Phase structures in Fuzzy geometries

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PLAN

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Motivations: Noncommutative geometry

- General relativity and Quantum theory together imply space-time structure at Planck scale do not conform to conventional notions of geometry.

Motivations: Noncommutative geometry

- General relativity and Quantum theory together imply space-time structure at Planck scale do not conform to conventional notions of geometry.
- Need for these changes were pointed out by Sergio Doplicher. mentioned by Podles in Lectures on quantum groups ascribing to Werner Nahm. Stems from existence of horizons. Extra dimensional spacetimes with higher dimensional 'Planck scale' $O(Tev)$ will make the spacetimes fuzzy.
- We will explore implications of these fuzzy geometries.

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From

"On the hypotheses which lie at the bases of geometry", Bernhard Riemann, 1854 (from the translation by W K Clifford).

Fuzzy $CP^1, CP^2, S^1 \otimes R$,, *Fuzzy torus...*

- $\sum X_i^2 = R^2$ along with $[X_i, X_j] = i\epsilon_{ijk}X_k$. Representations of $SU(2)$ algebra provide a basis to describe functions on fuzzy sphere.
- The fact $S^2 = CP^1$ is a coadjoint orbit is useful in quantising this space. This can be extended to $CP^2 = \dfrac{SU(3)}{SU(2) \otimes U(1)}$ and any CP^n.
- Fuzzy torus: $U, V, U\ V = e^{i\theta}\ V\ U$. Finite dimensional representations can be constructed for this algebra for rational θ.
- Fuzzy cylinder: $Z, e^{i\phi}; [Z, e^{i\phi}] = i\alpha\ e^{i\phi}$. Such geometry appears for a model for Noncommutative blackhole.

Higgs algebra

- Higgs algebra defined by: $[X_+, X_-] = \alpha Z + \beta Z^3, [X_\pm, Z] = \pm X_+$. Equation for Casimir can be given as:

$$\mathcal{C} = \frac{1}{2}\left[\{X_+, X_-\} + g(Z) + g(Z-1)\right], \quad (1)$$

where $g(Z)$ is:

$$g(Z) = C_0 + \frac{\alpha}{2} Z(Z+1) + \frac{\beta}{4} Z^2(Z+1)^2. \quad (2)$$

- For $C_0 = \mu^2$, $\alpha = -2(2\mu + 1)$, and $\beta = 4$ The Casimir reduces to the expression: $X^2 + Y^2 + (Z^2 - \mu)^2$.
- Equating the Casimir to 1 and plotting the function for different values of μ we see interesting topolgy change:

Higgs manifold and topology change

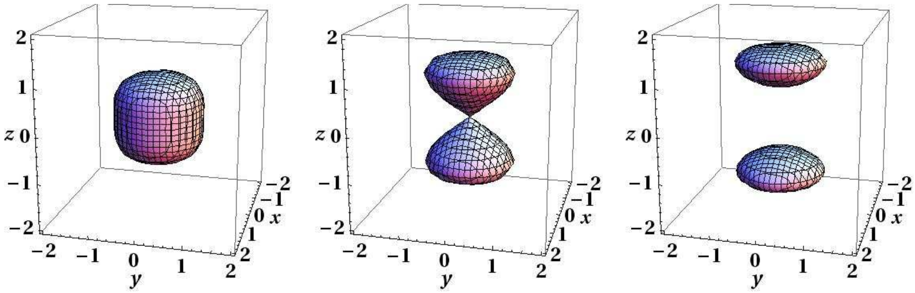


Figure: Surface plots depicting the change in topology for $\mu = 0, 1, 1/2$.

Fuzzy cylinder geometry

- NC cylinder is defined by the relations

$$\left[\, Z,\ e^{i\phi} \,\right] \;=\; \alpha e^{i\phi} \tag{3}$$

  where Z is hermitian and $e^{i\phi}$ is unitary.

- The geoemtry of BTZ black hole (which is also near horizon geometry of many blackholes) is obtained by quotionting $SL(2,R)$ by a discrete subgroup of its isometry.The noncommutative BTZ black hole is obtained by a deformation of $SL(2,R)$ which respects the quotienting. The coordinate operators r, ϕ and t satisfy the algebra

$$[Z, e^{i\hat{\phi}}] = \alpha e^{i\hat{\phi}} \qquad [\hat{r}, \hat{t}] = [\hat{r}, e^{i\hat{\phi}}] = 0\ , \tag{4}$$

- The noncommutative cylinder algebra also belongs to the class of the κ-Minkowski algebra

Matrix representation of NC Cylinder

- Since we are interested in simulations we have to discretise NC cylinder we consider the spin J irreducible representation (IRR) of the $SU(2)$ Lie algebra,

$$[X_+, X_-] = 2Z, \quad [X_\pm, Z] = \mp X_\pm. \tag{5}$$

But when we use the finite dimensional representations of $SU(2)$ we cannot have unitary $e^{i\phi}$. For this we decompose X_+ as product of hermitian and unitary operators:

$$X_+ = e^{i\phi} R \tag{6}$$

In the above, $e^{i\phi}$ is unitary, and R is a positive hermitian, necessarily singular, matrix which commutes with Z (and is thus diagonal).

- We have since, R commutes with Z,

$$[Z, X_+] = [Z, e^{i\phi}] R = e^{i\phi} R \tag{7}$$

Scalar fields on fuzzy spheres

- Let Φ be a scalar field on a fuzzy sphere defined by spin $j = \frac{N-1}{2}$ representation. It is given by a $N \times N$ matrix. The action is given by:
- $S = \frac{4\pi}{N} Tr \, \{\Phi[L_i[L_i, \Phi]\} + R^2\{r\Phi^2 + \lambda\Phi^4\}$
- Possible ground states characterised by $\Phi = 0, \Phi \neq 0, Tr \, \Phi = 0$. correseponding to uniform and nonuniform or stripe phases.
- Continuum limit: $N \longrightarrow \infty$. Planar limit: $R \longrightarrow \infty$. One gets commutative planar or noncommutative planar (Moyal) depending on $\frac{R^2}{N} \longrightarrow \infty \;\; or \;\; finite$.
- If we have a complex scalar field Φ then global $U(1)$ symmetry can be broken contrary to the expectation from Coleman-Mermin-Wagner theorem in the NC limit. This is due to the nonlocality of NC geometries.

O(3) fields on fuzzy spheres

- If we have (three) scalar fields Φ_i with global $O(3)$ symmetry then of topological 'hedgehog' solutions bring new stabilty.
- The action in this case is:

$$S(\Phi) = \frac{4\pi}{N}\,\mathrm{Tr}[\sum_i |\,[L_i,\Phi]\,|^2 + R^2(r|\Phi|^2 + i\beta\epsilon_{ijk}\Phi_i\Phi_j\Phi_k + \lambda(|\Phi|^2)^2 + \mu|\,[\Phi_i,\Phi_j]\,|^2)]$$

- The above action has metastable configurations with topological obstructions:

$$\Phi_i = \alpha L_i, \text{ with } \alpha = \sqrt{\frac{\frac{2|\mathrm{r}|}{\lambda}}{\mathrm{N}^2 - 1}}$$

In the above we have assumed $\beta = 0, \mu = 0$ for simplicity.

- We study through simulations the net effect of topological nature of the background configuration and non-locality on fluctuations with winding number one configuration.

Simulations on fuzzy geometries

- All the above geometries which are described finite size matrices are amenable for simulations. We use “pseudo-heat bath” updating method in our numerical simulations.
- In our simulations, for each choice of parameters, we choose an initial configuration.
- Fluctuations around this configuration are then generated by the above updating method. Since this configuration is a variational solution to minimising the classical action, it will thermalise as we update/include the thermal fluctuations.
- We also use over-relaxation to reduce the auto correlation of the configurations generated in the Monte-Carlo history.
- In simulations, the condensate will not maintain its exact form along the Monte Carlo history. The configuration can evolve into different random $SU(2)$ rotated configurations. To overcome this, one defines an observable made of Φ's which is invariant under the $SU(2)$ rotations.

Details of simulations for a field Φ

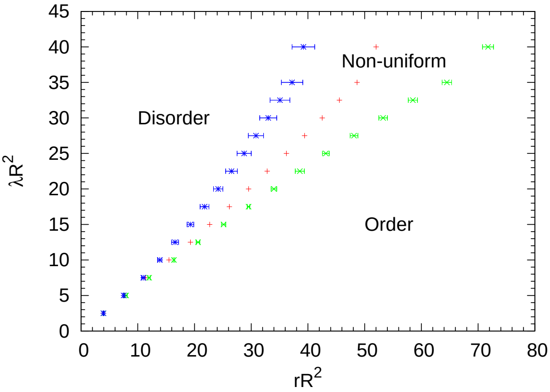
- In the "pseudo-heatbath" algorithm, given a Φ we update the elements of this matrix one at a time using the probability distribution,

$$\begin{aligned} P\left(\Phi_{ij}\right) &= e^{-S\left(\Phi_{ij}\right)} \\ \text{where} \qquad S\left(\Phi_{ij}\right) &= \alpha\left(\Phi_{ij} - A\right)^2 + \lambda B\left(\Phi_{ij} - C\right)^4, \end{aligned}$$

  A, B, C, α are constants suitably adjusted.
- To study the phase diagram and transitions we measure observables such as: $Tr(\Phi), \quad Tr(\Phi^2), \quad Tr(S)$ at various values rR^2 for different choices of $(N, \lambda R^2)$.

Phase diagram and triplepoint

We show the phase diagram for $N = 25$ in the λR^2 *vs* rR^2 plane with the triple point.



Details of simulations for $O(3)$ field Φ

- We define the observable,

$$A_{ij} = \frac{1}{N^2} Tr(L_i \Phi_j), M = \sqrt{A^\dagger A}$$

  M projects out the $l = 1$ angular momentum mode.

- Analysing the statistical behavior of M will show the stability of the initial configuration.
- For the commutative limit we fixed R^2 and considered higher values of N. We did not observe any change in the distribution of M. The average value, and the fluctuations of M remain the same (FIG 1). As for $N = 48$ the $l = 1$ configuration also decays for $N = 64$. This result suggests topological configuration is not stable in the commutative limit.
- For the noncommutative limit we fixed $\frac{R^2}{N}$ and increased N.Except for the lowest $N = 48$ the state did not decay during the entire run for higher N (FIG 2)

$O(3)$ field Φ Commutative and Noncommutative limits

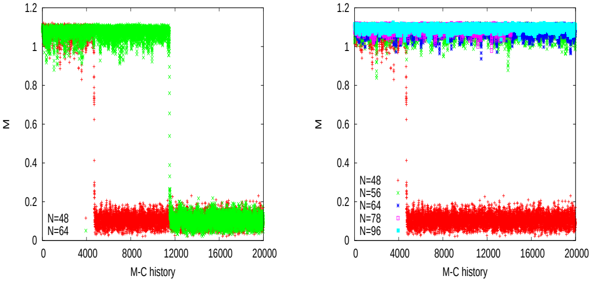


Figure: Commutative (FIG 1) and Noncommutative (FIG 2) limits

We also computed the fluctuations of M to see any possible scaling with the cut-off N and find $\chi = \langle M^2 \rangle - \langle M \rangle^2$ decrease with N like $\sim N^{-4}$.

Fields on fuzzy cylinder

- Scalar fields on such a background can be described by the action:

$$\begin{aligned} S &= \widetilde{Tr}\left(\mid [Z, \Phi] \mid^2 \; + \; \mid e^{-i\phi}[e^{i\phi}, \Phi] \mid^2 \; + \; V(\Phi) \right) \\ V(\Phi) &= r\Phi^2 + c\Phi^4 \end{aligned}$$

  for a hermitian field Φ.
- This action has a problem of instability. The source of this comes from trace $\widetilde{Tr}(\Phi^4) = Tr((P\Phi)\Phi^2(\Phi P))$ cannot contain any quartic (nor cubic) term for the variables $\Phi_{i\,2J} = \Phi_{2J\,i}$. It can be cured by constraining $\Phi = P\Phi P$
- With this new choice of the field the action becomes:

$$S \; = \; Tr_{J'}\left(\mid [\tilde{Z}, \Phi] \mid^2 \; + \mid [\widetilde{e^{i\phi}}, \Phi] \mid^2 + \; V(\Phi) \right)$$

  where $J' = J - 1/2$ is the reduced angular momentum, while $\widetilde{e^{i\phi}}$ and $\tilde{Z}$ are the matrices obtained from $e^{i\phi}$ and Z by removing the last line and column.

Simulation results for fuzzy cylinder

- The model has four parameters (μ, λ, r, N). The goal is to explore the parameter space for various phases of Φ. The temperature (T) is regulated by varying the parameter μ.
- $\mu \ll 1$ corresponds to low temperatures when the fluctuations are small and we get uniform phase.
- At high temperatures, $\mu \gg 1$, the thermal fluctuations lead the system to the disorder phase.
- At intermediate temperatures we get the non-uniform or stripe phases. Due to the non-trivial topology of the cylinder (the first homotopy group being non-trivial), one can have a more complex phase structure.
- The phases can be characterised by the observables $m_u = Tr(\Phi)$, $m_z = Tr(\Phi Z)$, $m_x = Tr(\Phi e^{i\phi})$.
- We did not observe the phase with stripes going along the cylinder as ground state for any choice of μ but phase with stripes going around the cylinder for some μ.

Stripes on fuzzy cylinder

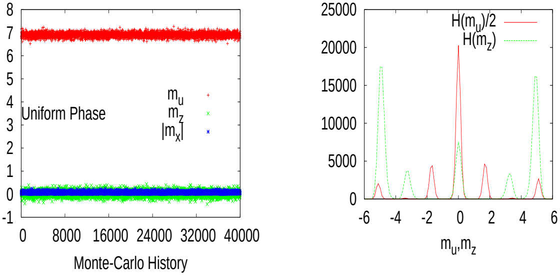


Figure: Uniform (FIG 3) and Nonuniform (FIG 4) phases on NCCYLINDER

Disorder and stripe on fuzzy cylinder

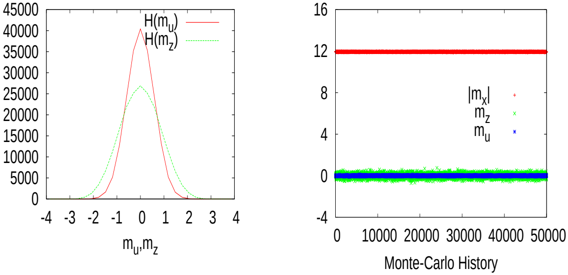


Figure: Disorder (FIG 5) and Stripe (FIG 6) phases on NCCYLINDER

Conclusions

- Fuzzy geometries allow nonuniform phases breaking symmetries like translations.
- Topological obstructions enhances the stabilty of such symmetry broken phases.
- Given extra dimensional spaces to be described by effective Planck scales which $O(TeV)$ the NC geometry and the phase structures will have interesting phenomenological implications.
- Based on the papers:
  - 0906.1660, 0801.4479, 0706.0695
  - Topological stability of broken symmetry on fuzzy spheres, S Digal, TRG, hep-th:1108.3320
  - Phase structure of fuzzy black holes, S Digal, TRG, K S Gupta, X Martin, hep-th: