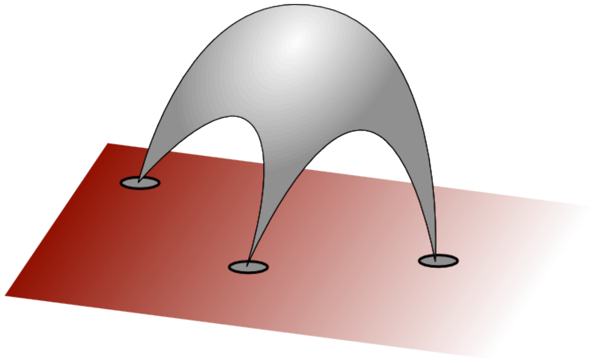
Holographic three-point functions



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Based on 1106.0495 with T. Klose

Conformal correlation functions

For a conformal field theory, symmetry is sufficient to fix the space-time dependence of the one-, two- and three-point correlation functions. E.g. scalar primary operators,

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \frac{\delta_{a,b}}{|x_1 - x_2|^{2\Delta}}$$

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\mathcal{O}_c(x_3)\rangle = \frac{c_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}|x_{31}|^{\Delta_c+\Delta_a-\Delta_b}}$$

The non-trivial dependence on the coupling comes via the anomalous dimension and the structure constants. In principle all other correlation functions can be determined via the OPE.

Integrability

- For N = 4 SYM in the planar limit there are additional hidden symmetries.
- Perturbative dilatation operator can be mapped to an integrable spin chain Hamiltonian...

[Minahan & Zarembo]

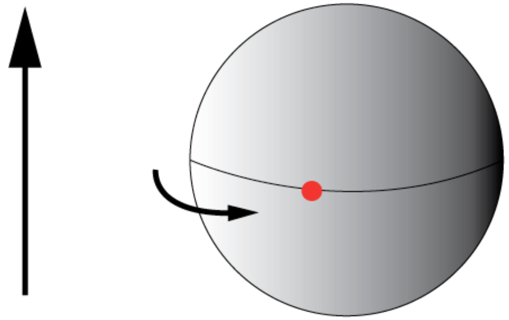
$$\mathcal{O} = \mathrm{Tr}[\Phi\Phi\Psi\Phi\ldots\Phi\Psi\Phi] \quad \longleftrightarrow \quad |\mathcal{O}\rangle = |\downarrow\downarrow\uparrow\downarrow\ldots\downarrow\downarrow\uparrow\downarrow\rangle$$

$$\mathfrak{D}\cdot\mathcal{O} = \Delta(\lambda)\mathcal{O} \quad \longleftrightarrow \quad H|\mathcal{O}\rangle = E(\lambda)|\mathcal{O}\rangle$$

...and via AdS/CFT the theory is dual to strings described by an integrable 2-d sigma-model.

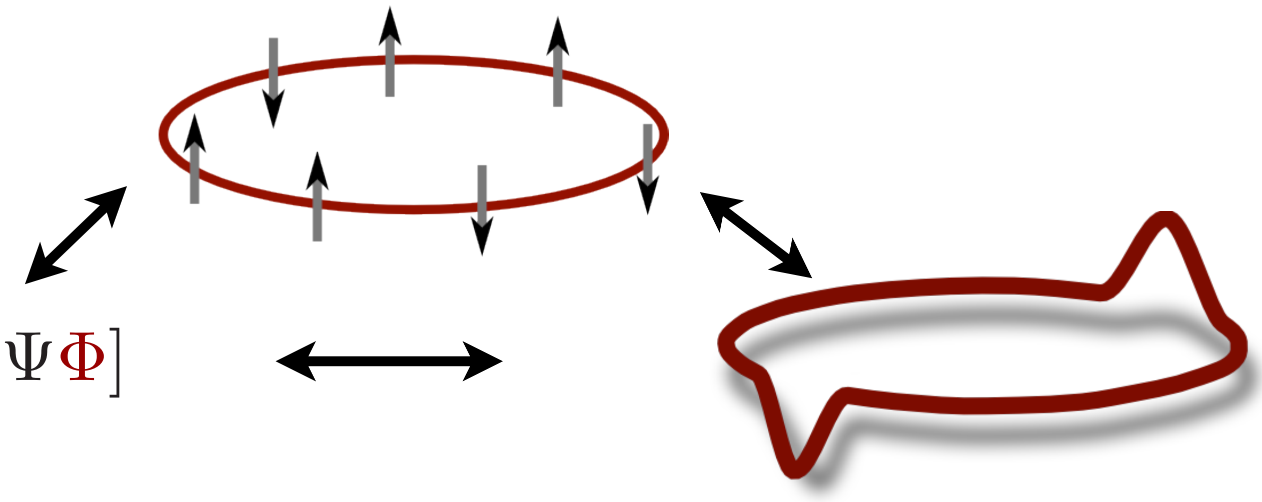
- Anomalous dimensions correspond to the energies of physical strings, e.g. CPO corresponds to a point-like BMN string

$$\mathrm{Tr}[\Phi\Phi\Phi\ldots\Phi\Phi] \quad \Leftrightarrow$$



$$\mathbb{R}\times S^2 \subset \mathrm{AdS}_5 \times S^5$$

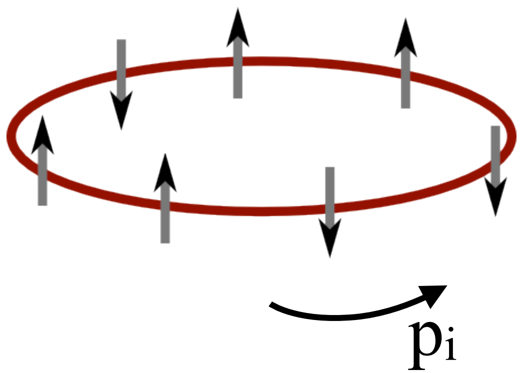
- In general we can then consider magnon like excitations about the vacuum,



$$\mathrm{Tr}[\Phi\Phi\Psi\Phi\ldots\Phi\Psi\Phi]$$

Anomalous Dimensions

For "long" single trace operators/infinite volume string states/ spin-chain states we can solve for the spectrum by means of the asymptotic Bethe ansatz.
A given operator is characterized by excitations with momenta $\{p_i\}$...



$$e^{ip_iL} = \prod_{i\neq j} S(p_i,p_j) \quad \& \quad E_{\rm Tot} = \sum_i E(p_i)$$

For short states we can also find all order results (though more technically complicated c.f. talk of S. Frolov).

Structure Constants

Can we find similar results for c_{abc}?

- The calculation of two-point functions made use of the relation to the eigenvalues of dilatation op./string energies.

- No similar relation for structure constants and so we need to directly calculate the correlation functions.

Progress in this direction at strong coupling using semiclassical methods: Janik et al 1002.4613, Buchbinder 1002.1716, Buchbinder and Tseytlin 1005.4516, Zarembo 1008.1059, Costa et al 1008.1070 ....

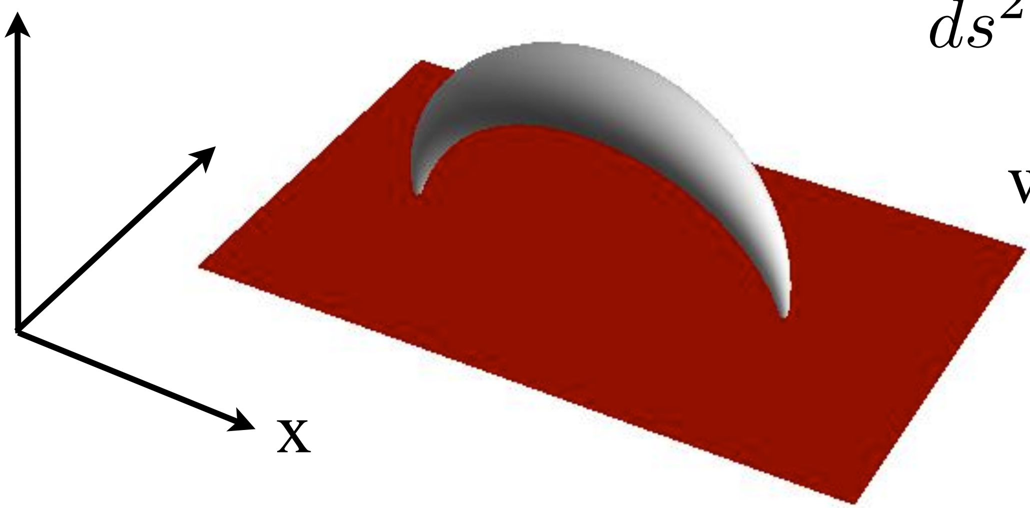
We will recast and extend some of these results and explain how to systematically go beyond the leading approximation

Key Idea: Make use of light-cone gauge formalism for Green-Schwarz string correlation function.

Holographic Correlation Functions

Simplest case: two string vertex operators located at the boundary of AdS_5 at positions x_1 and x_2 source a string which propagates into the bulk

$$|\mathbf{z}| = \sqrt{\mathbf{z}^2}$$



$$ds^2 = \frac{d\vec{x}^2 + d\mathbf{z}^2}{\mathbf{z^2}}$$

with $\vec{x} \in \mathbb{R}^4$ & $\mathbf{z} \in \mathbb{R}^6$

$$\langle V_{\{Q_1\}}(\tau_1, \vec{x}_1) V_{\{Q_2\}}(\tau_2, \vec{x}_2)\rangle = \int [\mathcal{D}X\mathcal{D}P\mathcal{D}h]\ V_1 V_2\ e^{-\mathcal{S}[h,X,P]}$$

with $$\mathcal{S} = \frac{\sqrt{\lambda}}{2\pi}\int d^2\sigma\ P\cdot X - \mathcal{H}_{\text{w.s.}}$$

String w.s. and target space both have Eucl. signature.

Holographic BMN Correlation Functions

For point-like, BMN, string:

Polyakov/Tseytlin

$$\mathrm{Tr}[\Phi\Phi\Phi\ldots\Phi\Phi](\vec{a}) \quad \Leftrightarrow$$

$$V_{\{\Delta,J\}}(\vec{a},\mathbf{n}) = \left(\frac{|\mathbf{z}|}{\mathbf{z}^2+(\vec{x}-\vec{a})^2}\right)^{\Delta}\left(\frac{\mathbf{n}\cdot\mathbf{z}}{|\mathbf{z}|}\right)^{J} U_{\text{quad. der.}}$$

$\mathbf{n}$: a 6-d vector which defines the plane of rotations in the S^5

$U_{\text{quad. der.}}$: prefactor quadratic in derivatives

- When $\Delta\sim\sqrt{\lambda}$ and $J\sim\sqrt{\lambda}$ we can interpret the vertex op. as contributing boundary terms to the worldsheet action at times τ_1 & τ_2.
- We can find the saddle point approx. to the path integral by solving the equations of motion including boundary terms: generically the w.s. is complex so we consider a complexified AdS space.
- This classical solution is not a physical string, rather it's a tunnelling amplitude. It can be found by analytically continuing the physical BMN string solution.

AdS Light-cone Gauge

- The string worldsheet has the usual diffeo- and Weyl-invariance and it is useful (particularly later) to fix a gauge.
- Pick two-boundary coordinates and make the complex combinations

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^3 \pm i x^0)$$

- Then the gauge we use is

$$x^+ = \tau \ , \quad p_- = \text{const}$$

the ws metric is diagonal $h^{ab} = \text{diag}(\mathbf{z}^2, \mathbf{z}^{-2})$.

Important point: going to light-cone gauge solves the Virasoro constraints, in the path-integral this implies that all vertex operators must describe physical, on-shell states or equivalently that the energy of the string states are related to their charges e.g. angular momentum

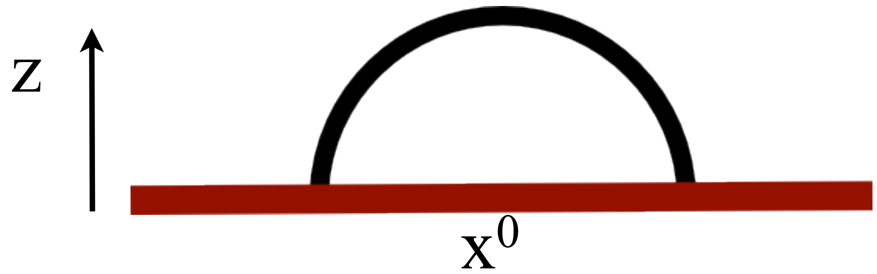
$$\Delta = E = E(J, S, ...) \quad \text{i.e.} \qquad \Delta_{\text{BMN}} = J$$

Saddle point for two BMN vertex operators is particularly simple in AdS light-cone gauge

$$x^- = \tau, \quad p_- = \frac{\Delta}{\sqrt{\lambda}} \frac{i\sqrt{2}}{|x_{12}|}$$

$$\mathbf{z} = \frac{1}{2}(x^0 - x_1^0)e^{-\phi}\mathbf{n}^* + \frac{1}{2}(x_2^0 - x^0)e^{\phi}\mathbf{n} \quad \text{with} \quad x^0 = -i\sqrt{2}\tau$$

This is simply an AdS geodesic



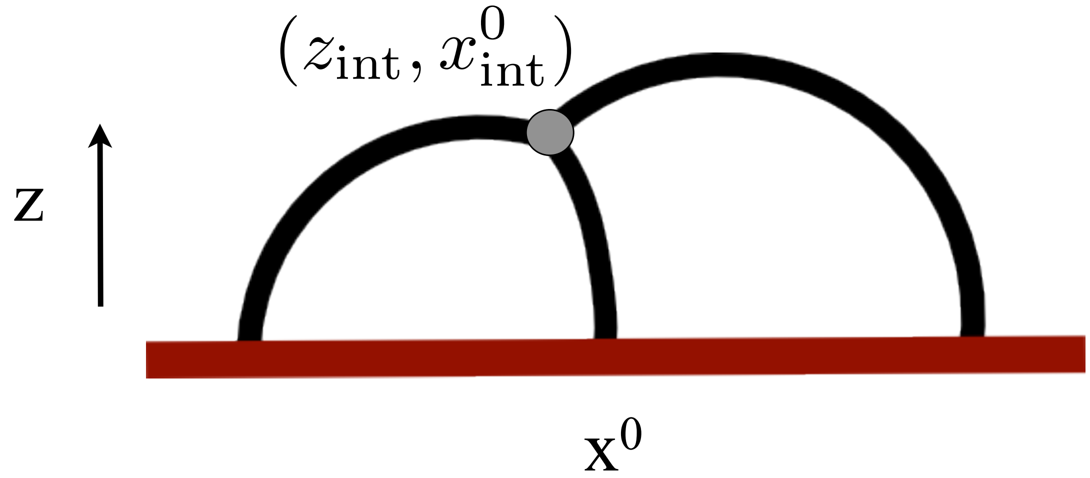
and for a solution it is required that:

$$\mathbf{n}_1 = \mathbf{n}_2 \ , \quad J_1 = \Delta_1 = \Delta = \Delta_2 = -J_2$$

Evaluating the exp. of the action on the solution the bulk action is zero and the boundary action is finite and equal to

$$\langle V_1 V_2 \rangle = \frac{1}{|x_{12}|^{2\Delta}}$$

Saddle point for three BMN vertex operators can be made from three two pt functions...



$$\langle V_1(\vec{x}_1, \mathbf{n}_1) V_2(\vec{x}_2, \mathbf{n}_2) V_3(\vec{x}_3, \mathbf{n}_3)\rangle$$

...which satisfy the e.o.m. everywhere (but is not smooth). Minimizing the full action including the boundary terms determines the intersection. For generic $\{n_1, n_2, n_3\}$ there is a solution if

$$\mathbf{n}_1 \cdot \mathbf{n}_2 \neq 0 \ , \quad \mathbf{n}_1 \cdot \mathbf{n}_3 \neq 0 \ , \quad \mathbf{n}_2 \cdot \mathbf{n}_3 \neq 0$$

If all three strings rotate in the same plane we find an additional constraint

i.e. extremal correlator $\quad \Delta_1 = \Delta_2 + \Delta_3$

In the non-extremal case we find (with $\mathbf{n}_i \cdot \mathbf{n}_j = 1$)

$$\langle V_1(\vec{x}_1, \mathbf{n}_1) V_2(\vec{x}_2, \mathbf{n}_2) V_3(\vec{x}_3, \mathbf{n}_3) \rangle$$

$$= \frac{1}{|\vec{x}_{12}|^{\alpha_3} |\vec{x}_{23}|^{\alpha_1} |\vec{x}_{13}|^{\alpha_2}} \left(\frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} (\alpha_1 + \alpha_2 + \alpha_3)^{\alpha_1+\alpha_2+\alpha_3}}{(\alpha_1+\alpha_2)^{\alpha_1+\alpha_2} (\alpha_2+\alpha_3)^{\alpha_2+\alpha_3} (\alpha_1+\alpha_3)^{\alpha_1+\alpha_3}} \right)^{\frac{1}{2}}$$

where $\alpha_1 = \Delta_2 + \Delta_3 - \Delta_1$ and cyclic perm.

In the our limit where the charges are large and generic,

$$J_i \sim \sqrt{\lambda} \,, \alpha_i \sim \sqrt{\lambda}$$

this is the same result found for CPOs at weak and strong coupling using supergravity. Lee, Minwalla, Rangamani and Seiberg, hep-th/9806074

To find the sphere wavefunction dependence we would need in principle to make the replacement in the vertex operators

$$\left(\frac{\mathbf{n} \cdot \mathbf{z}}{|\mathbf{z}|} \right)^J \to Y(\hat{\mathbf{z}}) \qquad \text{SO(6) spherical harm.}$$

Holographic non-BPS Correlation Functions

With caveats we can generalise this calculation to a wider class of strings states e.g. circular strings which wind around the sphere.
Starting from the physical string solution we can analytically continue

$$z_1 = \frac{c-b}{2\sqrt{2}\cosh\kappa\tilde{\tau}}\cosh(\omega\tilde{\tau}+im\sigma+\phi_1)\ , \qquad z_2 = \frac{c-b}{2\sqrt{2}\cosh\kappa\tilde{\tau}}\sinh(\omega\tilde{\tau}+im\sigma+\phi_1)\ ,$$
$$z_3 = \frac{c-b}{2\sqrt{2}\cosh\kappa\tilde{\tau}}\cosh(\omega\tilde{\tau}-im\sigma+\phi_2)\ , \qquad z_4 = \frac{c-b}{2\sqrt{2}\cosh\kappa\tilde{\tau}}\sinh(\omega\tilde{\tau}-im\sigma+\phi_2)\ ,$$
$$x^+ = \tau\ , \qquad x^- = -\tau$$

and interpret this as the saddle-point for the two-point function.

- We can write down vertex operators which source this solution - however this only determines exponentially large part and moreover is non-unique.
- There exist two proposals Buchbinder 1002.1716 & Ryang 1011.3573.
- Can find a three point solution which is sourced by these vertex ops., answer is essentially the same in both cases.
- Now the bulk action is non-vanishing however it combines with the boundary action to give a finite result.

Quantum Corrections

Another direction is to consider the corrections to the saddle-point approx. by studying the fluctuations about the classical solution. Furthermore this allows us to study near-BMN, or excited strings.
To do this we need to include

- corrections to vertex operators.
- corrections to the saddle-point evaluation of the path-integral.

Expanding about the BMN string solution (for the eight transverse l.c. coords.)

$$X^M = X^M_{\rm cl} + \frac{\tilde{X}^M}{\lambda^{1/4}}$$

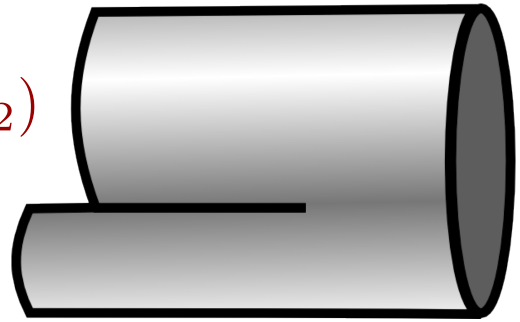
we find the fluctuation action describes eight massive bosons

$$S_{\rm fl} = \frac{1}{4\pi}\int d^2\sigma\ [\dot{\tilde{X}}^2 + \acute{\tilde{X}}^2 + \mu^2 \tilde{X}^2] \qquad \text{with} \qquad \mu = \frac{\Delta}{\sqrt{\lambda}}$$

which only depends on the total charge and not on the boundary positions or the orientation on the sphere. Similarly for the fermions.

- Thus the fluctuations correspond to massive harmonic oscillators.
- Necessary to use decompactification limit.

Functional Methods



We can apply the standard path-integral light-cone functional methods

$$\langle V_1(\tau_1)\dots V_N(\tau_N)\rangle = \mathcal{N} e^{-\mathcal{S}_{\rm cl}-\mathcal{B}_{\rm cl}} \int \prod dP_{i,n}\psi_i(P_{i,n}) \int D\tilde{X} e^{\sum H_{lc}|\tau_i|} exp\Big[i\sum_i \int P\tilde{X} - S_{\rm fl}\Big]$$

The corrections to the vertex operators are encoded in the wavefunctions

$$\psi_i(P_{i,n}) = \prod \langle 0|(\alpha_{i,n})^{k_{i,n}}|P_{i,n}\rangle$$

to leading order.

Integrate out the coordinate fluctuations by introducing the w.s. Greens function $N(\sigma,\tau_i;\sigma',\tau_j)$ or equivalently its Neumann coefficients N^{ij}_{mn}.

so that $$\langle V_1 V_2 \rangle = \frac{1}{|x_{12}|^{2\Delta}} \langle \{k_{1,n}\}, \{k_{2,n}\}| e^{-\sum \alpha^\dagger_{2,n} \alpha^\dagger_{1,n}} |0\rangle$$

and

$$\langle V_1 V_2 V_3 \rangle = e^{-\mathcal{S}_{\rm cl} - \mathcal{B}_{\rm cl}} C^{123}$$

with $$C^{123} = \langle \{k_{i,n}\}, \{k_{2,n}\}, \{k_{3,n}\}| exp\Big[\sum_{\substack{n,m \\ i<j}} N^{ij}_{nm} \alpha^\dagger_{i,n} \alpha^\dagger_{j,m}\Big] |0\rangle$$

similar terms from fermions.

Here we essentially find the same result as in plane-wave LCSFT [Spradlin & Volovich, Pankiewicz, Stefanski, Chu et al, also Dobashi,Shimada,Yoneya and many others] which are closely related to flat-space results but with

$$\omega_p = \sqrt{\mu^2 + p^2}$$

Neumann coefficients are given by

$$N^{ij}_{mn} = \delta^{ij} \delta_{mn} - \sqrt{\omega_{i,m} \omega_{j,n}} (X^{(i)T} \Gamma^{-1} X^{(j)})_{mn}$$

[Spradlin & Volovich, He et al, Lucietti et al]

Prefactor

It is long known that in l.c. gauge, string amplitudes can't be calculated as overlap functions of vertex operators but that insertions factors must be placed at the string interaction points (required for super-Poincare symmetry). For flat space Green-Schwarz string this is described by,

$$S = \sum |MN\rangle \partial X^M \bar{\partial} X^N$$ [Mandlestam '74, '86]

$$|MN\rangle = \delta^{MN} + \frac{1}{2}\lambda^a \lambda^b \gamma_{ab}^{MN} + \cdots + \frac{1}{8!}(\lambda^a \ldots \lambda^h)\epsilon_{ab\ldots gh}$$

Fermionic fluctuations

Analogous factor found in plane-wave LCSFT, [Spradlin&Volovich, Pankiewicz, Chu et al, Di Vecchia et al, Lee & Russo, Dobashi and Yoneya, Shimada, many others].
Constrained by plane-wave symmetries.
In our calculation the same object arises (due to gauge fixing), however

$$Q^- = \int d\sigma \; e^{-ix^-/2} \mathcal{Q}^-_{\mathrm{p.p}}$$

there is non-locality or non-trivial co-product structure.
In the strict decompactification limit this effect can be neglected.

Conclusions

- Considered the strong coupling/semiclassical string calculation of 3pt functions for certain operators with large charges: $\Delta\sim\sqrt{\lambda}$ and $J\sim\sqrt{\lambda}$.
- For BMN strings we reproduce the known results for three CPO and including fluctuations we produce the results of plane-wave LCSFT.
- Advantages of method:
  - — we can apply similar fluctuation analysis to more general classical solutions (e.g. not restricted to extremal correlators).
  - — can systematically include subleading corrections perturbatively
    - quartic terms in the fluctuation action
    - mixing of vertex operators
    - Corrections to prefactor

In principle we would like to make use of symmetries to make exact predictions e.g. it is known that the exact ws propagator is

$$\omega_p = \sqrt{1+p^2} \to \sqrt{1+4\sin^2\frac{p}{2}}$$

all-order Neumann coeff. in terms of generalised μ-deformed Gamma functions?

$$N^{ij}_{mn} = \delta^{ij}\delta_{mn} - \sqrt{\omega_{i,m}\omega_{j,n}}(X^{(i)T}\Gamma^{-1}X^{(j)})_{mn}$$