BPS Saturated Objects in String Theory: K3 Elliptic Genus and Igusa Cusp Form χ_{10}

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Work in collaboration with M. Gaberdiel (ETHZ), S. Stieberger (MPI Munich) and R. Volpato (ETHZ)
synopsis of 1006.0221, 1008.3778, 1106.4315, 1108.0323

BPS Saturated Quantities in String Theory

Studied from different viewpoints in string theory

target space	world-sheet
particular effective superstring couplings (top. amplitudes)	topological quantities in CFT (twisted correlators, BPS indices)

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superspace analyticity properties

- $\mathcal{N} = 2$: chirality
- $\mathcal{N} = 4$: G-analyticity in harmonic superspace

differential equations:

- holomorphicity eq. Bershadsky, Cecotti, Ooguri, Vafa 1993

  coupled first order relations

  Antoniadis, SH, Narain, Sokatchev 2011
- harmonicity relations

  Berkovits, Vafa 1994
  Ooguri, Vafa 1995
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• topological amplitudes, BPS saturated couplings	• topological objects in CFT
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mathematically: encode top. invariants of internal manifold	mathematically: related to BPS indices or index-like structures

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• topological invariants	• BPS indices
physically: window into dualities, black hole entropy, pheno. Ooguri, Strominger, Vafa 2004; Antoniadis, SH 2009 Dabholkar, Denef, Moore, Pioline 2005	physically: algebraic properties of the spectrum of BPS states Harvey, Moore 1995, 1996 Gaberdiel, SH, Persson 2011; SH, Persson 2011

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• dualities, symmetries, physical problems	• algebraic properties of BPS spectrum

Numerous examples found in different string setups!

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This talk focuses on one particular BPS quantity which has received a lot of attention recently: the elliptic genus of K3

Outline

Elliptic Genera of $\mathcal{N} = 2$ SCFT

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$$\phi(\tau, z) := \mathrm{Tr}_{RR}\left[(-1)^{F+\bar{F}} y^{J_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right], \qquad \begin{array}{l} q = e^{2\pi i \tau} \\ y = e^{2\pi i z} \end{array}$$

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$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^w \, e^{2\pi i m \frac{cz^2}{c\tau + d}} \, \phi(\tau, z)$$

$$\phi(\tau, z + \ell\tau + \ell') = e^{-2\pi i m(\ell^2 \tau + 2\ell z)} \phi(\tau, z) \qquad \ell, \ell' \in \mathbb{Z}$$

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- Fourier Expansion

$$\phi(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell \quad \text{with} \quad c(n, \ell) = (-1)^w c(n, -\ell)$$

Elliptic Genera of σ-models with $\mathcal{N} = (4, 4)$

Elliptic genus encodes basic topological information of SCFT-target *e.g.*

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Decomposition in $\mathcal{N} = 4$ characters for $K3$ target space

Eguchi, Taormina 1988
Eguchi, Hikami 2009

$$\phi_{K3} = 24\,\text{ch}_{h=\frac{1}{4},\ell=0} + \sum_{n=0}^{\infty} A_n \text{ch}_{h=n+\frac{1}{4},\ell=\frac{1}{2}}\,, \qquad \begin{aligned} \text{ch}_{h,\ell=\frac{1}{2}} &:= q^{h-\frac{3}{8}} \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3} \\ \text{ch}_{h=\frac{1}{4},\ell=0} &:= \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3}\,\mu(\tau,z) \end{aligned}$$

where $\mu(\tau, z)$ is an Appell-Lerch sum (mock modular theta function)

Mathieu Moonshine

$$\phi_{K3} = 24\,\mathrm{ch}_{h=\frac{1}{4},\ell=0} + \sum_{n=0}^{\infty} A_n \mathrm{ch}_{h=n+\frac{1}{4},\ell=\frac{1}{2}}$$

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First few A_n are the dimensions of the smallest irreducible representations of the largest Mathieu group $\mathbb{M}_{24}$ (Moonshine!). Eguchi, Ooguri, Tachikawa 2010

Suggests $\mathbb{M}_{24}$-symmetry of (sub)space of BPS states contributing to ϕ_{K3}

$$\mathcal{H}^{\text{BPS}}_{\phi_{K3}} = \bigoplus_n (H_n \otimes \mathcal{H}^{\mathcal{N}=4}_n)$$

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Leads to conjecture

$$\phi_{K3} = (\dim H_{00})\mathrm{ch}_{h=\frac{1}{4},l=0} - (\dim H_0)\mathrm{ch}_{h=\frac{1}{4},l=\frac{1}{2}} + \sum_{n=1}^{\infty} (\dim H_n)\mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}$$

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This might be a 'supersymmetric generalization' of monstrous moonshine!

McKay, Thompson 1978

Twining Genera

Idea of checking: twining genera: ($g \in \mathbb{M}_{24}$, 26 conjugacy classes)

$$\phi_g := \text{Tr}_{H_{00}}(g)\text{ch}_{h=\frac{1}{4},l=0} - \text{Tr}_{H_0}(g)\text{ch}_{h=\frac{1}{4},l=\frac{1}{2}} + \sum_{n=1}^{\infty}\text{Tr}_{H_n}(g)\text{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}$$

$\mathbb{M}_{24}$ Character Table

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
252	28	9	2	4	0	0	0	1	-1	-1	-1	0	0	-1	-1	0	0	0	0	12	2	0	0	4	1
253	13	10	3	1	1	1	-1	-2	0	0	0	-1	-1	0	0	1	1	1	1	-11	-1	1	1	-3	0
1771	-21	16	1	-5	0	0	-1	0	0	1	1	0	0	0	0	-1	-1	-1	7	11	1	0	0	3	0
3520	64	10	0	0	-1	-1	0	-2	0	0	0	1	1	1	1	0	0	0	-8	0	0	-1	-1	0	0
45	-3	0	0	1	e_7^+	e_7^-	-1	0	1	0	0	$-e_7^+$	$-e_7^-$	-1	-1	1	-1	1	3	5	0	e_7^-	e_7^+	-3	0
45	-3	0	0	1	e_7^-	e_7^+	-1	0	1	0	0	$-e_7^-$	$-e_7^+$	-1	-1	1	-1	1	3	5	0	e_7^+	e_7^-	-3	0
990	-18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	-1	-2	3	-10	0	e_7^-	e_7^+	6	0
990	-18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	-1	-2	3	-10	0	e_7^+	e_7^-	6	0
1035	-21	0	0	3	$2e_7^+$	$2e_7^-$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^-$	$-e_7^+$	3	0
1035	-21	0	0	3	$2e_7^-$	$2e_7^+$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^+$	$-e_7^-$	3	0
1035	27	0	0	-1	-1	-1	1	0	1	0	0	-1	-1	0	0	0	2	3	6	35	0	-1	-1	3	0
231	7	-3	1	-1	0	0	-1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0	-9	1	0	0	-1	-1
231	7	-3	1	-1	0	0	-1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0	-9	1	0	0	-1	-1
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	-2	-7	10	0	0	0	2	-1
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	-2	-7	10	0	0	0	2	-1
483	35	6	-2	3	0	0	-1	2	-1	1	1	0	0	0	0	0	0	3	0	3	-2	0	0	3	0
1265	49	5	0	1	-2	-2	1	1	0	0	0	0	0	0	0	0	0	-3	8	-15	0	1	1	-7	-1
2024	8	-1	-1	0	1	1	0	-1	0	-1	-1	1	1	0	0	0	0	0	8	24	-1	1	1	8	-1
2277	21	0	-3	1	2	2	-1	0	0	0	0	0	0	0	0	0	2	-3	6	-19	1	-1	-1	-3	0
3312	48	0	-3	0	1	1	0	0	1	0	0	-1	-1	0	0	0	-2	0	-6	16	1	1	1	0	0
5313	49	-15	3	-3	0	0	-1	1	0	0	0	0	0	0	0	0	0	-3	0	9	-1	0	0	1	1
5796	-28	-9	1	4	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	-4	-1
5544	-56	9	-1	0	0	0	0	1	0	-1	-1	0	0	1	1	0	0	0	0	24	-1	0	0	-8	1
10395	-21	0	0	-1	0	0	1	0	0	0	0	0	0	-1	-1	0	0	3	0	-45	0	0	0	3	0

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Conjectured transformation for $g \in \mathbb{M}_{24}$ (order N) Gaberdiel, SH, Volpato 2010

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$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | c \equiv 0 \bmod N \right\}\,, \qquad h \in \mathbb{N}$$

h determines multiplier system and differs for various conjugacy classes

$$h = 1 \iff g \in \mathbb{M}_{23} \subset \mathbb{M}_{24}$$

Status of Checks of $\mathbb{M}_{24}$ Moonshine

Available/conjectural information about twining genera

- first few Fourier coefficients Eguchi, Ooguri, Tachikawa 2010
- conjectured modular properties (particularly multiplier system)

allows for closed expressions for all 26 conjugacy classes Cheng 2010 Gaberdiel, SH, Volpato 2010 Eguchi, Hikami 2010

$$\phi_{2A}(\tau,z) = 8\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}\,, \quad \phi_{2B}(\tau,z) = -2\theta_4(2\tau,0)^2\frac{\theta_1(\tau,z)^2}{\eta(\tau)^6}\,, \quad \ldots$$

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Knowledge of the twining genera in turn allows to uniquely decompose all H_n in terms of irreducible representations of $\mathbb{M}_{24}$.

For $n \leq 1000$: H_n valid $\mathbb{M}_{24}$ representations (decomposition into irreps with non-negative integer coefficients) Gaberdiel, SH, Volpato 2010 Eguchi, Hikami 2010

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No proof, but very strong evidence in favor of the conjecture!

Explanation of Mathieu-Moonshine?

Challenge: Given the massive evidence in favour of the conjecture, what is the reason for the appearance of $\mathbb{M}_{24}$?

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So far no satisfactory explanation, but progress in partial understanding

- symplectic automorph. of K3 are subgroup of $\mathbb{M}_{23} \subset \mathbb{M}_{24}$ Mukai 1988 Kondo 1998
- at given point in moduli space generically no enhancement to subgroup of $\mathbb{M}_{24}$ through quantum symmetries of $K3$ σ-model, instead for distinct cases. $\mathcal{N} = 4$ preserving sym. Gaberdiel, SH, Volpato 2011
  - subgroup of $G'.G''$, where $G' \subset \mathbb{Z}_2^{11}$ and $G'' \subset \mathbb{M}_{24}$
  - subgroup of $5^{1+2}.\mathbb{Z}_4$
  - subgroup of $\mathbb{Z}_3^4.A_6$
  - subgroup of $3^{1+4}.\mathbb{Z}_2.G''$, where G'' is either trivial, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_4$.

  (p^{1+2n} is the extra special group of order p^{1+2n})
- all symmetries subgroups of Conway Co_1

New insights (different approach?) needed to explain Mathieu moonshine.

Connection to Target Space Physics

Attempts at understanding so far have focused on the world-sheet.

Question: How much symmetry maps over to target space physics?

- Any connection of $\mathbb{M}_{24}$ to algebra of BPS states? Harvey, Moore 1995, 1996; Gaberdiel, SH, Persson 2011; SH, Persson 2011
- Which target space structures are sensible to $\mathbb{M}_{24}$?
- Can we think of $\mathbb{M}_{24}$ as symmetry of some effective action couplings?
- Are there $\mathcal{N} = 4$ (1/4 BPS) topological amplitudes which capture the elliptic genus?

Connection to Target Space Physics

Attempts at understanding so far have focused on the world-sheet.

Question: How much symmetry maps over to target space physics?

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Will report on first steps in anwering these questions in the following.

I.e. I will discuss a particular BPS saturated amplitude in type II string theory on $K3 \times T^2$ which is related to the elliptic genus of $K3$.

($\mathcal{N} = 2$ results for heterotic on $K3 \times T^2$ Kawai 1995)

Massive External States

Consider (carefully arranged) amplitude of the form SH, Stieberger 2011

$$\mathcal{F}_N = \left\langle \int d^2z_1 V_R(z_1) \int d^2z_2 V_R(z_2) \prod_{a=1}^{N} \int d^2x_a \; V_M^{(-1,-1)}(p_a, x_a) \prod_{b=1}^{N} V_{\text{PCO}} \right\rangle$$

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Key indgredient are massive external scalar fields (first massive level)

$$V_M^{(-1,-1)}(p,x) = \, : e^{-\varphi}\, \psi_3\, \partial H(x) \; e^{-\tilde{\varphi}}\, \tilde{\psi}_3\, \bar{\partial} X_3(\bar{x}) \; e^{ipX} : \; .$$

$X_3 \ldots$ torus coordinate

$\varphi \ldots$ bosonization of superghost

$\psi_3, \tilde{\psi}_3 \ldots$ free torus fermion

$\partial H(x) \ldots$ bosonization of J_{K3}

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Explicit result

$$\mathcal{F}_N = \int \frac{d^2\tau}{\tau_2}\, \tau_2^{2N} \left\langle \prod_{a=1}^{N} \partial H(x_a) \right\rangle_{K3} \sum_{\Gamma^{2,2}} (P_L)^N (P_R)^{2N} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} =$$

$$= (\mathcal{D}_{\bar{U}})^N \int \frac{d^2\tau}{\tau_2} \left[\frac{\partial^N}{\partial z^N} \phi_{K3}(\tau, z) \right]_{z=0} \sum_{\Gamma^{2,2}} (\tau_2 P_R)^N\, q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2}$$

Analysis of Amplitude

Consider 1-loop expression in more detail

$$\int \frac{d^2\tau}{\tau_2} \left[\frac{\partial^N}{\partial z^N}\,\phi_{K3}(\tau,z)\right]_{z=0} \sum_{\Gamma^{2,2}} (\tau_2 P_R)^N\, q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2}$$

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Can we make the symmetry properities of this expression more manifest?

- compute the world-sheet torus integral
- make T-duality invariance manifest by writing it in terms of modular forms of $SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U$
- any additional (hidden?) structures or symmetries?

Generating Functional

Define generating functional

$$\mathbb{F}(\lambda, T, U) := \int \frac{d^2\tau}{\tau_2} \sum_{N=0}^{\infty} \frac{\lambda^N (\partial_z^N \phi_{K3}(\tau, z))_{z=0}}{(2T_2 U_2)^{N/2} N!} \sum_{\Gamma^{2,2}} (\tau_2 P_R)^N \, q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2}$$

$$= \int \frac{d^2\tau}{\tau_2} \sum_{\Gamma^{2,2}} \phi_{K3}\left(\tau, \frac{\lambda \tau_2 P_R}{\sqrt{2T_2 U_2}}\right) \, q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2}$$

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Use Fourier-expansion of the elliptic genus

$$\phi_{K3}(\tau, z) = \sum_{n,\ell} c(n,\ell) q^n e^{2\pi i z \ell} \,, \qquad \text{with} \qquad c(n,\ell) = c\left(n - \frac{\ell^2}{4}\right)$$

with the first few coefficients given explicitly

$$c(0) = 20\,, \quad c(1) = 216\,, \quad c(2) = 1616\,, \quad c(3) = 8032$$

$$c(-1/4) = 2\,, \quad c(3/4) = -128\,, \quad c(7/4) = -1026\,, \quad c(11/4) = -5504$$

World-sheet Torus Integral

The world-sheet torus integral for the generating functional can be directly evaluated ($Y = T_2 U_2 - \lambda^2$) Dixon, Kaplunovsky, Louis 1991
Harvey, Moore 1995

$$\mathbb{F}(\lambda, T, U) \sim \ln\left[Y^{10}\left|e^{2\pi i(T+U+i\lambda)}\prod_{r,n',\ell>0}\left(1-e^{2\pi i(rT+n'U+i\ell\lambda)}\right)^{c(n'r,\ell)}\right|\right]$$

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Igusa cusp form of weight 10 of $Sp(4,\mathbb{Z})$.

(Surprising) results

- generating functional is manifestly invariant under the T-duality group
- expansion parameter λ enhances invariance to $Sp(4,\mathbb{Z})$
- correct behaviour in the limit $\lambda \to 0$

Conclusions

In this talk I discussed the elliptic genus of $K3$ from several (complementary) perspectives

- world-sheet/CFT: conjectured action of $\mathbb{M}_{24}$ on the space of states contributing to ϕ_{K3} (Mathieu moonshine)
  - overwhelming 'observational' evidence
  - no satisfactory explanation
- space-time: identified 1/4 BPS saturated one-loop coupling which is related to ϕ_{K3} in type II string theory on $K3 \times T^2$

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Several open questions

- world-sheet/CFT:
  - Why $\mathbb{M}_{24}$? Could there be a larger group, *i.e.* Co_1?
  - Is there an analogue of the 'Monster Module'?
- space-time:
  - Can we make the $\mathbb{M}_{24}$ action visible in target space?
  - How does the dual heterotic side look like?