underbrace{\sum_{m\geqslant 1} m}_{-\frac{1}{12}}$$

- Again the absence of anomaly in the Lorentz algebra requieres $D = 10$

Quantizing the superstring IV

In the NS sector the ground state is defined as $|p\rangle_{NS} := e^{ip\cdot x}|\Omega\rangle_{NS}$

$$\alpha^i_n|p\rangle_{NS} = b^i_{r+\frac{1}{2}}|p\rangle_{NS} = 0, \qquad n, r > 0$$

its mass is $\alpha' M^2 = -\frac{D-2}{16}$

- The first excited level is a $SO(D-2)$ vector $b^i_{-\frac{1}{2}}|k\rangle$ with mass $\alpha' M^2 = \frac{10-D}{16}$ which is massless for $D = 10$

SPACE-TIME SUPERSYMMETRY I

The ground state is defined in Ramond sector as

$$\alpha_n^i|\Omega\rangle = d_n^i|\Omega\rangle = 0, \qquad n > 0$$

We consider the ground state with momentum

$$|p\rangle := e^{ip\cdot x}|\Omega\rangle$$

Since $[L_0, d_0^i] = 0$ then $d_0^i|p\rangle$ and $|p\rangle$ have the same mass
From the commutation relations $\{d_0^i, d_0^j\} = \delta^{ij}$ we build of Fock space composed of creation and annhilation operators

$$\delta_i := \frac{1}{\sqrt{2}}(d^{2i-1} + id^{2i}); \quad (\delta_i)^\dagger \ := \frac{1}{\sqrt{2}}(d^{2i-1} - id^{2i}); \quad 1 \leqslant i \leqslant 4$$

$$\{\delta_i, (\delta_j)^\dagger\} = \delta_{ij}; \qquad \{\delta_i, \delta_j\} = 0$$

The ground state is now defined as

$$\delta_i|p\rangle_R = 0; \qquad 1 \leqslant i \leqslant 4 \Longrightarrow |p\rangle_R = \delta_1 \cdots \delta_4 \, e^{ip\cdot x}|\Omega\rangle$$

We consider the following set of states setting $\delta^i = (\delta_i)^\dagger$

$$|p\rangle_R, \quad \delta^i|p\rangle_R, \quad \delta^i\delta^j|p\rangle_R, \quad \delta^i\delta^j\delta^k|p\rangle_R, \quad \delta^1\cdots\delta^4|p\rangle_R$$

Since $[L_0, d_0^i] = 0$ and the normal ordering constant $a_{\mathrm{R}} = 0$ we have 16 states of zero mass $\alpha' M^2 = 0$. We can split these into the 2 chiral representation of $SO(8)$ fermions massless states $|\mathbf{8}_s, p\rangle_R$ and $|\mathbf{8}_c, p\rangle_R$

$$|\mathbf{8}_s, p\rangle_R = |p\rangle_R + \sum_{ij} \quad \delta^i\delta^j|p\rangle_R + \quad \delta^1\cdots\delta^4|p\rangle_R$$

$$|\mathbf{8}_c, p\rangle_R = \sum_i \delta^i|p\rangle_R + \sum_{ijk}\delta^i\delta^j\delta^k|p\rangle_R$$

SPACE-TIME SUPERSYMMETRY III

This builds the following spectrum of the theory

$\alpha' M^2$	Neveu-Schwarz	Degen.	Ramond	Degen.
$-\frac{1}{2}$	$\|p\rangle_{NS}$	**1**	None	**0**
0	$b^i_{-\frac{1}{2}}\|p\rangle_{NS}$	**8**	$\|\mathbf{8}_{s,c}, p\rangle_R$	$\mathbf{8}_s \oplus \mathbf{8}_c$
$\frac{1}{2}$	$b^j_{-\frac{1}{2}} b^i_{-\frac{1}{2}}\|p\rangle_{NS}$	**28**	None	**0**
	$\alpha^i_{-1}\|p\rangle_{NS}$	**8**	None	**0**
1	$b^k_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^i_{-\frac{1}{2}}\|p\rangle_{NS}$	**56**	$d^i_{-1}\|\mathbf{8}_{s,c}, p\rangle_R$	$\mathbf{64}_s \oplus \mathbf{64}_c$
	$b^i_{-\frac{3}{2}}\|p\rangle_{NS}$	**8**		
	$\alpha^j_{-1} b^i_{-\frac{1}{2}}\|p\rangle_{NS}$	**64**	$\alpha^i_{-1}\|\mathbf{8}_{s,c}, p\rangle_R$	$\mathbf{64}_s \oplus \mathbf{64}_c$

We can match the spectrum of bosons and fermions by introducing a operator $(-)^F$ such that $((-)^F)^2 = 1$ and

$$\{(-)^F, b^i_r\} = 0, \forall r \text{ in the NS sector}$$
$$\{(-)^F, d^i_n\} = 0, \forall n \text{ in the R sector}$$

- The operator $P_{GSO} = (1 + (-)^F)/2$ is the GSO projector.
- If we make the choice $P_{GSO}|p\rangle_{NS} = 0$ the states with *even* number of fermionic oscillators are projected out.
- If we make the chirality choice in the Ramond sector choice $P_{GSO}|\mathbf{8}_c\rangle_R = 0$ the states with *even* number of fermionic oscillators on $|\mathbf{8}_c\rangle_R$ are projected out, or the states with an *odd* number of fermionic oscillators on $|\mathbf{8}_s\rangle_R$ are projected out

- leaving this time a supersymmetric spectrum

$\alpha' M^2$	Neveu-Schwarz	Degen.	Ramond	Degen.
0	$b^i_{-\frac{1}{2}}\vert p\rangle_{NS}$	**8**	$\vert\mathbf{8}_s, p\rangle_R$	$\mathbf{8}_s$
1	$b^k_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^i_{-\frac{1}{2}}\vert p\rangle_{NS}$	**56**	$d^i_{-1}\vert\mathbf{8}_c, p\rangle_R$	$\mathbf{64}_c$
	$b^i_{-\frac{3}{2}}\vert p\rangle_{NS}$	**8**		
	$\alpha^j_{-1} b^i_{-\frac{1}{2}}\vert p\rangle_{NS}$	**64**	$\alpha^i_{-1}\vert\mathbf{8}_s, p\rangle_R$	$\mathbf{64}_s$

- This is the field content of $\mathcal{N}=1$ SYM in $D=10$ so far we can have any gauge group G of type $SO(N)$ for unoriented string and $U(N)$ for oriented strings

The closed superstring I

As before one gets the closed string spectrum by tensor product $|\Phi\rangle_{closed} = |\Phi\rangle_L \otimes |\tilde{\Phi}\rangle_R$, the GSO projector acts one each sector, we have the follwing massless spectrum. We have two type of theories differentiated by the relative chirality in the R-R sector

$$\begin{aligned} &\textit{type IIA} \quad |\mathbf{8}_s \otimes \mathbf{8}_c, p\rangle := |\mathbf{8}_s, p\rangle_R \otimes |\mathbf{8}_c, p\rangle_R \\ &\textit{type IIB} \quad |\mathbf{8}_s \otimes \mathbf{8}_s, p\rangle := |\mathbf{8}_s, p\rangle_R \otimes |\mathbf{8}_s, p\rangle_R \end{aligned}$$

	type IIA	fields	type IIB	fields
NS-NS	$b^i_{-\frac{1}{2}} \tilde{b}^j_{-\frac{1}{2}} \lvert p\rangle$	G_{ij}, B_{ij}, Φ	$b^i_{-\frac{1}{2}} \tilde{b}^j_{-\frac{1}{2}} \lvert p\rangle$	G_{ij}, B_{ij}, Φ
R-R	$d^i_{-1} \lvert \mathbf{8}_s \otimes \mathbf{8}_c, p\rangle$	$C^{(1)}_i, C^{(3)}_{ijk}$	$d^i_{-1} \lvert \mathbf{8}_s \otimes \mathbf{8}_s, p\rangle$	$C^{(0)}, C^{(2)}_{ij}, C^{(4)}_{ijkl}$
NS-R	$\tilde{b}^i_{-\frac{1}{2}} \lvert \mathbf{8}_c, p\rangle$	$\tilde{\Psi}^i_{\dot{\alpha}}, \tilde{\lambda}_{\dot{\alpha}}$	$\tilde{b}^i_{-\frac{1}{2}} \lvert \mathbf{8}_s, p\rangle$	$\tilde{\Psi}^i_{\alpha}, \tilde{\lambda}_{\alpha}$
R-NS	$b^i_{-\frac{1}{2}} \lvert \mathbf{8}_s, p\rangle$	$\Psi^i_{\alpha}, \lambda_{\alpha}$	$b^i_{-\frac{1}{2}} \lvert \mathbf{8}_s, p\rangle$	$\Psi^i_{\alpha}, \lambda_{\alpha}$

In the RR sector we have in the spectrum a 1-form and and 3-form for the type IIA string and a 0-,2-,4-form for the type IIB string which together with the NS-NS fields build the massless bosonic sector of $\mathcal{N} = 2$ type IIA and type IIB supergravity in $D = 10$. The NS-R and R-NS sector provide the fermions

We have discussed the construciton of the spectrum of

1. open string type I with group G
2. closed type IIA
3. closed type IIB

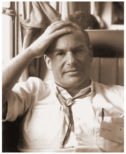
- On can consider having right moving fermions given by Lorentz scalars but world-sheet fermions $\lambda^a \in G$: heteorotic string

$$S_{het} = T_F \int d^2\xi \, (\partial X^2 + \psi_L \partial_- \psi_L + \lambda_R \partial_+ \lambda)$$

1. open string type I G
2. closed heterotic G
3. closed type IIA
4. closed type IIB

- Anomaly cancellation will restrict the gauge group to be $SO(32)$ for the type I case and $SO(32)$ or $E_8 \times E_8$ for the heterotic string
- There are 5 perturbative string theories in $D = 10$ dimensions corresponds to various perturbative vacua of M-theory

Green-Schwarz formulation



$$S_{GS} = -\frac{1}{2}\int d^2\xi\ [T_F \partial_a X^i \partial^a X^i - \frac{i}{\pi}\bar{S}_L^\alpha \partial_- S_L^\alpha - \frac{i}{\pi}\bar{S}_R^\alpha \partial_+ S_R^\alpha]$$

- matter fields X^i with $i = 1, \ldots, 8$ in the $\mathbf{8}_v$ and S^a in $\mathbf{8}_{s,c}$ of $SO(8)$
- S^a are world-sheet scalar
- advantages: explicitly supersymmetric
- disadvantage: difficult to use beyond 1-loop due to the light-cone gauge. Need to take care of contact terms that could lead to unphysical singularities inside the moduli space

RAMOND-NEVEU-SCHWARZ FORMULATION

$$S_{RNS} = -T_F \int d^2\xi \, [\partial_+ X^\mu \partial_- X^\mu + i\bar{\psi}^\mu_L \partial_- \psi^\mu_L + i\bar{\psi}^\mu_R \partial_+ \psi^\mu_R]$$

- matter fields X^μ and ψ^μ with $\mu = 0, \ldots, 9$ in the **10** of $SO(1,9)$
- ψ^μ world-sheet fermions $\psi_L = \psi_z \, dz^{1/2}$ – spin structures
- most popular, easier to use
- susy is not explicit: need the sum over the spin structure
- Perturbative formalism well under control up to and including 2-loop [D'hoker, Phong]
- interesting recent proposal for higher loop measure of integration

Pure spinor formalism

$$S_{ps} = T_F \int d^2\xi \, [\partial_+ X^\mu \partial_- X^\mu + p_a \partial_- \theta^a + w\partial_-\lambda + c.c.]$$

$$\lambda\gamma^m\lambda = 0 \qquad Q = \oint \lambda^\alpha \left(\frac{\partial}{\partial\theta^\alpha} + \cdots\right); \qquad Q^2 = 0$$

- matter fields X^μ with $\mu = 0, \ldots, 9$ in the **10** of $SO(1,9)$ and θ^a in **16** spinor of $SO(1,9)$
- θ^a is a world-sheet scalar. λ, w are the pure spinor ghost
- most promising, make susy cancellation explicit, is well adpated for AdS backgrounds
- Difficulties are due to the lack of proper understanding of the origin of the formalism and the treatement of the pure spinor constraints in the integral is still very delicate

Part III

Interactions and Effective action

STRING THEORY AS A S-MATRIX THEORY

- So far we have constructed the spectrum of the string theories. We need to see how their field are dynamical.
- One approach is the use the constraint from conformal invariance that forces the background field to satisfy their equations of motion
- another one is to compute amplitudes.
- String theory is a unitary theories with massive spectrum we can construct amplitudes and S-matrix elements
- For external momenta $\alpha' \, k_i \cdot k_j \ll 1$ one can do a derivative expansion leading to higher dimension operators to the lagrangian

Effective action I

- The string amplitudes lead to an infinite set of massive string exchange leading to an infinit set of higher derivative corrections to the SYM action

$$\mathcal{S}^{eff} \propto \frac{1}{\alpha'^3} \int d^{10}x e^{-\varphi} \sqrt{-g^{\sigma}} \, [\mathrm{tr}(F^2) + b_1(\varphi)\, t_8 \mathrm{tr}(F^4) + b_2(\varphi)\, \partial^2 t_8 \mathrm{tr}(F^4) +$$

- The coefficients b_i depend on the spectrum of the string theory and on the parameters of the perturbative string vaccum this is the low-energy effective action $\mathcal{S}^{eff}$

- These corrections provides new vertices that one needs to use in amplitudes computations in order to reproduce the unitary string S-matrix elements in a perturbative fashion.

- One first constructs an effective action $\mathcal{S}^{eff}_{2pt}$ that describes the massless free particles of the theory.

Effective action II

- One then adds cubic terms that describe their three-point couplings, as given by the string vertex, thus obtaining $\mathcal{S}^{eff}_{3pt}$.
- One then considers the four-point scattering amplitude. Unitarity guarantees that the massless poles will be those generated by the tree graphs of $\mathcal{S}^{eff}_{3pt}$.
- The remainder is due to massive particle exchange, has no singularities for small external momenta and can thus be expanded in a power series in $\alpha' p^2$. Each term in this expansion can be reproduced by some local vertex, V_{4pt}, (which starts out quartic in the massless fields) and thus one can construct $\mathcal{S}^{eff}_{3pt}$, the effective action correct through quartic order. This procedure can then be repeated for the five-, six-, etc. point scattering amplitudes, thereby yielding, in principle, the effective lagrangian to all orders.

Effective action III

- The task of constructing $\mathcal{S}^{eff}$ can be greatly simplified by exploiting the local and global symmetries (ie supersymmetry, dualities) of the theory to generate terms at a given order that must emerge in higher orders as a consequence of these symmetries. In the last lecture we will discuss these constraints.

Amplitudes I

- We have constructed a theory with the massless spectrum of non-Abelian super-Yang-Mills we can intriduce the gauge group G by adding chan-paton factors at the end of the opens string

- The *classical* space-time lagrangian reproducing the equations of motion and gauge and supersymmetry invariance is $\mathcal{N} = 1$ SYM in $D = 10$

$$S^{\mathrm{YM}} = \frac{1}{g_{\mathrm{YM}}^2} \int d^{10}x \, (\mathrm{tr}(F^2) + \mathrm{tr}(\bar{\Psi}\Gamma^{\mu} D_{\mu} \Psi))$$

- From this classical Lagrangian allows to deduce the Feynman rules and compute *amplitudes* and S-matrix elements

AMPLITUDES II

External states are given by vertex operators creating the state with helicity ζ and momentum k at $\sigma = 0$

$$V(\zeta, k) = \zeta_\mu \; : \partial X^\mu + ik \cdot \psi\psi^\mu \, e^{ik\cdot X} :$$

for the emission of an Abelian gauge boson A_μ

The propagators is the inverse world-sheet propagator

$$\Delta = \frac{1}{L_0 - 1} = \int_0^{+\infty} d\tau \, e^{-\tau\,(L_0 - 1)}$$

AMPLITUDES III

By computing the 3-point amplitude we derive structure of the cubic vertex

$$\mathfrak{A}^{\rm tree}_{3pt} = e^{-\varphi}\,\langle V(1)\Delta V(2)\Delta V(3)\rangle$$

$$= e^{-\varphi}\,\alpha'^3\,\delta^{(10)}(\sum_i k_i)\prod_{i=1}^{3}\zeta_i^{\mu_i}\,(k^1_{\mu_3}\eta_{\mu_1\mu_2} + k^2_{\mu_1}\eta_{\mu_1\mu_3} + k^3_{\mu_1}\eta_{\mu_3\mu_2})$$

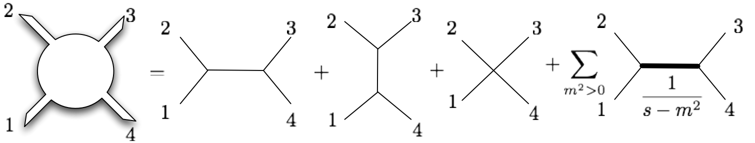
$$\times\,\mathrm{tr}([\lambda^1,\lambda^2]\lambda^3)$$

- This takes the same form as the field theory 3-point coupling $A^{\rm tree}_{3pt} = \mathfrak{A}^{\rm tree}_{3pt}$ and working out the complete normalisations we deduce that (introducting $g_s = \exp(\varphi)$)

$$g^2_{\rm YM} = \sqrt{8}\,(2\pi)^7\,g_s\,\alpha'^3$$

- and that the non-Abelian interaction are specified by the structure constant $f^{abc} \propto \mathrm{tr}([\lambda^a,\lambda^b]\lambda^c)$

Amplitudes IV



$$\mathfrak{A}^{\rm tree}_{4pt}(1,2,3,4) = e^{-\varphi}\,\langle V(1)\Delta V(2)\Delta V(3)\Delta V(4)\rangle$$
$$= e^{-\varphi}\,\frac{K_{kin}(1234)}{\alpha' s\alpha' t}\,\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)}{\Gamma(1+\alpha' u)}\,\delta^{(10)}(\sum_i k_i)$$

- where $K_{kin}(1234) = t_8 F_{a_1}\cdots F_{a_4}\mathrm{tr}(\lambda^{a_1}\cdots\lambda^{a_4})$ where t_8F^4 is the unique lorentz scalar determined by supersymmetry
- The field theory answer is

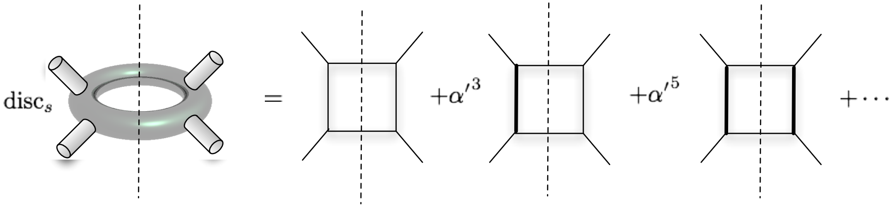
$$A^{\rm tree}_4(1,2,3,4) = g^2_{\rm YM}\,\frac{K_{kin}(1234)}{\alpha' s\alpha' t}\,\delta^{(10)}(\sum_i k_i)$$

Amplitudes V

- Higher point amplitudes in string theory display the propagation of massive string states which lead to an *infinite* set of higher-derivative operators

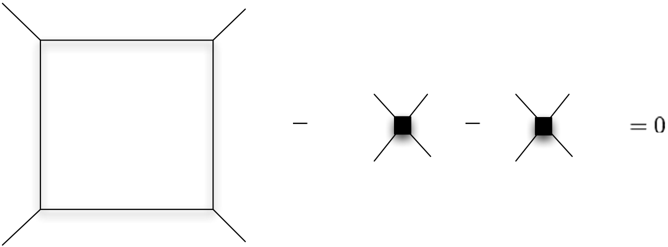
$$\mathfrak{A}^{\rm tree}_{4pt} = A^{\rm tree}_{4pt} + e^{-\varphi}\,\frac{K_{kin}}{\alpha' s \alpha' t}\left(\sum_{p,q\geqslant 1} c_{p,q}\,(\alpha' s)^p (\alpha' t)^q\right)$$

- At one-loop order



Ultraviolet divergence I

- The loop amplitudes in field theory have ultraviolet divergences



$$\Lambda^2 \times (c_1\, t_8 \mathrm{tr}(F^4) + c_2\, t_8 \mathrm{tr}(F^2)) \qquad \alpha' \times c_1\, t_8 \mathrm{tr}(F^4) \qquad \alpha' \times c_2\, t_8 \mathrm{tr}(F^2)^2$$

- The UV divergences need counter-terms that are higher-derivative local correction to the effective action. These counter-terms have to satisfy all the symmetries of the theory.

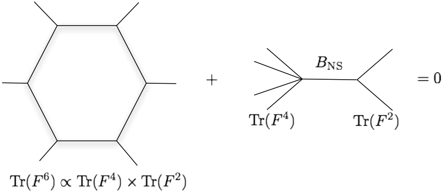
Ultraviolet divergence II

$$\mathcal{S} = \frac{1}{\alpha'^3} \int d^{10}x \, [e^{-\varphi} \mathrm{tr}(F^2) + \alpha'^2 \, c_1 t_8 \mathrm{tr}(F^4) + \alpha'^2 \, c_2 t_8 (\mathrm{tr} F^2)^2]$$

- The coefficients c_i are determined from the expression of the one-loop amplitude in string theory.
- We will present in a minute the argument for the perturbative finiteness of string perturbation and at the end of the lecture we will discuss the issues when removing the string cut-off

Anomalies

- Since $D = 10$ $\mathcal{N} = 1$ SYM has chiral adjoint fermion. The loop amplitudes in field theory have anomalies



- This anomaly can be cancelled by the [Green, Schwarz] mechanism if the gauge group is such that

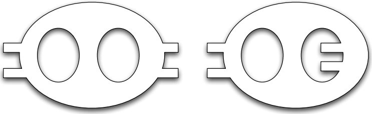
$$\mathrm{Tr}(\lambda^1_{adj} \cdots \lambda^6_{adj}) \propto \mathrm{Tr}(\lambda^1_{adj} \cdots \lambda^4_{adj})\mathrm{Tr}(\lambda^5_{adj}\lambda^6_{adj})$$

- the introduction of the coupling to supergravity
- restriction of the gauge group to be $SO(32)$ and $E_8 \times E_8$

SYM+SUGRA I

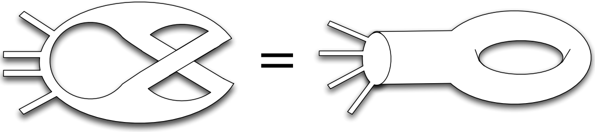
- the anomaly cancellation mechanism showed that we need to couple SYM to massless B-field this field is part of a supergravity multiplet

- Another reason where we see the mixing of SYM and SUGRA is to look at the non-planar $1/N$ corrections.



SYM+SUGRA II

- For instance the $1/N^2$ correction to the $L = 3$ loop 4 gluon amplitude



- is equivalent to adding an handle and there is no way and non meaning into separating gravity and YM contributions.

- The leading low-energy limit $\ell_s \to 0$ of the L-loop open string amplitude is given by

$$\mathfrak{A}_L^4 \sim g_s^{L-1}\, N^L \,(1 + \frac{c_1}{N^2} + \frac{c_2}{N^4} + \cdots)\, \partial^2 t_8 \mathrm{tr}(F^4)$$

- c_i is given by pure SYM and mixed SUGRA+SYM contributions

Supergravity I

The massless spectrum for type II string contained the degrees of freedom of the $\mathcal{N} = 2$ supergravity massless multiplet in $D = 10$
The fields satisfied the linearized eom and gauge invariance of supergravity.
The *classical* reproducing the gauge + local susy invariance is

$$S^{\mathrm{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \, \sqrt{-g^E} \, (\mathcal{R}^{(10)} + (\partial\Phi)^2 - H^2 + \sum_p F^2 + \text{fermions})$$

Newton's constant is determined from the 3 point coupling. Type IIA and Type IIB have effective action SUGRA. We will return to this point later.

Supergravity II

- The cubic 3-point amplitudes

$$\begin{aligned}\mathfrak{M}^{\text{tree}}_{3pt} &= e^{-2\phi}\,\langle V(1)\Delta V(2)\Delta V(3)\rangle \\ &= \mathfrak{A}^{\text{tree}}_{3pt}(123)\,\tilde{\mathfrak{A}}^{\text{tree}}_{3pt}(123)\end{aligned}$$

This implies the following relation for the 10D Newton' constant is with $g_s = \exp(\varphi)$

$$2\kappa_{10}^2 = (2\pi)^7\,\alpha'^4\,g_s^2$$

One important point about the relation that $\mathfrak{M}_{closed} = \mathfrak{A}_{open} \otimes \mathfrak{A}_{open}$ is the relation between the g_{YM} coupling constant and the gravitational coupling

$$\frac{g_{10}^4}{\kappa_{10}^2} = 16\,(2\pi)^7\,\alpha'^2$$

- This is the relation that is precisely needed for the anomaly cancellation mechanism

Supergravity III

For string compactified a torus T^d of volume v_d to $10-d$ dimensions Newton's constant and YM coupling constant in $D=10-d$ dimension are given by

$$\mathcal{L}^{\mathrm{Ym}} = \frac{1}{2g_{\mathrm{YM}}^2}\int d^D x\,\mathrm{tr}(F^2)+\cdots$$

$$\mathcal{L}^{\mathrm{Grav}} = \frac{1}{2\kappa_D^2}\int d^D x\,\sqrt{-G}\,\mathcal{R}_{(D)}+\cdots$$

$$\frac{1}{2\kappa_D^2}=\frac{v_d}{2\kappa_{10}^2};\qquad \frac{1}{2g_D^2}=\frac{v_d}{2g_{10}^2}$$

SUPERGRAVITY IV

- therefore

$$\kappa_D \sim g_D^2 \, \frac{\sqrt{v_d}}{\alpha'}$$

- For $D = 4$ the YM coupling constant is dimensionless, and assuming $g_4 \sim 1$ we have $\sqrt{v_d}/\alpha' \sim 10^{-33} cm$
- The constrains on the Regge recurrences of elementary particles implies $\alpha' \leqslant (10^{-15} cm)^2$ therefore $R = v_6^{1/6} \leqslant 10^{-21} cm$
- In general $D = 4$ Newton's constant is a function of the moduli

$$\frac{1}{2\kappa_4^2} = f(R_i^2/\alpha', g_s, \cdots)$$

- we will discuss at the end of this lecture what happens when one tries to send $v_d \to 0$ and $\alpha' \to 0$ and decouple the string excitations when relating string theory to $\mathcal{N} = 8$ supergravity

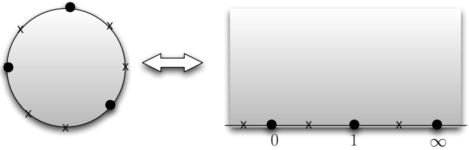
Reformulation of QFT amplitudes I

- We have explained that string theory provides a *unitary* S-matrix approch to gauge and gravity amplitudes.
- The field theory limit of string amplitudes reproduces the field theory amplitudes but this reorganizes the field theory perturbation by resumming all the Feynman graph contribution into a single unified formula
- Therefore some properties of field theory difficult to see using traditional approach are easily shown using string theory
- On example is the reformulation of the YM and gravity amplitudes in terms unified building block inherited from the fact that spectrum was $|\mathrm{phys}\rangle_{closed} = |\mathrm{phys}\rangle_{open} \otimes |\widetilde{\mathrm{phys}}\rangle_{open}$

Reformulation of QFT amplitudes II

As well at the level of field theory amplitudes this implies that all QCD and Gravity amplitudes can be build from a elementary building block

[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove]



$$\mathfrak{A}(1,\dots,n) \sim \sum_{\sigma\in\mathfrak{S}_{n-1}/\mathbb{Z}_2} \mathrm{Tr}\left(\lambda^{a_1}\lambda^{a_{\sigma(2)}}\cdots\lambda^{a_{\sigma(n)}}\right)\, \mathcal{A}(1,\sigma(2,\dots,n))$$

- λ^a are generator in the fundamental representation
- $\mathcal{A}(1,\sigma(2,\dots,n))$ are the color ordered open string amplitudes
- $PSL(2,\mathbb{R})$ invariance $z_1=0$, $z_{n-1}=1$ and $z_n=+\infty$. (3 marked points)

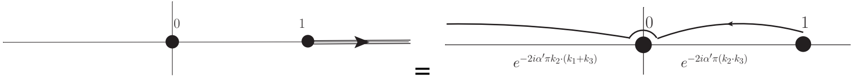
Reformulation of QFT amplitudes III

$$\mathcal{A}(1,\dots,n) = \int_{x_1<\cdots<x_n} d^{n-3}x\, f(x_i - x_j) \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j}$$

- The function $f(x)$ does not have branch cut but has poles. Depends on the polarisation of the external states.
- The precise form of $f(x)$ depends on the type of string we use

Contour deformation

Reformulation of QFT amplitudes IV



$$\mathcal{A}_n(\beta_1,\dots,\beta_r,1,\alpha_1,\dots,\alpha_s,n)=(-1)^r\times$$

$$\times\,\Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)}\sum_{\sigma\subset\mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\{\sigma\},n)\Big]$$

$$0=\Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)}\sum_{\sigma\subset\mathrm{OP}\{\alpha\}\cup\{\beta^T\}}\prod_{i=1}^{r}\prod_{j=1}^{s}e^{(\alpha_i,\beta_j)}\mathcal{A}_n(1,\{\sigma\},n)\Big]$$

- $\exp(\alpha,\beta)=\exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta-z_\alpha)>0$ or 1 otherwise

Reformulation of QFT amplitudes V

- For closed string amplitudes the fact that spectrum is the product of open string spectrum allows holormorphic factorization

$$\mathfrak{M}(1,\ldots,n) = \int_{C_x} d^{n-3}x \int_{C_y} d^{n-3}y \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{\frac{\alpha' k_i\cdot k_j}{2}} (y_i - y_j)^{\frac{\alpha' k_i\cdot k_j}{2}} f(x_{ij}) g(y_{ij})$$
$$= \sum_{left\,/\,right\ ordering} e^{\Pi(k_i\cdot k_j)} \mathfrak{A}_n \mathfrak{A}_n$$

- The countours C_x and C_y have to be ordered due to choice of the holomorphic factorization $|z|^{\alpha' k_i\cdot k_j} \to z^{\frac{\alpha' k_i\cdot k_j}{2}} \bar{z}^{\frac{\alpha' k_i\cdot k_j}{2}}$

Reformulation of QFT amplitudes VI

- Leading to

$$\mathfrak{M}_n \sim \sum_{\sigma\in\mathfrak{S}_{n-3}} \sum_{\gamma\in\mathfrak{S}_j} \sum_{\beta\in\mathfrak{S}_{n-3-j}} \mathcal{S}_{\alpha'}[\gamma\circ\sigma|\sigma]_{k_1} \mathcal{S}_{\alpha'}[\beta\circ\sigma|\sigma]_{k_{n-1}}$$
$$\times \mathcal{A}_n(1,\sigma(\dots),n-1,n)\,\widetilde{\mathcal{A}}_n(\gamma\circ\sigma,1,n-1,\beta\circ\sigma,n)$$

- with the momentum kernel $\mathcal{S}$

$$\mathcal{S}_{\alpha'}[i_1,\dots,i_k|j_1,\dots,j_k]_p \equiv \prod_{t=1}^{k} \frac{1}{\alpha'} \sin(p\cdot k_{i_t} + \sum_{q>t}^{k} \theta(i_t,i_q)\, k_{i_t}\cdot k_{i_q})$$

- All color ordered amplitudes satisfy the annihilation relation

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}_{\alpha'}[\sigma(2,\dots,n-1)|\beta(2,\dots,n-1)]_{k_1} \mathcal{A}_n(n,\sigma(2,\dots,n-1),1) = 0$$

Reformulation of QFT amplitudes VII

- Taking the field theory limit $\alpha' \to 0$ we get

$$\begin{aligned}\mathcal{A}_n^{\text{YM}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{scalar}} \\ \mathcal{M}_n^{\text{Grav}} &= A^{\text{vector}} \otimes \mathcal{S} \otimes A^{\text{vector}}\end{aligned}$$

- with the momentum kernel $\mathcal{S}$

$$\mathcal{S}[i_1, \ldots, i_k | j_1, \ldots, j_k]_p \equiv \prod_{t=1}^{k} \Big(p \cdot k_{i_t} + \sum_{q>t}^{k} \theta(i_t, i_q)\, k_{i_t} \cdot k_{i_q} \Big)$$

- This is representation of the YM and gravity tree-level amplitudes that is independent of any particular parametrization one would use of the given amplitudes.

Part IV

Duality symmetries

D-BRANES I

The C_{p+1} forms from the RR spectrum couple to extended solitons of 10D type IIA/B SUGRA

$$S^{\mathrm{IIA,B}} = -\frac{1}{2\kappa^2_{(10)}}\int d^{10}x\,\sqrt{-G}\,\Bigg[\Big(R - \frac{1}{2}(\partial\Phi)^2 - \frac{e^{-\Phi}}{12}(dB)^2\Big) - \sum_p \frac{e^{-\frac{(3-p)}{2}\Phi}}{2(p+2)!}F^2_{(p+2)}\Bigg]$$

where $p = 0, 2, 4$ for type-IIA theory, $p = -1, 1, 3, 5$ for type-IIB in the Einstein frame

D-BRANES II

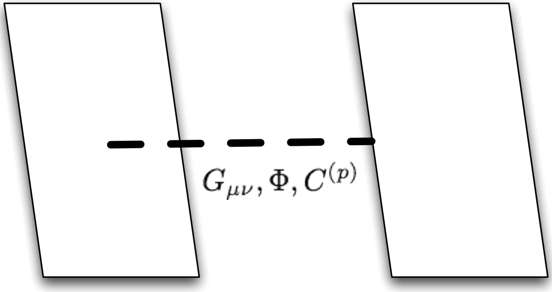
The Dp-brane action has world-volume theory coupling to the pull-back fo the space-time metric $\hat{g}_{ab}$ and Ramond-Ramond p-form $\hat{C}_{a_1\cdots a_{p+1}}$

$$I_{\mathrm{D}p} = \int d^{p+1}\xi \left(T_p\, e^{\frac{p-3}{4}\Phi} \sqrt{-\hat{g}} + \rho_p\, \hat{C}^{(p+1)} \right).$$

- The world-volume theory shows the tension T_p and the charge ρ_p
- Linearizing the worldvolume action gives the coupling to the graviton $h_{\mu\nu}$, dilaton Φ or Ramond-Ramond field C

$$j_\Phi = \frac{p-3}{4}\, T_p\, \delta(x^\perp)$$
$$j_C = \rho_p\, \delta(x^\perp)$$
$$j_h = T_{\mu\nu} = \frac{1}{2} T_p\, \delta(x^\perp)\eta_{\mu\nu}; \quad 0 \leqslant \mu, \nu \leqslant p$$

D-BRANES III



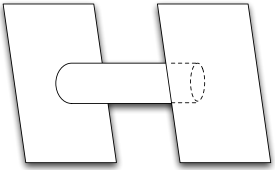
D-BRANES IV

- The leading order interaction energy from the exchange between two static branes localized at distance r away in the transverse plane

$$
\begin{aligned}
\mathcal{E}(r) &\sim -2\kappa_{(10)}^2 \int d^{10}x \int d^{10}\tilde{x}\, \left[j_\Phi \Delta \tilde{j}_\Phi - j_C \Delta \tilde{j}_C + j_h \Delta \tilde{j}_h \right] \\
&= 2\, V_{(p)}\, \kappa_{(10)}^2\, [\rho_p^2 - T_p^2]\, \Delta^E_{(9-p)}(r)
\end{aligned}
$$

- where Δ are the propagator of the fields in ten dimensions where V_p is the p-brane volume and $\Delta^E_{(9-p)}(r)$ is the propagator in the $9-p$ transverse dimensions.
- The net force is the result of the difference between Ramond-Ramond repulsion and gravitational plus dilaton attraction

D-BRANES V



- **closed string exchange** is given by the cylinder diagram dual of a one-loop (annulus) open string diagram
- The one-loop vacuum energy of oriented open strings reads

$$\mathcal{E}(r) = -\frac{V_p}{2} \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \int_0^\infty \frac{dt}{t} \,\mathrm{Str}\, e^{-\pi t(k^2+M^2)/2}$$

- Strings stretching between the two D-branes have mass

$$\alpha' M^2 = r^2 + N$$

D-BRANES VI

- The zero point energy is changed by the configuration is still supersymmetric. The fermionic and bosonic contribution compensate exactly with the result

$$\mathcal{E}(r) = V_{(p)}\,(1-1)\,2\pi(4\pi^2\alpha')^{3-p}\,\Delta^E_{(9-p)}(r)\;+\;o(e^{-r/\sqrt{\alpha'}})$$

- Ramond-Ramond repulsion cancels exactly the gravitational and dilaton attraction

- we deduce the relation between charge and tension typical of supersymmetry

$$T_p^2 = \rho_p^2 = \frac{\pi}{\kappa^2_{(10)}}(4\pi^2\alpha')^{3-p}$$

D-BRANES VII

The previous computation indicates that the tension of the Dp-brane is

$$T_p = \frac{\sqrt{\pi}}{\kappa_{(10)}} (2\pi\ell_s)^{3-p} = \frac{2\pi}{g_s\,(2\pi\ell_s)^{1+p}}$$

where we used $\alpha' = \ell_s^2$, to be compared to the tension of the fundamental string

$$T_F = \frac{1}{2\pi\ell_s^2}$$

Using $g_s = \exp(\varphi)$ and $\kappa_{(10)} = (2\pi\ell_s)^4 g_s/\sqrt{4\pi}$ that

$$T_p T_{6-p} = \frac{\pi}{\kappa_{(10)}^2}$$

There is another non-perturbative object: the NS5-brane of tension

$$T_{NS5} = \frac{\pi}{\kappa_{(10)}^2}\, T_F^{-1} = \frac{2\pi}{g_s^2\,(2\pi\ell_s)^6}$$

T-DUALITY I

The string interactions lead to corrections to the effective action, the coefficient of the correction depend on the string spectrum and the parameters of the vaccum of the theory

Let's consider the case of type II string compactified on tori T^d to $D = 10 - d$ dimensions

The perturbative string will depend on the moduli of the tori: g_{ij}, B_{ij}

Because of the non-trivial background the zero mode of the world-sheet coordinates have non trivial dependence on the parameters of the torus.

Let's consider the case of a circle of radius R in the direction $\mu = 9$, the closed string coordinates satisfy

$$X^9(\tau, \sigma + \pi) = X^9(\tau, \sigma) + 2\pi R\, m \qquad m \in \mathbb{Z}$$

the mode expansion now takes the form

$$X^9(\tau, \sigma) = x^9 + 2\alpha' \frac{n}{R}\tau + 2\pi R\, m\sigma + \text{oscillators} \qquad m \in \mathbb{Z}$$

T-DUALITY II

where the momentum along the circle is quantized as well as the winding of the string around the circle, with Kaluza-Klein n and Winding modes m The closed string spectrum is

$$H_{closed} = L_0 + \bar{L}_0 = 2\alpha' p^2 + \frac{1}{2}\left[\left(\frac{\sqrt{\alpha'}}{R}n\right)^2 + \left(m\frac{R}{\sqrt{\alpha'}}\right)^2\right] + \text{oscillator}$$

$$L_0 - \bar{L}_0 = 0 = N_L - N_R + 2mn$$

The mass spectrum depend on the radius and is invariant under the transformation

$$R \to \frac{\alpha'}{R}; \qquad (m, n) \to (n, m)$$

The level matching condition is modified as well by the product of the winding modes times the KK modes mn which is *independent* of the moduli of the circle.

The interactions are as well preserved by this symmetry.

T-DUALITY III

For general torus compactification with metric g_{ij}The mass is of wrapped string is given by

$$\alpha' M^2 = \frac{1}{2} \sum_{i,j=1}^{d} n_i \mathcal{M}^{ij} m_j + \text{oscillators}$$

is a function of the moduli of the toru $\mathcal{M}$

$$\mathcal{M} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

The string spectrum is invariant under the discrete symmetry group $SO(d, d; \mathbb{Z})$

The angular momentum of the string is independent of the moduli

$$L_0 - \bar{L}_0 = 2 \sum_{i=1}^{d} n_i m^i + \text{oscillators} \in 2\mathbb{Z}$$

S-DUALITY I

In type IIB string we have see that there is a scalar field C_0 in the spectrum. One can consider the complexified coupling constant

$$\tau = C_0 + i\, e^{-\varphi}$$

where string coupling constant $g_s = \exp(\varphi)$
This scalar field parametrizes the vaccum structure of the type IIB theory in $D = 10$ dimension

$$\tau \in SL(2,\mathbb{R})/U(1)$$

The $\mathcal{S}$-matrix elements of type IIb string a parametrized by the classical value of τ_o, and the theory is classically invariant under the $SL(2,\mathbb{R})$ action on τ

$$\mathcal{S}\left(\frac{a\tau + b}{c\tau + d}\right) = \mathcal{S}(\tau); \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

S-DUALITY II

The S-duality does not preserve the string length $\sqrt{\alpha'}$ and the string massive perturbative spectrum but preserves the 10-dimensional Newton's constant

$$\frac{1}{2\kappa_{10}^2} \sim \frac{1}{\alpha'^4 g_s^2}$$

The S-duality transformation has $g_s \to 1/g_s$ from $\tau \to -1/\tau$ therefore to preserve the Planck length one needs

$$\alpha' \to \alpha' g_s$$

Therefore we see that this transformation relates the F-string to D-string

$$T_F = \frac{1}{2\pi\alpha'} \leftrightarrow T_1 = \frac{1}{2\pi\alpha' g_s}$$

S-DUALITY III

this transformation related F-string to D-brane, and because this inverts the coupling constant this maps weak to strong coupling.

- In $D = 10$ for type IIb superstring the $SL(2,\mathbb{R})$ the $U(1)$ symmetry is broken by the massive string states but it is as well anomalous in the field theory

- The anomaly in the $U(1)$ needs a anomaly counter-term written down by [Gaberdiel, Green]

$$\delta S^{IIb} = \int F \wedge t_8 R^4$$

$$F = dQ = i\frac{d\tau \wedge d\bar{\tau}}{4\tau_2^2}$$

- Only $SL(2,\mathbb{Z})$ is preserved at the quantum level

U-DUALITY I

- The S-duality $SL(2,\mathbb{Z})$ and T-duality group $SO(d,d;\mathbb{Z})$ do not commutte

$$G(\mathbb{Z}) := \langle SO(d,d;\mathbb{Z}), SL(2,\mathbb{Z})\rangle = E_d(\mathbb{Z})$$

D	$E_{11-D(11-D)}(\mathbb{R})$	K_D	$E_{11-D(11-D)}(\mathbb{Z})$
10A	$\mathbb{R}^+$	1	1
10B	$Sl(2,\mathbb{R})$	$SO(2)$	$Sl(2,\mathbb{Z})$
9	$Sl(2,\mathbb{R})\times\mathbb{R}^+$	$SO(2)$	$Sl(2,\mathbb{Z})$
8	$Sl(3,\mathbb{R})\times Sl(2,\mathbb{R})$	$SO(3)\times SO(2)$	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$
7	$Sl(5,\mathbb{R})$	$SO(5)$	$Sl(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5)\times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$USp(8)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$SO(16)$	$E_{8(8)}(\mathbb{Z})$

- $E_{11-D(11-D)}$ real split forms, K_D maximal compact subgroup.
- The vacuum of the theory is the scalar manifold $\vec{\varphi}\in\mathcal{M}_D = G_D/K_D$

Duality of the effective action I

- This is a symmetry of string perturbation and the effective action is invariant under this symmetry

- We still have maximally 32 supercharges supergravity in these dimensions and string theory will induce higher-derivative that are function of these moduli

- Half of the supersymmetries are realized on-shell:
  the S-matrix element is proportional to the tree-amplitude

$$\mathcal{A}_{4pt} = \mathcal{A}_{4pt}^{tree} \times \ell_s^6 stu\, \mathcal{F}(\vec{\varphi}; \ell_s^2 s, \ell_s^2 t, \ell_s^2 u)$$

- For the four-graviton amplitudes $\mathcal{A}_{4pt}^{tree} = \mathcal{R}^4/(\ell_s^6\, stu)$

Duality of the effective action II

- The low-energy energy $\ell_s^2 k_i \cdot k_j \ll 1$ expansion of S-matrix elements has analytic (local higher derivative operators) and non-local (branch cut needed for unitarity)

- The analytic contributions: $\sigma_n \equiv (s^n + t^n + u^n)$

$$\mathcal{A}_{4pt}^{anal.} = \ell_s^{8+4p+6q} \sum_{p \geqslant 0, q \geqslant -1} \mathcal{E}_{(p,q)}(\vec{\varphi})\, \sigma_2^p \sigma_3^q\, \mathcal{R}^4$$

- non-analytic contributions corresponding to massless thresholds needed by unitarity $\mathrm{disc}\, \mathcal{A}_{4pt}^{nonanal.} \sim (\mathcal{A}_{4pt}^{anal.})^2$

Duality of the effective action III

$$\mathcal{A}_{4pt}^{nonanal.} = \ell_s^8\,\mathcal{R}^4\Big(\ell_s^2\,s\log(-\ell_s^2 s) + \mathcal{E}_{(0,0)}\,(\ell_s^2\,s)^4\log(-\ell_s^2 s)$$
$$+\,\mathcal{E}_{(1,0)}\,(\ell_s^2\,s)^6\,\log(-\ell_s^2 s)$$
$$+\,(\ell_s^2 s)^5\,(\log^2(-\ell_s^2 s) + \log(-\ell_s^2 s)\log(y_{10})) + \cdots\Big)$$

- The coefficients are function of the moduli are invariant under the action of the duality transformations
- They are as well constrained by supersymmetry invariance

$$(\delta_{(0,-1)} + \sum_{2p+3q\geqslant 0}\alpha'^{2p+3(q+1)}\delta_{(p,q)})\,(S_{(0,-1)} + \sum_{2p+3q\geqslant 0}\alpha'^{2p+3(q+1)}S_{(p,q)}) = 0$$

- Constraining the coefficients $\mathcal{E}_{(p,q)}$ in the effective action

Non-decoupling of string states I

- The couping to the effective action receives perturbative contributions from string oscillators, winding and Kaluza-Klein momenta but as well non-perturbative contributions from D-brane and NS-brane instantons

- Under the action of the dualities these contributions are exchanged

- One can ask the question about sending to infinity all string states and keeping QFT contributions?

- Since this means sending the UV cut-off to $\Lambda^2 \sim \alpha'^{-1} \to \infty$ one needs to be careful

Non-decoupling of string states II

- For string theory compactified to D dimensions on a torus of size $10-d$ and radii R. The tension of wrapped branes satisfy

$$T_p T_{D-p-4} = \frac{\pi}{\kappa^2_{(D)}}$$

- Where

$$\kappa^2_{(D)} = \ell_D^{D-2} = \frac{\kappa^2_{(10)}}{R^{10-D}}$$
$$\frac{1}{\ell_D^{D-2}} = \frac{R^{10-D}}{\ell_s^8 g_s^2}$$

- is Newton's constant in D dimensions and Planck length

Non-decoupling of string states III

- Wrapping Dp-brane on a p-cycle with $p \leqslant 5$

$$M_p = \frac{R^p}{g_s \ell_s^{p+1}} = \left(\frac{R}{\ell_s}\right)^{p-3} \frac{1}{\ell_4}$$

- Wrapped NS5 brane on a 5-cycle

$$M_{NS5} = \frac{R^5}{\ell_s^6 g_s^2} = \frac{\ell_s^2}{R} \frac{1}{\ell_4^2}$$

- Wrapped F-string have a mass (KK states)

$$M_F = \frac{R}{\ell_s^2}$$

- One checks that

$$M_p M_{6-p} = M_{NS5} M_F = \frac{1}{\ell_4^2}$$

NON-DECOUPLING OF STRING STATES IV

- Decoupling the string states requieres

$$1/R, 1/\ell_s, M_F \gg 1/\ell_4$$

- [Green, Ooguri, Schwarz] have pointed out that since $M_F M_{NS5} = 1/\ell_4^2$ in this limit the NS5 brane becomes massless there is no more mass gap and the limit is singular

- In perturbative string ℓ_s determines a minimal length

$$R \to \tilde{R} = \frac{\ell_s^2}{R}; \qquad \ell_s = \text{fixed}$$

- Spectrum of wound string spectrum has a minimum: exchanging winding and momentum

Non-decoupling of string states V

$$(\ell_s\, m)^2 = \frac{\ell_s^2}{R^2} + \frac{R^2}{\ell_s^2} \geqslant 2$$

- Planck length ℓ_D determines a minimal length in the 'decoupling' limit

$$\frac{1}{R} \leftrightarrow \frac{1}{\tilde{R}} = \frac{R}{\ell_D^2}, \qquad \ell_s \leftrightarrow \frac{\ell_D^2}{\ell_s}, \qquad g_s = \frac{1}{\tilde{g}_s}, \qquad \ell_D = \text{fixed}$$

- In the 'decoupling' limit $\ell_s \to 0$ with $\ell_D =$fixed the theory dualizes to a strong coupling limit $g_s \to \infty$ decompactified to ten-dimensions $\tilde{R} \to \infty$

Non-decoupling of string states VI

- The effect couplings are [Green,Russo,Vanhove]
- The $D^4\mathcal{R}^4$ in 7 dimensions $g_7 \to \infty$

$$\mathcal{E}^{(7)}_{(1,0)} = \frac{\pi}{30} g_7^3 \mathbf{E}^{SL(4)}_{[001];\frac{5}{2}} + \frac{1}{2}\, g_7 \mathbf{E}^{SL(4)}_{[100];\frac{5}{2}} + \frac{2}{\pi^2} \hat{\mathbf{E}}^{SL(4)}_{[010];2} - \frac{4\pi^2}{5} \log g_7^2 + O(e^{-(g_7^2 v_3)^{\frac{1}{2}}}, e^{-(g_7^2 \ell_s/r_i)^{\frac{1}{2}}})$$

- To be compared with the weak coupling expansion $g_7 \to 0$

$$\mathcal{E}^{(7)}_{(1,0)} = \frac{\zeta(5)}{g_7^4} + \frac{3}{\pi^3 g_7^2}\, \mathbf{E}^{SL(4)}_{[010];\frac{5}{2}} + \frac{2}{3} (\widehat{\mathbf{E}}^{SL(4)}_{[100];2} + \widehat{\mathbf{E}}^{SL(4)}_{[001];2}) + \frac{8\pi^2}{15} \log g_7^2 + O(e^{-(g_7^2 v_3)^{-\frac{1}{2}}}, e^{-(g_7^2 \ell_s/r_i)^{-\frac{1}{2}}})$$

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