Fuzzy spaces from tensor models, cyclicity condition, and n-ary algebras

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§ 1 Rank-three tensor models

Dynamical variable : a tensor with three indices

$$C_{abc} \qquad\qquad a,b,c = 1,2,\ldots,N$$

Generalized hermiticity condition

$$C_{abc} = C_{bca} = C_{cab} = C^*_{bac} = C^*_{acb} = C^*_{cba}$$

Symmetry

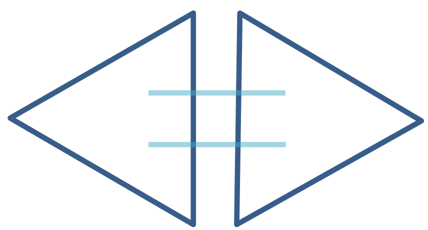
$$C_{abc} \to T_a{}^{a'} T_b{}^{b'} T_c{}^{c'} C_{a'b'c'} \qquad\qquad T_a{}^{a'} \in O(N)$$

(cf. Hermitian matrix models: unitary groups)

Originally considered to describe simplicial quantum gravity with D>2.

Ambjorn-Durhuus-Jonsson, NS, Godfrey-Gross '91

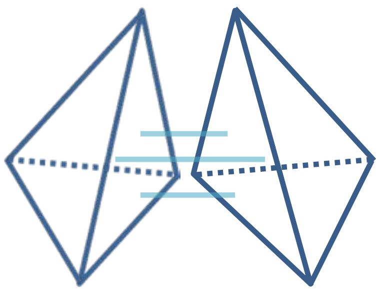
2-dim simplicial quantum gravity



$$M_{ab}$$

Random matrix model

3-dim simplicial quantum gravity



$$C_{abc}$$

Random tensor model
(rank-three)

It seems that the rank-three tensor models can only deal with 3-dim gravity. <u>Not true</u>.

An interpretation of tensor models in terms of fuzzy spaces

Fuzzy space : defined by algebra of functions ϕ_a on it

$$\phi_a \phi_b = f_{ab}{}^c \phi_c$$

dynamical theory of fuzzy spaces $\backsimeq$ rank-three tensor models

$$f_{ab}{}^c \sim C_{abc}$$

This interpretation may be applicable in
- semi-classical limits
- susy

of the tensor models

classical solutions = background fuzzy spaces
perturbations = effective field theories on fuzzy spaces

A bonus : the rank-three tensor models can deal with various dimensional fuzzy spaces, <u>not limited to 3-dim</u>.

§ 2 Fuzzy spaces from tensor models

(bosonic)

Precise correspondence

$$C_{abc} = f_{ab}{}^{d} h_{dc}$$

Reality of ϕ_a

$$\phi_a^* = \phi_a$$

Algebra of ϕ_a

$$\phi_a \phi_b = f_{ab}{}^{c} \phi_c$$

Inner product

$$\langle \phi_a | \, \phi_b \rangle = h_{ab}$$

Real, bi-linear, symmetric $h_{ab} = h_{ba}$

Generalized hermiticity condition of C_{abc} $\quad (C_{abc} = C_{bca} = C^*_{bac} = \cdots)$

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Cyclicity condition

$$\langle \phi_a \phi_b | \phi_c \rangle = \langle \phi_a | \phi_b \phi_c \rangle$$
$$= \langle \phi_b | \phi_c \phi_a \rangle$$

Complex conjugate of product

$$(\phi_a \phi_b)^* = \phi_b \phi_a \qquad (\phi_a^* = \phi_a)$$

By taking $h_{ab} = \delta_{ab}$ (this will be assumed hereafter),

symmetry of tensor models = orthogonal transformations of basis ϕ_a, which conserve the inner product.

§ 3 Fuzzy spaces from tensor models and quantum mechanics

The algebras of functions ϕ_a are *nonassociative* in general. Tractable ?

$$(\phi_a\phi_b)\phi_c \neq \phi_a(\phi_b\phi_c)$$

Thanks to the cyclicity condition, some basic notions in quantum mechanics turn out to be *applicable*.

Basic objects

Observables

$$\mathcal{O} = v^a\phi_a \qquad v^a\text{: real} \qquad (\mathcal{O}^* = \mathcal{O})$$

States

$$|s\rangle = |s^a\phi_a\rangle \qquad s^a\text{: complex}$$

(*complex* functions on *real* manifolds, e.g. e^{ipx})

Applicable notions

- No ambiguity in defining matrix elements

$$\langle \phi_a | \mathcal{O} | \phi_b \rangle \equiv \langle \phi_a \mathcal{O} | \phi_b \rangle = \langle \phi_a | \mathcal{O} \phi_b \rangle$$

- The matrix elements compute the ordered products (time ordering,…)

$$\hat{\mathcal{O}}_{ab} \equiv \langle \phi_a | \mathcal{O} | \phi_b \rangle$$

$$\begin{aligned}\left(\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \cdots \hat{\mathcal{O}}_K\right)_{ab} &= \langle (\cdots(\phi_a \mathcal{O}_1)\mathcal{O}_2)\cdots)\mathcal{O}_K | \phi_b \rangle \\ &= \langle \phi_a | \mathcal{O}_1(\mathcal{O}_2(\cdots(\mathcal{O}_K \phi_b)\cdots)) \rangle \\ &= \langle (\cdots(\phi_a \mathcal{O}_1)\cdots)\mathcal{O}_p | \mathcal{O}_{p+1}(\cdots(\mathcal{O}_K \phi_b)\cdots)) \rangle\end{aligned}$$

- Mean value of an observable is real

$$\langle \mathcal{O} \rangle^* = \langle \mathcal{O} \rangle$$

$$\because \langle s^* | \mathcal{O} | s \rangle^* = \langle s^* | \mathcal{O} s \rangle^* = \langle s^* \mathcal{O} | s \rangle = \langle s^* | \mathcal{O} | s \rangle$$

- An observable can be diagonalized with orthogonal states.

$$\mathcal{O}|s_i\rangle \equiv |\mathcal{O}s_i\rangle = e_i|s_i\rangle \qquad \langle s_i^*|s_j\rangle = \delta_{ij}$$

($\because$ $\langle\phi_a|\mathcal{O}|\phi_b\rangle$: hermitian)

- Uncertainty relation

$$\triangle\mathcal{O}_1\ \triangle\mathcal{O}_2 \geq \frac{1}{2}\left|\langle s^*|[\mathcal{O}_1,\mathcal{O}_2;s]\rangle\right|$$

$$[\mathcal{O}_1,\mathcal{O}_2;s] \equiv \underline{\mathcal{O}_1(\mathcal{O}_2 s)-\mathcal{O}_2(\mathcal{O}_1 s)}$$

Noncommutativity of ordered product

Contributions *both from noncommutativity and nonassociativity*

Uncertainty exists even when commutative but nonassociative.

$$(\because \left||(\mathcal{O}_1-\langle\mathcal{O}_1\rangle+i\lambda(\mathcal{O}_2-\langle\mathcal{O}_2\rangle))s\rangle\right|^2 \geq 0)$$

- Condition for full diagonalization of a fuzzy space algebra

$$\text{If and only if} \quad \left\{ \begin{array}{l} [\phi_a, \phi_b] = \phi_a \phi_b - \phi_b \phi_a = 0 \\ \text{and} \\ [\phi_a, \phi_b; \phi_c] = \phi_a(\phi_b \phi_c) - \phi_a(\phi_b \phi_c) = 0 \end{array} \right.$$

then

$$\left\{ \begin{array}{l} \phi'_a = M_a{}^b \phi_b \\ \phi'_a \phi'_b = e_a \delta_{ab} \phi'_a \end{array} \right. \qquad {}^\exists M_a{}^b \in O(N)$$

If and only if algebra is commutative and associative,
fuzzy space becomes just a collection of *independent* “points”.

§ 4 Truncation of algebras of functions

General procedure

$$A \equiv \mathrm{span}_R\{\phi_a | a = 1, 2, \ldots, N\}$$

Consider a subspace $\tilde{A} \subset A$

$$\tilde{A} = \mathrm{span}_R\{\tilde{\phi}_a | a = 1, 2, \ldots, M\} \qquad M < N$$

and define

$$\tilde{C}_{abc} \equiv \langle \tilde{\phi}_a \tilde{\phi}_b | \tilde{\phi}_c \rangle \qquad a, b, c = 1, 2, \ldots, M$$
$$\tilde{h}_{ab} \equiv \langle \tilde{\phi}_a | \tilde{\phi}_b \rangle$$

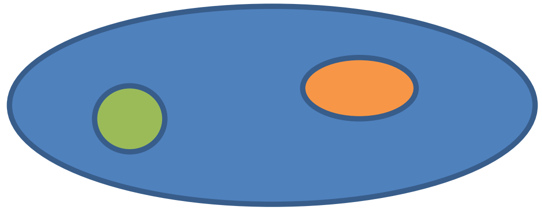
then the algebra defined by

$$\tilde{\phi}_a \tilde{\phi}_b = \tilde{f}_{ab}{}^c \tilde{\phi}_c \qquad (\tilde{f}_{ab}{}^c \equiv \tilde{C}_{abd} \tilde{h}^{dc}, \ \tilde{h}^{ab} \tilde{h}_{bc} = \delta^a_c)$$

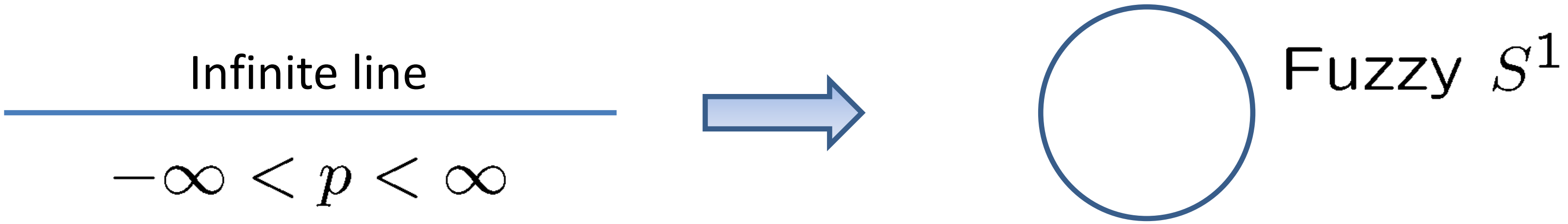
satisfies the cyclicity condition (and others).

This general procedure gives systematically

(i) Fuzzy subspaces

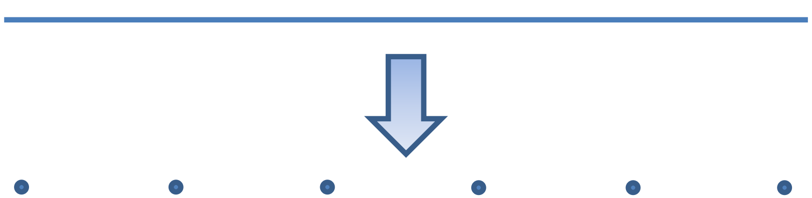


(ii) Compactification



$$p = 0, \pm 1, \pm 2, \ldots, \pm \Lambda$$

(iii) Lattice theory

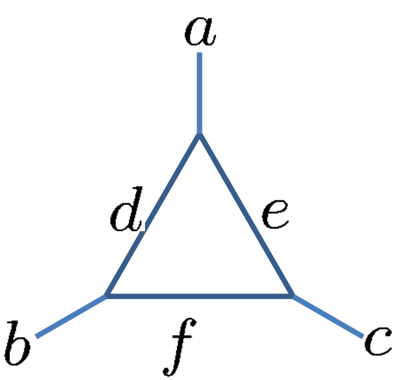


$$-\infty < x < \infty$$

$$x = 0, \pm 1, \pm 2, \ldots, \pm L$$

(iv) Coarse graining

$$\overline{C}_{abc} \equiv C_{ade} C_{bfd'} C_{ce'f'} h^{dd'} h^{ee'} h^{ff'}$$



Define an averaged algebra $\bar{A}$ by

$$\overline{\phi}_a \overline{\phi}_b = \overline{f}_{ab}{}^c \overline{\phi}_c \qquad \overline{f}_{ab}{}^c \equiv \overline{C}_{abd} h^{dc}$$

Then construct $\tilde{A} \subset \bar{A}$, which may be obtained by discarding "unimportant" elements.

Unimportant elements have relatively smaller lengths with respect to the hermitian metric

$$H_{ab} \equiv h^{cd} \langle \overline{\phi}_c \overline{\phi}_a | \overline{\phi}_b \overline{\phi}_d \rangle$$

§ 5 N-ary transformations as symmetry

Symmetry of tensor models

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Linear transformations of ϕ_a which conserve the inner product
(and are real) $\langle \phi_a | \phi_b \rangle = h_{ab} \ (= \delta_{ab})$

The cyclicity condition enables systematic constructions of such linear transformations in terms of n-ary transformations.

$$\delta\phi_b = [\phi_{a_1}, \ldots, \phi_{a_{n-1}}; \phi_b]$$

cf. Nambu '73: extension of Hamilton dynamics
Bagger-Lambert, Gustavsson

General procedure of construction

$(\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}, s; \phi_b) \equiv$ A product of $\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}, \phi_b$ in way s

Obtain the transpose by using the cyclicity condition,

$$\langle (\phi_{a_1}, \ldots, \phi_{a_n}, s; \phi_b) | \phi_c \rangle = \langle \phi_b | (\phi_{a_1}, \ldots, \phi_{a_n}, \overline{s}; \phi_c) \rangle$$

in terms of a product of $\phi_{a_1}, \ldots, \phi_{a_n}, \phi_c$.

Then

$$[\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}, s; \phi_b]$$
$$\equiv (\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}, s; \phi_b) - (\phi_{a_1}, \phi_{a_2}, \ldots, \phi_{a_n}, \overline{s}; \phi_b)$$

satisfies

$$\langle [\phi_{a_1}, \ldots, \phi_{a_n}, s; \phi_b] | \phi_c \rangle = -\langle \phi_b | [\phi_{a_1}, \ldots, \phi_{a_n}, s; \phi_c] \rangle$$

$\therefore \delta_{a_1 \ldots a_n} \phi_b \equiv [\phi_{a_1}, \ldots, \phi_{a_n}, s; \phi_b]$ conserves the inner product.

(Reality condition can also easily be satisfied.)

Examples

(i) $(\phi_a, s; \phi_b) = \phi_a \phi_b$

Then

$$\langle (\phi_a, s; \phi_b) | \phi_c \rangle = \langle \phi_a \phi_b | \phi_c \rangle = \langle \phi_b | \phi_c \phi_a \rangle$$

$$\therefore (\phi_a, \overline{s}; \phi_b) = \phi_b \phi_a$$

$$\therefore [\phi_a, s; \phi_b] = \phi_a \phi_b - \phi_b \phi_a$$

(ii) $(\phi_a, \phi_b, s; \phi_c) = \phi_a(\phi_b \phi_c)$

$$\langle (\phi_a, \phi_b, s; \phi_c) | \phi_d \rangle = \langle \phi_a(\phi_b \phi_c) | \phi_d \rangle = \langle \phi_c | (\phi_d \phi_a) \phi_b \rangle$$

$$\therefore [\phi_a, \phi_b, s; \phi_c] = \phi_a(\phi_b \phi_c) - (\phi_c \phi_a) \phi_b$$

Comments

- the cyclicity condition is essentially used.

- any higher n-ary transformation conserving the inner product.

- the Lie algebraic structure of the symmetry of tensor models is incorporated in the hierarchical structure of the n-ary algebras.

- Symmetries of tensor models
  unbroken or spontaneously broken
  by a classical solution representing a background fuzzy space

  (i) Unbroken symmetries: the n-ary transformations respect Leibnitz rule & the fundamental identity, and form a Lie n-algebra.

  (ii) Broken symmetries: the n-ary transformations may represent local gauge symmetries such as diffeomorphism.

c.f. Borisov-Ogievetsky '74 , Ferrari-Picasso '71, Brandt-Ng '74

Example

Fuzzy *D*-dim flat space, obtained by deformation with Gaussian

$$\phi_{p^1}\phi_{p^2} = e^{-\alpha((p^1)^2+(p^2)^2+(p^1+p^2)^2)}\phi_{p^1+p^2} \qquad (\phi_p \sim e^{ipx})$$

$p^i = (p^i_1, p^i_2, \ldots, p^i_D)$: D-dim momentum

$(p)^2 = g^{\mu\nu}p_\mu p_\nu$ $\quad g^{\mu\nu}$: constant symmetric tensor

α : deformation parameter ($\alpha \to 0$: usual space)

(Cf. Moyal space $\phi_{p^1}\phi_{p^2} = e^{i\theta^{\mu\nu}p^1_\mu p^2_\nu}\phi_{p^1+p^2}$)

- nonassociative (commutative)
- the cyclicity condition satisfied. $\langle\phi_{p^1}|\phi_{p^2}\rangle = \delta^D(p^1+p^2)$
- *D*-dim Poincare symmetry respected.
- truncation to a finite system may be taken appropriately.
- they are fixed configurations of coarse-graining processes
- they can be obtained as classical solutions of some tensor models, but the actions take complicated/unnatural forms.

The coordinates of the flat fuzzy space form a 3-Lie algebra

$$x^\mu \equiv \frac{1}{i}\frac{\partial \phi_p}{\partial p_\mu}\bigg|_{p=0} \qquad \left(x^\mu \sim \frac{1}{i}\frac{\partial e^{ipx}}{\partial p_\mu}\bigg|_{p=0} \right)$$

satisfies

$$[x^\mu, x^\nu; x^\rho] = 4\alpha(g^{\mu\rho}x^\nu - g^{\nu\rho}x^\mu)$$ Rotations

$$[x^\mu, \phi_0; x^\nu] = 4\alpha g^{\mu\nu}\phi_0$$ Translations

$$([\phi_a, \phi_b; \phi_c] \equiv (\phi_a\phi_c)\phi_b - \phi_a(\phi_c\phi_b), \;\; \phi_0 = \phi_p|_{p=0})$$

D-dim Poincare symmetry of the fuzzy flat space is represented by 3-ary transformations, which form a Lie 3-algebra of Poincare symmetry.

A usage of spontaneously broken transformations

Gaussian-deformed fuzzy flat space in *x*-representation

$$\phi_x \equiv \int d^D p \; e^{ipx} \phi_p$$

$$\phi_{x_1}\phi_{x_2} = \int d^D x_3 \, e^{-\beta((x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2)} \phi_{x_3}$$

Diffeomorphism is represented by a 3-ary transformation.

$$\Phi = v^a \phi_a = \int d^D x \;\, v(x)\, \phi_x$$

$$\begin{aligned}\delta\Phi &= \int d^D x \;\, [\phi_x, \phi_{x+\varepsilon(x)}; \Phi] \\ &= \int d^D x \left(\varepsilon^\mu(x)\partial_\mu v(x) + \frac{1}{2}(\partial_\mu \varepsilon^\mu(x)) v(x) + \cdots \right) \phi_x\end{aligned}$$

Scalar field $v(x)$ transforms as a scalar “half” density.

$C_{x_1 x_2 x_3}$ transforms as well : metric changes by $\delta g_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$

§ 6 Scalar field action

$$S = -\langle \Phi^* \phi^a | \phi_a \Phi \rangle + m_0^2 \langle \Phi^* | \Phi \rangle$$
$$= -\langle \Phi^* | \phi^a \phi_a \Phi \rangle + m_0^2 \langle \Phi^* | \Phi \rangle$$

Kinetic term

$\Phi = v^a \phi_a$ $\quad$ v^a : complex

takes invariant form under the symmetry of tensor models
(described background independently)

(i) Unbroken symmetry
The action automatically respects the symmetry of a fuzzy space.
e.g. Poincare symmetry, spherical symmetry, SUSY

(ii) Invariant under spontaneously broken symmetry
e.g. Diffeomorphism invariance automatically incorporated.
The action should also be applicable to curved fuzzy space.

Examples

(i) Gaussian-deformed fuzzy flat space

$$K(p) = m_0^2 - c_0 \exp(-3\alpha p^2) = (m_0^2 - c_0) + 3\alpha c_0\, p^2 + \cdots$$

(ii) Fuzzy 2-dim sphere (commutative nonassociative)

$$C_{j_1 m_1\ j_2 m_2\ j_3 m_3} = \left[\frac{\prod_{i=1}^{3}(2j_i+1)D(j_i)}{4\pi}\right]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$K(j) = c_0 + c_1\, \underline{j(j+1)} + \cdots$$

Laplacian on two sphere correctly appears.

Summary

Studied the fuzzy spaces generated from the rank-three tensor models

- Algebras are noncommutative and nonassociative in general
- the cyclicity condition

Seem tractable and physically interesting :

- some basic notions in quantum mechanics applicable
- Truncation straightforward

  fuzzy subspace, compactification, lattice theory, coarse graining
- n-ary transformations as symmetry
- fuzzy spaces rather freely constructible e.g. D-dim fuzzy flat space
- background independent description of scalar field theory
- SUSY straightforward

Outlook

Gauge fields

Fuzzy compact dimensions

A "practical" application

D-dim fuzzy SUSY flat space $\xrightarrow[\text{truncation}]{}$ SUSY lattice theories

Resembles string/M theory ?

A natural extension : $\phi(x), g_{\mu\nu}(x), B_{\mu\nu}(x)$

$$C_{p^1p^2p^3} = \phi\, e^{-\alpha((p_1)^2+(p^2)^2+(p^3)^2)+B^{\mu\nu}p^1_\mu p^2_\nu}\delta^D(p^1+p^2+p^3)$$

SUSY   R-R fields,…

N-ary algebras : M-theory