Complex Geometry and Sigma Models

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Outline

- Formulations of Generalized Kähler Geometry
- The corresponding Sigma Models

M. Göteman, C. Hull, M. Roček, I. Ryb, R. von Unge, M. Zabzine.

$$(M, g, J_{(\pm)}, H)$$

$$J^2_{(\pm)} = -\mathbf{1}\ , \quad J^t_{(\pm)} g J_{(\pm)} = g\ , \quad \nabla^{(\pm)} J_{(\pm)} = 0$$

$$\Gamma^{(\pm)} = \Gamma^0 \pm \tfrac{1}{2} g^{-1} H\ , \quad H = dB\ .$$

$$E := g + B$$

$$(M, g, J_{(\pm)})$$

$$J^2_{(\pm)} = -\mathbf{1}\ , \quad J^t_{(\pm)} g J_{(\pm)} = g\ , \quad \omega_{(\pm)} := g J_{(\pm)}$$

$$d^c_{(+)}\omega_{(+)} + d^c_{(-)}\omega_{(-)} = 0\ , \quad dd^c_{(\pm)}\omega_{(\pm)} = 0\ ,$$

$$H := d^c_{(+)}\omega_{(+)} = -d^c_{(-)}\omega_{(-)}$$

Description on $T \oplus T^*$

$$\mathcal{J}^2_{(1,2)} = -\mathbf{1}\ , \quad [\mathcal{J}_{(1)}, \mathcal{J}_{(2)}] = 0\ , \quad \mathcal{J}^t_{(1,2)}\mathcal{I}\mathcal{J}_{(1,2)} = \mathcal{I}\ , \quad \mathcal{G} := -\mathcal{J}_{(1)}\mathcal{J}_{(2)}$$

$$\mathcal{J}_{(1,2)} =$$

$$\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_{(+)} \pm J_{(-)} & -(\omega^{-1}_{(+)} \mp \omega^{-1}_{(-)}) \\ \omega_{(+)} \mp \omega_{(-)} & -(J^t_{(+)} \pm J^t_{(-)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

$$(M, J_{(\pm)})$$

Locally, $\exists$ "symplectic" two-forms $\mathcal{F}_{(\pm)}$ such that

$$\mathcal{F}_{(\pm)}(v, J_{(\pm)}v) > 0 \ , \quad d(\mathcal{F}_{(+)}J_{(+)} - J^t_{(-)}\mathcal{F}_{(-)}) = 0 \ .$$

$$\mathcal{F}_{(\pm)} = \tfrac{1}{2}i(B^{(2,0)}_{(\pm)} - B^{(0,2)}_{(\pm)}) \mp \omega_{(\pm)}$$

$$\mathcal{F}_{(+)} = -\tfrac{1}{2}E^t_{(+)}J_{(+)} \ , \quad \mathcal{F}_{(-)} = -\tfrac{1}{2}J^t_{(-)}E^t_{(-)}$$

Summary:

Geometric data: $(M, g, H, J_{(\pm)})$ or $(M, g, J_{(\pm)})$ or $(M, \mathcal{F}_{(\pm)}, J_{(\pm)})$. In each case, there is a complete description in terms of a Generalized Kähler potential K. Unlike the Kähler case, the expressions are non-linear in second derivatives of K. E.g.,

$$J_{(+)} = \begin{pmatrix} J & 0 \\ (K_{LR})^{-1}[J, K_{LL}] & (K_{LR})^{-1}JK_{LR} \end{pmatrix}$$

$$g = \Omega[J_{(+)}, J_{(-)}]$$

$$\mathcal{F}_{(+)} = d\lambda_{(+)} \ , \quad \lambda_{(+)\ell} = iK_R J(K_{LR})^{-1}K_{L\ell} \ , \ldots$$

$d = 2\ , \ \ N = (2,2)$

Algebra:

$$\{\bar{\mathbb{D}}_{\pm}, \bar{\bar{\mathbb{D}}}_{\pm}\} = i\partial_{\pm\!\!\!\!=}$$

Constrained superfields:

$$\bar{\mathbb{D}}_{\pm}\phi^{a} = 0\ ,$$
$$\bar{\mathbb{D}}_{+}\chi^{a'} = \mathbb{D}_{-}\chi^{a'} = 0\ ,$$
$$\bar{\mathbb{D}}_{+}\mathbb{X}^{\ell} = 0\ ,$$
$$\bar{\mathbb{D}}_{-}\mathbb{X}^{r} = 0\ .$$

Notation: $c := a, \bar{a}\ , \quad t := a', \bar{a}'\ , \quad L := \ell, \bar{\ell}\ , \quad R := r, \bar{r}\ .$

Superspace I

The $(2,2)$ formulation uses the generalized Kähler Potential.

$$S = \int \mathbb{D}_+\bar{\mathbb{D}}_+\mathbb{D}_-\bar{\mathbb{D}}_- K(\phi^c, \chi^t, \mathbb{X}^L, \mathbb{X}^R)$$

$K \to K(\mathbb{X}^L, \mathbb{X}^R)$

Reduction to $(2,1)$ superspace

$$\mathbb{D}_- =: D_- - iQ_- \ , \quad \mathbb{X}| =: X \ , \quad Q_-\mathbb{X}^L| =: \Psi^L_-$$

$$S = \int \mathbb{D}_+\bar{\mathbb{D}}_+ D_- \left(K_L \Psi^L_- + K_R J D_- X^R \right)$$

$$S = i \int \mathbb{D}_+\bar{\mathbb{D}}_+ D_- (\lambda_{(+)\alpha} D_- \varphi^\alpha + c.c.)$$

which uses the “Liouville form” $(\mathcal{F}_{(+)} = d\lambda_{(+)})$

Reduction to $(1,1)$ finally yields

$$S = \int D_+ D_- \left(D_+ X E D_- X \right) \ .$$

The reduction goes via $\mathbb{D}_+ =: D_+ - iQ_+ \ , \quad Q_+ \mathbb{X}^R| =: \Psi_+^R$ and both the auxiliary spinors Ψ_-^L and Ψ_+^R have been eliminated.

The $(1,1)$ formulation uses $E = g + B$ directly.