Minimal length in quantum space and integrations of the line element in noncommutative geometry

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Spectral distance in the Moyal plane,

E. Cagnache, F. D'Andrea, P.M., J.-C. Wallet; J. Geo. Phys. **61** (2011).

Minimal length in quantum space and integrations of the line element in ncg,

F. Mercati, P.M., L. Tomassini; arXiv:1106.0261.

Translation isometries in the Moyal plane: spectral distance between coherent states,

P.M., L. Tomassini; arXiv:1109.XXXX.

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Introduction: how to define distance on a "quantum space" ?

- Length operator in Doplicher-Fredenhagen-Roberts [DFR] model,

$$[q_\mu, q_\nu] = i\theta_{\mu\nu}\mathbb{I} \quad \Longrightarrow L \doteq \sqrt{\sum_{\mu=1}^{2N} dq_\mu^2} \quad \text{with} \quad dq_\mu = q_\mu \otimes \mathbb{I} - \mathbb{I} \otimes q_\mu,$$

  q_μ acts on $\mathcal{H}$. The set of outputs of a distance measurement is $\mathrm{Sp}(L)$.
  A couple of "quantum points" is a 2-point state vector $\phi \in \mathcal{H} \otimes \mathcal{H}$.
  The associated mean value of the distance is $\langle \phi, L\phi \rangle$.

- Spectral triple $(\mathcal{A} \xrightarrow{\pi} \mathcal{H}, D)$. Connes' *spectral (pseudo)-distance* between states ω, ω' of $\mathcal{A}$ (i.e normalized, positive linear applications $\mathcal{A} \to \mathbb{C}$)

$$d_D(\omega, \omega') \doteq \sup_{a \in \mathcal{A}} \{ |\omega(a) - \omega'(a)|, \ \|[D, \pi(a)]\| \leq 1 \}.$$

Consider a separable (i.e. non-entangled) 2-point state vector $\phi = \psi_1 \otimes \psi_2$, and the associated vector states $\omega_{\psi_i}(a) = \langle \psi_i, \pi(a)\psi_i \rangle$. Define the *quantum length* as

$$d_L(\omega_{\psi_1}, \omega_{\psi_2}) \doteq \langle \psi_1 \otimes \psi_2, L(\psi_1 \otimes \psi_2) \rangle.$$

Question: to what extent does the spectral distance and the quantum length measure the same thing: $d_L(\omega_{\psi_1}, \omega_{\psi_2}) = d_D(\omega_{\psi_1}, \omega_{\psi_2})$?

I. Commutative case

II. The Moyal plane as a common framework

- quantum length in the DFR model
- spectral distance in the Moyal plane

Obvious discrepancies: does the quantization of the coordinates necessarily imply a minimal length ?

III. Spectral triple doubling: $\mathcal{A} \to \mathcal{A} \otimes \mathbb{C}^2$

- coherent states
- stationary states

IV. Integrations of the line element in noncommutative geometry

- geodesic equation in the Moyal plane

I. Commutative case

Comparing Connes spectral distance d_D to the quantum length d_L is idiotic:

$d_D(\psi, \psi) = 0$ while $d_L(\psi, \psi) = \langle \psi \otimes \psi, L(\psi \otimes \psi) \rangle$ has no reason to vanish !

Nevertheless, the comparison makes sense, for $d_L = d_D$ in the commutative case:

- The coordinate operator q_μ acts on $\psi \in L^2(\mathbb{R}^{2N})$ as $(q_\mu \psi)(x) = x_\mu \psi(x)$.

  The universal differential dq_μ acts on $\psi_1 \otimes \psi_2$ as

  $$(dq_\mu(\psi_1 \otimes \psi_2))(x, y) = (x_\mu - y_\mu)\,((\psi_1 \otimes \psi_2))(x, y).$$

  The length operator L acts as multiplication by $d_{\text{geo}}(x, y) = \sqrt{\sum_\mu (x_\mu^2 - y_\mu^2)}$.

  $$d_L(\delta_x, \delta_y) = d_{\text{geo}}(x, y)$$

- Let $\mathcal{M}$ be a Riemannian spin manifold, and

$$\mathcal{A} = C_0^\infty(\mathcal{M}),\ \mathcal{H} = L_2(\mathcal{M}, S),\ D = \partial\!\!\!/.$$

Then the spectral distance d_D coincides with the Wassertein distance of optimal transport theory. In particular on pure states, that is evaluations at points: $\delta_x(f) = f(x)$, one has

$$d_D(\delta_x, \delta_y) = d_{\text{geo}}(x, y) = d_L(\delta_x, \delta_y).$$

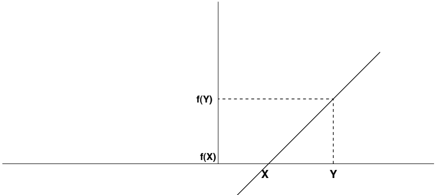
Real line:

$$\text{representation} : (\pi(f)\psi)(x) = f(x)\,\psi(x), \quad f \in C_0^\infty(\mathbb{R}),\ \psi \in L^2(\mathbb{R}),$$

$$\text{norm} : [D, \pi(f)]\psi = [\frac{d}{dx}, f]\psi = \left(\frac{df}{dx}\right)\psi \Longrightarrow \|[D, f]\| = \sup_{x\in\mathbb{R}}|f'(x)|,$$

$$\text{pure states} : \delta_x(f) = f(x),$$

$$\text{spectral distance} : \sup_{f\in C_0^\infty(\mathbb{R})}\{|f(x) - f(y)| \,/\, \|f'\| \leq 1\} = |x - y|.$$



On the Euclidean space, the quantum length d_L and the spectral distance d_D are two equivalent ways of "algebrizing" the usual notion of distance.

How much do they differ in the noncommutative case ?

II. The Moyal plane as a common framework

DFR model

The q_μ's are affiliated to the algebra of compact operators $\mathbb{K}$. Furthermore, there is an action of the Poincaré group which we do not take into account here, meaning we fix once for all the symplectic form (in dimension 2),

$$\Theta = \{\theta_{\mu\nu}\} = \lambda_P \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with } \lambda_P \text{ the Planck length.}$$

Spectral triple for Moyal plane

$$\mathcal{A} = (\mathcal{S}(\mathbb{R}^2), \star), \quad \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2, \quad D = -i \sum_{\mu=1}^{2} \sigma^\mu \partial_\mu$$

where $$(f \star g)(x) = \frac{1}{(\pi\theta)^2} \int d^2s \, d^2t \; f(x+s) g(x+t) e^{-i2s\Theta^{-1}t}.$$

The left regular representation of $f \in \mathcal{A}$ on $\mathcal{H}$ is

$$\pi(f) = L(f) \otimes \mathbb{I}_2 : \; \pi(f)\psi = \begin{pmatrix} f \star \psi_1 \\ f \star \psi_2 \end{pmatrix}.$$

$$D = -i\sqrt{2} \begin{pmatrix} 0 & \bar{\partial} \\ \partial & 0 \end{pmatrix} \quad \text{with} \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \; \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2).$$

- $\bar{\mathcal{A}} = \mathbb{K}$: common framework to compare d_L and d_D.

Pure states

The evaluation at x is not a state of $\bar{\mathcal{A}}$ for $(f^* \star f)(x)$ may not be positive. Keeping in mind the analogy *pure states = points*, let's take the pure states of $\bar{\mathcal{A}}$ as "quantum (or fuzzy) points". By a well known result of operator algebra: the pure states of $\bar{\mathcal{A}} = \mathbb{K}$ are the vector states in an *irreducible* representation.

The left regular representation $\mathcal{L}$ is not irreducible, it is a multiple of the Schrödinger representation π_S. Intertwiner:

$$W : h_{mn} \to h_m \otimes h_n \qquad m, n \in \mathbb{N},$$

where the h_{mn}'s are Wigner transition functions (orthonormal basis of $L^2(\mathbb{R}^2)$), h_m's are the eigenfunctions of the quantum h.o. (orthonormal basis of $L^2(\mathbb{R})$).

$$W\mathcal{L}(f)W^* = \pi_S(f) \otimes \mathbb{I} \Longrightarrow \begin{cases} W\mathcal{L}(x_1)W^* = X \otimes \mathbb{I} & W\mathcal{L}(z)W^* = \mathfrak{a}^* \otimes \mathbb{I} \\ W\mathcal{L}(x_2)W^* = P \otimes \mathbb{I} & W\mathcal{L}(\bar{z})W^* = \mathfrak{a} \otimes \mathbb{I} \end{cases}$$

with $X, P, \mathfrak{a}, \mathfrak{a}^*$ the position, momentum, creation and annihilation operators.

- The set of pure states of $\overline{\mathcal{A}}$ is thus the set of vector states,

$$\omega_\psi(f) \doteq \langle \psi, \pi_S(f)\psi \rangle,$$

  where $\psi = \sum_m \psi_m h_m$ is a unit vector in $L^2(\mathbb{R})$.

Proposition

E. Cagnache, F. D'Andrea, P.M., J.C. Wallet (J. Geo. Phys. 2009)

1. The spectral distance on the Moyal plane is not bounded, neither above nor below from zero. Furthermore it is not quantized: for any $r \in \mathbb{R}^+ \cup \{\infty\}$ there exist pure states ω, ω' such that

$$d_D(\omega, \omega') = r.$$

2. The Moyal plane $(\mathcal{A}, \mathcal{H}, D)$ is *not* a spectral metric space in the sense of Rieffel, i.e. the metric topology on the state space is *not* the weak-$*$ topology.

3. Restricting to stationary states, that is eigenstates of the quantum harmonic oscillator,

$$\omega_m(f) \doteq \langle h_m, \pi_S(f) h_m \rangle,$$

the spectral distance is quantized,

$$d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

The stationary states form a 1-dimensional lattice within the space of pure states of the Moyal algebra $\mathcal{A}$.

Proposition

D. Bahns, S. Doplicher, K. Fredenhagen, G. Piacitelli (2010) & F. Mercati, P.M, L. Tomassini (2011)

1. The quantum length is discrete and bounded above from zero,

$$\mathrm{Sp}(L) = \{\ 2\lambda_P\sqrt{j+\frac{1}{2}}, \quad j \in \mathbb{N}\ \}.$$

2. The ground state, with eigenvalue $2\lambda_P^2$, of

$$L^2 = 2(H \otimes \mathbb{I} + \mathbb{I} \otimes H - \mathfrak{a} \otimes \mathfrak{a}^* - \mathfrak{a}^* \otimes \mathfrak{a}),$$

is infinitely degenerate. There is only one *separable* ground state $h_0 \otimes h_0$.

3. Caution when to take the square root:

$$d_L(\omega_m, \omega_n) \leq \lambda_P\sqrt{2(m+n+1)} = \sqrt{d_{L^2}(\omega_m, \omega_n)},$$

with equality only for $m = n = 0$,

$$d_L(\omega_0, \omega_0) = \sqrt{2}\lambda_P = \sqrt{d_{L^2}(\omega_0, \omega_0)}.$$

Even restricting to stationary states, there are obvious discrepancies.

Spectral distance: no minimal length,

$$d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

Quantum length: minimal length,

$$\sqrt{2}\lambda_P \leq d_L(\omega_m, \omega_n) \leq \sqrt{d_{L^2}(\omega_m, \omega_n)} = \lambda_P \sqrt{2(m+n+1)}.$$

The spectral distance is “fermionic”,

$$\psi \otimes \psi \to d_D(\psi, \psi) = 0.$$

The quantum length is“bosonic”,

$$d_L(\psi, \psi) = \langle \psi \otimes \psi, L\, \psi \otimes \psi \rangle \neq 0.$$

- We need to “bosonify” the spectral distance, that is allowing the emergence of a minimal length

$$d_{D_b}(\omega_\psi, \omega_\psi) \geq \sqrt{2}\lambda_P,$$

  and/or to we need to “fermionify” the quantum length, that is to turn it into a true distance,

$$d_{L_f}(\omega_\psi, \omega_\psi) = 0.$$

  This is achieved thanks to a standard procedure in noncommutative geometry, consisting in doubling the spectral triple.

III. Spectral triple doubling

$$\mathcal{A}' \doteq \mathcal{A} \otimes \mathbb{C}^2,\ \mathcal{H}' \doteq \mathcal{H} \otimes \mathbb{C}^2,\ D_b \doteq \partial\!\!\!/ \otimes \mathbb{I} + \gamma^5 \otimes \begin{pmatrix} 0 & \Lambda_b^{-1} \\ \Lambda_b^{-1} & 0 \end{pmatrix},\ \Lambda_b = \text{ const.}$$

Pure states: $\omega_i \doteq \omega \otimes \delta_i$, with $\delta_i, i = 1, 2$, are the pure states of $\mathbb{C}^2$.

Pythagoras equality: $d_{D_b}(\omega_1, \omega'_2) = \sqrt{d_D^2(\omega, \omega') + \Lambda_b^2}$ (in progress with F. D'Andrea)

Proposition

P.M., L. Tomassini (2011)

Assuming Pythagoras, the quantum square-length identifies to the spectral distance in the doubled Moyal space,

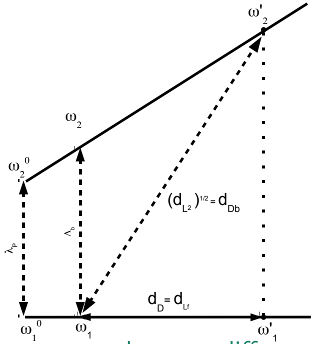
$$\sqrt{d_{L^2}(\omega, \omega')} = d_{D_b}(\omega_1, \omega'_2)$$

with $\Lambda_b = \min(d_L(\omega, \omega), d_L(\omega', \omega'))$, iff

$$d_D(\omega, \omega') = d_{L_f}(\omega, \omega') \qquad (1)$$

where

$$d_{L_f}(\omega, \omega') \doteq \sqrt{d_{L^2}(\omega, \omega') - \Lambda_b^2}.$$



(1) captures the true difference between the spectral distance and the quantum length, once solved the obvious discrepancies.

Condition (1) does not hold for stationary states since, assuming $m \leq n$,

$$d_{L_f}(\omega_m, \omega_n) = \lambda_P \sqrt{2(n-m)} \quad \text{while} \quad d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

But it holds true for coherent states, that is the quantum states such that the mean values of the quantum observables $X(t), P(t)$ and H take the value of the position, momentum and energy of a classical oscillator.

Coherent states:

Coherent states are translations of the ground state ω_0 of the quantum harmonic oscillator,

$$\omega_\kappa(f) \doteq \omega_0 \circ \alpha_{\lambda_p \sqrt{2}\kappa}(f)$$

where, for any $\kappa \in \mathbb{C} \simeq \mathbb{R}^2$,

$$(\alpha_\kappa f)(x) = f(x+\kappa).$$

ω_κ mimics a classical oscillator, with amplitude $|\kappa|$, and phase $\arg(\kappa)$.

ω_κ is a pure state of the Moyal algebra $\mathcal{A}$, since it is a vector state,

$$\omega_\kappa = \omega_\psi \quad \text{for} \quad \psi = \sum_{m\in\mathbb{N}} e^{-\frac{|\kappa|^2}{2}} \frac{\kappa^m}{\sqrt{m!}} h_m \in L^2(\mathbb{R}).$$

The Dirac operator commutes with translation so the spectral distance is invariant by translation

$$d_D(\omega \circ \alpha_\kappa, \omega' \circ \alpha_\kappa) = d_D(\omega, \omega').$$

Theorem

P.M., L. Tomassini (2011)

For any states ω,

$$d_D(\omega, \omega \circ \alpha_\kappa) = |\kappa|.$$

Therefore, considering the ground state ω_0 and any coherent state ω_κ, condition (1) between the spectral distance and the fermionified quantum length is satisfied:

$$d_D(\omega_0, \omega_\kappa) = \lambda_P \sqrt{2} |\kappa| = d_{L_f}(\omega_0, \omega_\kappa).$$

- Coherent states are good candidates as "quantum points", not only from DFR optimal localisation perspective, but also from Connes distance formula.

IV. Integrations of the line element in NCG

Condition (1) does not hold for stationary states since, assuming $m \leq n$,

$$d_{L_f}(\omega_m, \omega_n) = \lambda_P \sqrt{2(n-m)},$$

while

$$d_D(\omega_m, \omega_n) = \frac{\lambda_P}{\sqrt{2}} \sum_{k=m+1}^{n} \frac{1}{\sqrt{k}}.$$

The same line element $\frac{\lambda_P}{\sqrt{2x}} dx$ is integrated along a continuous geodesic (quantum length), or along a discrete geodesic (spectral distance),

$$d_{L_f}(\omega_0, \omega_n) = \int_0^n \frac{\lambda_P}{\sqrt{2x}}\, dx, \qquad d_D(\omega_0, \omega_n) = \sum_{k=0}^{m} \frac{\lambda_P}{\sqrt{2k}}.$$

- Both the spectral distance and the quantum length quantize the coordinates, hence the line element. The spectral distance also quantizes the geodesic.

- The difference vanishes at high energy: for fixed m,

$$\lim_{n\to\infty} \frac{d_D(\omega_m, \omega_n) - d_{L_f}(\omega_m, \omega_n)}{d_{L_f}(\omega_m, \omega_n)} = 0.$$

Let us call optimal element the element of the algebra that attains the supremum in the spectral distance formula.

On the Euclidean plane, the geodesic distance function $l(x_\mu) \doteq \sqrt{x_1^2 + x_2^2}$ yields both the length operator $L = l(dq_\mu)$ and - up to a regularization at infinity - the optimal element $l(q_\mu)$.

The quantum length supposes that the function l is known a priori: quantization of the geometry. The spectral distance formula is an equation whose solution is the function l: geometrization of the quantum (i.e. starting from algebraic objects and build a distance).

- Two distinct points of view, which coincide on the Euclidean plane because the length operator is the optimal element.
- This is no longer true in the Moyal case.

Geodesic equation in the Moyal plane

Writing $d\mathfrak{a} \doteq \mathfrak{a} \otimes \mathbb{I} - \mathbb{I} \otimes \mathfrak{a}$, with $\mathfrak{a} = \pi_S(z)$, $z \doteq \frac{x+iy}{\sqrt{2}}$, one obtains the length operator as $L = l_i(d\mathfrak{a})$ with

$$l_1(z) \doteq \sqrt{2(z\bar{z} - \lambda_P^2)} \text{ or } l_2(z) \doteq \sqrt{z\bar{z} + \bar{z}z} \text{ or } l_3(z) \doteq \sqrt{2(\bar{z}z + \lambda_P^2)}.$$

The optimal element is - up to regularization - $l_0(a)$ where l_0 is solution of

$$(\partial_z l_0 \star z) \star (\partial_z l_0 \star z)^* = \frac{1}{2}\bar{z} \star z. \tag{2}$$

- $l_i, i = 1, 2, 3$, are not solution of (2).
- For the Moyal plane, eq.(2) plays the role of the equation of the geodesic between stationary states. Notice that in the commutative limit (2) gives

  $$|\partial_z l_0|^2 = \frac{1}{2},$$

  which is satisfied by $l_0(z) = \sqrt{2}|z|$.

Conclusion

- The quantum length d_L in the DFR model and Connes spectral distance d_D are two ways to treat the metric aspect of a quantum space: points are "talking to each other" either through the interacting part

$$H_{\text{int}} \doteq -\mathfrak{a} \otimes \mathfrak{a}^* - \mathfrak{a}^* \otimes \mathfrak{a}$$

  of the square L^2 of the length operator, or through the Dirac operator.
- Both d_L and d_D coincide with the geodesic distance in the commutative case.
- In the noncommutative case, assuming some Pythagoras equalities, d_L and d_D can be compared after a doubling of the spectral triple. This gives a quantum taste to the spectral distance, and also allows to turn the quantum length into a true distance.
- The quantum length and the spectral distance coincide between the ground state and any coherent states.
- On stationary states, the quantum length and the spectral distance no longer coincide. The difference can be interpreted as two different ways of integrating the same quantum line element.