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T-DUALITY OF EXCITED STRING STATES

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LET US RECAPITULATE SOME OF THE KNOWN RESULTS ON T-DUALITY FOR A CLOSED BOSONIC STRING IN PRESENCE OF CONSTANT BACKGROUNDS- $\mathbf{G}^{(\mathbf{0})}_{\hat\mu\hat\nu}$ AND $\mathbf{B}^{(\mathbf{0})}_{\hat\mu\hat\nu}$

$$\mathbf{S} = \frac{\mathbf{1}}{\mathbf{2}}\int \mathbf{d}\sigma\mathbf{d}\tau\Big(\mathbf{G}^{(\mathbf{0})}_{\hat\mu\hat\nu}\partial_{\mathbf{a}}\mathbf{X}^{\hat\mu}\partial^{\mathbf{a}}\mathbf{X}^{\hat\nu} + \epsilon^{\mathbf{ab}}\mathbf{B}^{(\mathbf{0})}_{\hat\mu\hat\nu}\partial_{\mathbf{a}}\mathbf{X}^{\hat\mu}\partial_{\mathbf{b}}\mathbf{X}^{\hat\nu}\Big)$$

IS THE ACTION AND THE CORRESPONDING HAMILTONIAN DENSITY IS

$$\mathbf{H_c} = \mathbf{Z^T}\mathcal{M}_{\mathbf{0}}(\mathbf{G^{(0)}}, \mathbf{B^{(0)}})\mathbf{Z}$$

WHERE

$$\mathbf{Z} = \begin{pmatrix} P \\ X' \end{pmatrix}, \quad \mathcal{M}_{\mathbf{0}} = \begin{pmatrix} {G^{(0)}}^{-1} & -{G^{(0)}}^{-1}B^{(0)} \\ B^{(0)}{G^{(0)}}^{-1} & {G^{(0)}}^{-1} - B^{(0)}{G^{(0)}}^{-1}B^{(0)} \end{pmatrix}$$

UNDER THE INTERCHANGE $\mathbf{P} \leftrightarrow \mathbf{X'}$ THE HAMILTONIAN REMAINS INVARIANT IF $\mathcal{M}_{\mathbf{0}} \leftrightarrow \mathcal{M}_{\mathbf{0}}^{-\mathbf{1}}$.

THE HAMILTONIAN DENSITY IS ALSO INVARIANT UNDER THE GLOBAL
$\mathbf{O(\hat D, \hat D)}$ TRANSFORMATION

$$\mathbf{Z} \rightarrow \mathbf{\Omega_0 Z}, \quad \mathcal{M}_{\mathbf{0}} \rightarrow \mathbf{\Omega_0}\mathcal{M}_{\mathbf{0}}\mathbf{\Omega_0^T}, \ \eta_{\mathbf{0}} \rightarrow \eta_{\mathbf{0}}, \quad \mathbf{\Omega_0} \in \mathbf{O(\hat D, \hat D)}$$

WHERE η_0 IS THE $\mathbf{O(\hat D, \hat D)}$ METRIC.

$$\eta_{\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1 IS $\mathbf{\hat D \times \hat D}$ UNIT MATRIX. Z IS $\mathbf{2\hat D}$ DIMENSIONAL VECTOR AND $\mathcal{M}_{\mathbf{0}}$ IS $\mathbf{2\hat D \times 2\hat D}$ SYMMETRIC MATRIX.

IN GENERAL THE BACKGROUNDS MAY DEPEND ON THE SPACETIME COORDINATES $\mathbf{X}^{\hat{\mu}}$. THE WORLDSHEET ACTION IS A σ-MODEL ACTION. THESE BACKGROUNDS SATISFY β-FUNCTION EQUATIONS - THE EQUATIONS OF MOTION.

CONSIDER A SCENARIO WHERE THE BACKGROUNDS DEPEND ONLY ON SOME OF THE SPACE TIME COORDINATES.

$$\mathbf{X}^{\hat{\mu}} = \{\mathbf{X}^{\mu}, \mathbf{Y}^{\alpha}\}, \ \mu = \mathbf{0}, \mathbf{1}, \mathbf{2}..\mathbf{D} - \mathbf{1}, \alpha = \mathbf{D}, .\hat{\mathbf{D}} - \mathbf{1}$$

WITH $\hat{\mathbf{D}} = \mathbf{D} + \mathbf{d}$.

BACKGROUNDS ARE DECOMPOSED AS FOLLOWS (HASAN-SEN):

$$\mathbf{G}^{(\mathbf{0})}_{\hat{\mu}\hat{\nu}}(\mathbf{X}^{\mu}) = \begin{pmatrix} g_{\mu\nu}(X^{\mu}) & 0 \\ 0 & G_{\alpha\beta}(X^{\mu}) \end{pmatrix},$$

$$\mathbf{B}^{(\mathbf{0})}_{\hat{\mu}\hat{\nu}}(\mathbf{X}^{\mu}) = \begin{pmatrix} b_{\mu\nu}(X^{\mu}) & 0 \\ 0 & B_{\alpha\beta}(X^{\mu}) \end{pmatrix}$$

INTRODUCE A PAIR OF VECTORS $\mathcal{V}$ AND $\mathcal{W}$ OF DIMENSIONS 2D AND 2d RESPECTIVELY.

$$\mathcal{V} = \begin{pmatrix} \tilde{P}_{\mu} \\ X'^{\mu} \end{pmatrix}, \quad \mathcal{W} = \begin{pmatrix} P_{\alpha} \\ Y'^{\alpha} \end{pmatrix}$$

THE CANONICAL HAMILTONIAN DENSITY IS

$$\mathcal{H}_{\mathbf{c}} = \frac{\mathbf{1}}{\mathbf{2}}\left(\mathcal{V}^{\mathbf{T}}\tilde{\mathbf{M}}\mathcal{V} + \mathcal{W}^{\mathbf{T}}\mathbf{M}\mathcal{W}\right)$$

WHEREAS $\tilde{\mathbf{M}}(\mathbf{X})$ IS A$\mathbf{2D} \times \mathbf{2D}$ MATRIX $\mathbf{M}(\mathbf{X})$ IS ANOTHER $\mathbf{2d} \times \mathbf{2d}$ MATRIX GIVEN BY

$$\tilde{\mathbf{M}} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}b_{\rho\nu} \\ b_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - b_{\mu\rho}g^{\rho\lambda}b_{\lambda\nu} \end{pmatrix}$$

AND

$$\mathbf{M} = \begin{pmatrix} G^{\alpha\beta} & -G^{\alpha\gamma}B_{\gamma\beta} \\ B_{\alpha\gamma}G^{\gamma\beta} & G_{\alpha\beta} - B_{\alpha\gamma}G^{\gamma\delta}B_{\delta\beta} \end{pmatrix}$$

LET US FOCUS ON THE SECOND TERM AND DEFINE

$$\mathbf{H_2} = \frac{\mathbf{1}}{\mathbf{2}}\mathcal{W}^{\mathbf{T}}\mathbf{M}\mathcal{W}$$

WHICH WILL BE IMPORTANT FOR US LATER.

UNDER GLOBAL $\mathbf{O(d,d)}$ TRANSFORMATIONS

$$\mathbf{M} \rightarrow \mathbf{\Omega M \Omega^T}, \ \mathcal{W} \rightarrow \mathbf{\Omega}\mathcal{W}, \ \mathbf{\Omega^T}\eta\mathbf{\Omega} = \eta, \ \mathbf{\Omega} \in \mathbf{O(d,d)}$$

WHERE

$$\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

NOW 1 BEING $\mathbf{d \times d}$ UNIT MATRIX
AND $\mathcal{W}$ IS $\mathbf{O(d,d)}$ VECTOR.

SINCE $\tilde{\mathbf{M}}$ AND $\mathcal{V}$ ARE INERT UNDER THIS DUALITY TRANSFORMATION, $\mathbf{H_c}$, THE FULL HAMILTONIAN DENSITY, IS $\mathbf{O(d,d)}$ INVARIANT.

REMARKS:

IF WE CONSIDER TOROIDAL COMPACTIFICATION, ON $\mathbf{T^d}$, AND $\mathbf{Y}^\alpha$ ARE THE COMPACT COORDINATES, THEN THE DUALITY GROUP IS $\mathbf{O(d, d, Z)}$.

THE MODULI G AND B PARAMETRIZE THE COSET $\frac{\mathbf{O(d,d)}}{\mathbf{O(d)\times O(d)}}$

IN GENERAL FOR COMPACTIFICATION ON $\mathbf{T^d}$, THE S-S PROCEDURE IS

$$\hat{\mathbf{e}}_{\mathbf{M}}^{\mathbf{A}} = \begin{pmatrix} e_\mu^r(X) & \mathcal{A}_\mu^\beta(X) E_\beta^a(X) \\ 0 & E_\alpha^a(X) \end{pmatrix}$$

THE D-DIMENSIONAL SPACETIME METRIC IS:

$\mathbf{g}_{\mu\nu}(\mathbf{X}) = \mathbf{e}_\mu^\mathbf{r} \mathbf{e}_\nu^\mathbf{s} \mathbf{g}_\mathbf{rs}^\mathbf{0}$, $\mathbf{g}_\mathbf{rs}^\mathbf{0}$ BEING D-DIMENSIONAL LORENTZIAN SIGNATURE FLAT METRIC.

$\mathbf{G}_{\alpha\beta} = \mathbf{E}_\alpha^\mathbf{a} \mathbf{E}_\beta^\mathbf{b} \delta_\mathbf{ab}$ IS THE METRIC ALONG COMPACT DIRECTIONS $\mathcal{A}_\mu^\beta$ ARE THE GAUGE FIELDS ASSOCIATED WITH THE d-DIMENSIONAL TORUS.

$$\mathbf{B}_{\hat{\mu}\hat{\nu}}^{(\mathbf{0})}(\mathbf{X}^\mu) = \begin{pmatrix} b_{\mu\nu}(X^\mu) & B_{\mu\alpha}(X^\mu) \\ B_{\nu\beta}(X^\mu) & B_{\alpha\beta}(X^\mu) \end{pmatrix}$$

- **INTRODUCTION**
- **T-DUALITY FOR FIRST EXCITED LEVEL**
- **HIGHER LEVELS AND $\mathbf{O(d,d)}$ SYMMETRY**
- **SUMMARY AND CONCLUSIONS**

EXCITED MASSIVE STRINGY STATES ARE INTERESTING.

◇ AT PLANCKIAN ENERGY SCATTERING, STRINGY STATES ARE IMPORTANT. IN THE $\alpha' \to \infty$ LIMIT, THERE IS CONJECTURED SYMMETRY ENHANCEMENT. GROSS, MENDE, AMATI, CIAFALONI, VENEZIANO...,

◇ THERE ARE EVIDENCES FOR EXISTENCE OF GAUGE SYMMETRY ASSOCIATED WITH MASSIVE STATES. EVANS, OVRUT, KUBOTA, VENEZIANO...

◇ HIGHER SPIN MASSLESS FIELD THEORY - CLUE FROM STRING HIGHER SPIN STATES. SAGNOTI, TARONNA....

◇ ROLE OF HIGHER MASS STATES WHEN β-FUNCTION IS COMPUTED FOR MASSLESS SECTOR IN HIGHER LOOPS IN CASE OF CLOSED BOSONIC STRING. DAS, SATHIAPALAN, ITOI, WATABIKI

LET US DISCUSS T-DUALITY PROPERTIES OF EXCITED MASSIVE STATES OF CLOSED BOSONIC STRING IN THE CONTEXT OF THE PRECEDING REMARKS.

WHEN THE 'VERTEX TENSORS' ARE INDEPENDENT OF SPACETIME COORDINATES - THE SIMPLEST CASE.

TO HAVE THE DUALITY SYMMETRY, WHEN WE INTERCHANGE $\mathbf{P} \leftrightarrow \mathbf{X}'$ THESE TENSORS (BACKGROUNDS) MUST TRANSFORM IN A CERTAIN MANNER JUST LIKE $\mathbf{G}^{(0)} \leftrightarrow \mathbf{G}^{(0)^{-1}}$ FOR CONSTANT $\mathbf{G}^{(0)}$.

IN GENERAL THE VERTEX OPERATORS WILL DEPEND ON SPACETIME COORDINATES $\mathbf{X}^{\hat{\mu}}, \hat{\mu} = \mathbf{0}, \mathbf{1}, ..\hat{\mathbf{D}} - \mathbf{1}$ - HOWEVER, WHEN THEY DEPEND ONLY ON A SUBSET OF COORDINATES $\mathbf{X}^{\mu}, \mu = \mathbf{0}, \mathbf{1}, ..\mathbf{D} - \mathbf{1}$ AND INDEPENDENT OF $\mathbf{Y}^{\alpha}, \alpha = \mathbf{D}, \mathbf{D} + \mathbf{1}, ..\hat{\mathbf{D}} - \mathbf{1} : \mathbf{D} + \mathbf{d} = \hat{\mathbf{D}}-$ THEY CAN BE CAST IN AN $\mathbf{O(d, d)}$ SYMMETRIC FORM

THE CHECK IS BY EXPLICIT CALCULATIONS.

- FIRST RECALL SOME OF THE ESSENTIAL

PROPERTIES OF THE VERTEX OPERATORS FOR OUR PURPOSE.

• WE WORK IN THE WEAK FIELD APPROXIMATION. THE VERTEX OPERATORS, $\hat{\Phi}_n(X^{\hat{\mu}}), n$, REFERRING TO THE LEVEL OF EXCITED STATE, ARE REQUIRED TO BE (1,0) AND (0,1) PRIMARIES WITH RESPECT TO
T_{++} AND T_{--} RESPECTIVELY.
WHERE

$$T_{++} = \frac{1}{2}(\hat{G}^{(0)}_{\hat{\mu}\hat{\nu}}\partial X^{\hat{\mu}}\partial X^{\hat{\nu}}), \;\; T_{--} = \frac{1}{2}(\hat{G}^{(0)}_{\hat{\mu}\hat{\nu}}\bar{\partial} X^{\hat{\mu}}\bar{\partial} X^{\hat{\nu}})$$

WHERE $(\hat{G}^{(0)}_{\hat{\mu}\hat{\nu}} = (1, -1, -1...)$ AND TRESS ENERGY MOMENTUM TENSORS ARE DEFINED FOR FLAT TARGET SPACE METRIC WITH
$\partial X^{\hat{\mu}} = \dot{X}^{\hat{\mu}} + X'^{\hat{\mu}}, \;\; \bar{\partial} X^{\hat{\mu}} = \dot{X}^{\hat{\mu}} - X'^{\hat{\mu}}$

• THEY ARE CONSTRAINED AND SATISFY 'EQUATIONS OF MOTION' AND CERTAIN TRANSVERSALITY CONDITIONS.
IN CASE OF MASSLESS GRAVITON IN WEAK FIELD APPROXIMATION: $G_{\hat{\mu}\hat{\nu}}(X^{\hat{\mu}}) = G^{(0)}_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}}(X^{\hat{\mu}})$.
THE CONSTRAINTS ARE:
$\nabla^2 h_{\hat{\mu}\hat{\nu}} = 0$, and $\partial^{\hat{\mu}} h_{\hat{\mu}\hat{\nu}} = 0$

• THE FIRST EXCITED MASSIVE STATE OF CLOSED BOSONIC STRING.
THE VERTEX OPERATOR IS

$$\hat{\Phi}_1 = \hat{V}^{(1)}_1 + \hat{V}^{(2)}_1 + \hat{V}^{(3)}_1 + \hat{V}^{(4)}_1$$

WHERE

$$\hat{V}_1^{(1)} = A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}(X)\partial X^{\hat{\mu}}\partial X^{\hat{\nu}}\bar{\partial} X^{\hat{\mu}'}\bar{\partial} X^{\hat{\nu}'}$$

$$\hat{V}_1^{(2)} = A^{(2)}_{\hat{\mu}\hat{\nu},\hat{\mu}'}(X)\partial X^{\hat{\mu}}\partial X^{\hat{\nu}}\bar{\partial}^2 X^{\hat{\mu}'}, \ \ \hat{V}_1^{(3)} = A^{(3)}_{\hat{\mu},\hat{\mu}'\hat{\nu}'}(X)\partial^2 X^{\hat{\mu}}\bar{\partial} X^{\hat{\mu}'}\bar{\partial} X^{\hat{\nu}'}$$

$$\hat{V}_1^{(4)} = A^{(4)}_{\hat{\mu},\hat{\mu}'}(X)\partial^2 X^{\hat{\mu}}\bar{\partial}^2 X^{\hat{\mu}'}$$

WE DEFINE $\hat{V}_1^{(i)}, i = 1, 2, 3, 4$ AS VERTEX FUNCTIONS

UNPRIMED INDICES AND PRIMED INDICES CORRESPOND TO RIGHT MOVING AND LEFT MOVING SECTORS RESPECTIVELY ($\partial X^{\hat{\mu}}$ AND $\partial X^{\hat{\mu}'}$).

WE DEMAND $\hat{\Phi}_1$ TO BE (1,1) WITH RESPECT TO $T_{\pm\pm}$, THEN $\hat{V}_1^{(i)}$(i.e.$A^{(i)}$) ARE CONSTRAINED - THEY ARE NOT INDEPENDENT.

NOTE (i) ONLY $\hat{V}_1^{(1)}$ IS (1,1) ON ITS OWN. HOWEVER, THE OTHER THREE VERTEX FUNCTIONS $\hat{V}_1^{(2)} - \hat{V}_1^{(4)}$ ARE RELATED TO $\hat{V}_1^{(1)}$

(ii) WHEN WE DEMAND $\hat{\Phi}_1$ TO BE (1,1) THERE ARE TWO TYPES OF CONSTRAINTS ON THE VERTEX FUNCTIONS.

(a) EACH SATISFIES A MASS SHELL CONDITION

$$(\hat{\nabla}^2 - 2)A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}(X) = 0, \quad (\hat{\nabla}^2 - 2)A^{(2)}_{\hat{\mu}\hat{\nu},\hat{\mu}'}(X) = 0,$$

AND

$$(\hat{\nabla}^2 - 2)A^{(3)}_{\hat{\mu},\hat{\mu}'\hat{\nu}'}(X) = 0, \quad (\hat{\nabla}^2 - 2)A^{(4)}_{\hat{\mu},\hat{\mu}'}(X) = 0$$

$\hat{\nabla}^2$ IS $\hat{D}$ DIMENSIONAL LAPLACIAN DEFINED IN TERMS OF FLAT SPACETIME METRIC.

(b) TRANSVERSALITY CONDITIONS:

$$\mathbf{A}^{(2)}_{\hat\mu\hat\nu,\hat\mu'} = \partial^{\hat\nu'}\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'},\ \ \mathbf{A}^{(3)}_{\hat\mu,\hat\mu'\hat\nu'} = \partial^{\hat\nu}\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'},\ \ \mathbf{A}^{(4)}_{\hat\mu,\hat\mu'} = \partial^{\hat\nu'}\partial^{\hat\nu}\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'}$$

NOTE HOW THREE VERTEX FUNCTIONS RELATED $\mathbf{A}^{(2)}_{\hat\mu\hat\nu,\hat\mu'}, \mathbf{A}^{(3)}_{\hat\mu,\hat\mu'\hat\nu'}$ AND $\mathbf{A}^{(4)}_{\hat\mu,\hat\mu'}$ $\mathbf{A}^{(2)}_{\hat\mu\hat\nu,\hat\mu'}, \mathbf{A}^{(3)}_{\hat\mu,\hat\mu'\hat\nu'}$ AND $\mathbf{A}^{(4)}_{\hat\mu,\hat\mu'}$ ARE RELATED TO $\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'}$ THE OTHER SET OF CONSTRAINTS ARE

$$\mathbf{A}^{(1)\hat\mu}_{\hat\mu}{}_{,\hat\mu'\hat\nu'} + 2\partial^{\hat\mu}\partial^{\hat\nu}\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'} = 0$$

AND

$$\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'}{}^{\hat\mu'} + 2\partial^{\hat\mu'}\partial^{\hat\nu'}\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'} = 0$$

CONSIDER THE CASE WHEN TENSORS $\hat{V}_1^{(i)}$ (i.e. $A^{(i)}$) DO NOT DEPEND ON SPACETIME COORDINATES. THIS THIS ANALOG OF THE CASE WHEN G AND B ARE INDEPENDENT OF X.

LET US FOCUS ON $\hat{V}_1^{(1)}$

$$\hat{V}_1^{(1)} = A^{(1)}_{\hat{\mu}\hat{\nu},\hat{\mu}'\hat{\nu}'}\partial X^{\hat{\mu}}\partial X^{\hat{\nu}}\bar{\partial} X^{\hat{\mu}'}\bar{\partial} X^{\hat{\nu}'}$$

WE RAISE AND LOWER THE INDICES BY FLAT SPACE METRIC AND THEREFORE $\partial X^{\hat{\mu}} = P^{\hat{\mu}} + X'^{\hat{\mu}}$

THUS THE ABOVE EQUATION WHEN EXPRESSED IN TERMS OF $P^{\hat{\mu}}$ AND $X'^{\hat{\mu}}$ HAVE FOLLOWING FORMS.

(I) $G^{(1)}_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\lambda}}P^{\hat{\mu}}P^{\hat{\nu}}P^{\hat{\rho}}P^{\hat{\lambda}}$ **- PRODUCT OF MOMENTA ONLY.**

(II) $G^{(2)}_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\lambda}}X'^{\hat{\mu}}X'^{\hat{\nu}}X'^{\hat{\rho}}X'^{\hat{\lambda}}$ **- PRODUCT OF X' ONLY**

(III) $G^{(3)}_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\lambda}}P^{\hat{\mu}}P^{\hat{\nu}}P^{\hat{\rho}}X'^{\hat{\lambda}}$ **- PRODUCT OF THREE MOMENTA AND ONE X'; THERE ARE FOUR SUCH TERMS**

(IV) $G^{(4)}_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\lambda}}X'^{\hat{\mu}}X'^{\hat{\nu}}X'^{\hat{\rho}}P^{\hat{\lambda}}$ **- PRODUCT OF THREE X' AND ONE P; THERE ARE ALSO FOUR SUCH TERMS**

(V) $G^{(5)}_{\hat{\mu}\hat{\nu},\hat{\rho}\hat{\lambda}}P^{\hat{\mu}}P^{\hat{\nu}}X'^{\hat{\rho}}X'^{\hat{\lambda}}$ **- THERE SIX TERMS LIKE THIS WHICH IS PRODUCT OF A PAIR OF P'S AND PAIR OF X'**

WE CONCLUDE FROM CAREFUL INSPECTION OF THE TOTAL 16 TERMS IN (I)-(V) THAT 'VERTEX' $\mathbf{A}^{(1)}$ REMAINS INVARIANT UNDER THE INTERCHANGE $\mathbf{P} \leftrightarrow \mathbf{X}'$ IF WE ALSO INTERCHANGE $\mathbf{G}^{(1)} \leftrightarrow \mathbf{G}^{(2)}, \mathbf{G}^{(3)} \leftrightarrow \mathbf{G}^{(4)}$ AND THE SIX TERMS IN (V) REARRANGE THEMSELVES TO REMAIN INVARIANT.

THIS IS THE ANALOG OF $\mathbf{G} \leftrightarrow \mathbf{G}^{-1}$ UNDER $\tau \leftrightarrow \sigma$ DUALITY FOR CONSTANT TENSOR $\mathbf{A}^{(1)}$.

LET US CONSIDER THE SCENARIO WHEN THE VERTEX FUNCTION, $\mathbf{A}^{(1)}_{\hat\mu\hat\nu,\hat\mu'\hat\nu'}(\mathbf{X})\partial\mathbf{X}^{\hat\mu}\partial\mathbf{X}^{\hat\nu}\bar\partial\mathbf{X}^{\hat\mu'}$ DEPENDS ON $\mathbf{X}^{\mu}, \mu = \mathbf{0}, \mathbf{1}, ..\mathbf{D}-\mathbf{1}$ AND IS INDEPENDENT OF INTERNAL COORDINATES $\mathbf{Y}^{\alpha}$. THE VERTEX FUNCTION WILL BE DECOMPOSED INTO THE FOLLOWING FORMS:

(i) A TENSOR $\mathbf{A}^{(1)}_{\mu\nu,\mu'\nu'}$ ONE WHICH HAS ALL LORENTZ INDICES.

(ii) ANOTHER: THREE LORENTZ INDICES AND ONE INTERNAL INDEX.

(iii) A TENSOR WITH TWO LORENTZ INDICES AND TWO INTERNAL INDICES

(iv)A TENSOR WITH ONE LORENTZ INDEX AND THREE INTERNAL INDICES.

(v) A TENSOR WITH ALL INTERNAL INDICES: $\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}$ WHICH WILL CONTRACT WITH $\partial\mathbf{Y}^{\alpha}\partial\mathbf{Y}^{\beta}\bar\partial\mathbf{Y}^{\alpha'}\bar\partial\mathbf{Y}^{\beta'}$.

CONSIDER THE VERTEX FUNCTION (v). WE SHALL TAKE UP OTHERS LATER.
JUST LIKE THE CASE OF CONSTANT $\mathbf{A}^{(1)}$, THERE ARE 16 TERMS.

(I) $\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{P}^{\beta}\mathbf{P}^{\alpha'}\mathbf{P}^{\beta'}$

(II) $\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{Y}'^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{Y}'^{\beta'}$.

(III) $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{P}^{\beta}\mathbf{P}^{\alpha'}\mathbf{Y}'^{\beta'}$, $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{P}^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{P}^{\beta'}$,
$+\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{Y}'^{\beta}\mathbf{P}^{\alpha'}\mathbf{P}^{\beta'}$, $+\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{P}^{\beta}\mathbf{P}^{\alpha'}\mathbf{P}^{\beta'}$
(IV) $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{Y}'^{\beta}\mathbf{P}^{\alpha'}\mathbf{Y}'^{\beta'}$ $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{Y}'^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{P}^{\beta'}$
$+\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{P}^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{P}^{\beta'}$
$+\mathbf{A}^{(1)}_{\alpha\beta,\alpha'\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{Y}'^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{Y}'^{\beta'}$

(V) $+\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{P}^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{Y}'^{\beta'}$ $+\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{Y}'^{\beta}\mathbf{P}^{\alpha'}\mathbf{P}^{\beta'}$
$+\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{P}^{\alpha}\mathbf{Y}'^{\beta}\mathbf{P}^{\alpha'}\mathbf{Y}'^{\beta'}$ $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{P}^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{P}^{\beta'}$
$-\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{P}^{\beta}\mathbf{Y}'^{\alpha'}\mathbf{P}^{\beta'}$ $-\mathbf{A}^{(1)}_{\alpha\beta,\alpha\beta'}(\mathbf{X})\mathbf{Y}'^{\alpha}\mathbf{P}^{\beta}\mathbf{P}^{\alpha'}\mathbf{Y}'^{\beta'}$

OUR GOAL IS TO CAST THE VERTEX FUNCTION IN AN $\mathbf{O(d,d)}$ INVARIANT FORM. WE WOULD LIKE TO COMBINE VARIOUS TERMS IN (I)-(V) TO ACHIEVE THIS. RECALL

$$\mathcal{W} = \begin{pmatrix} P \\ Y' \end{pmatrix}$$

EXPRESSIONS IN (I) AND (II) CAN BE COMBINED TO CONSTRUCT TENSORS OF $\mathbf{O(d,d)}$

TO CONTRACT WITH THE VECTORS $\mathcal{W}$.

THE COMBINED TERMS IN (III) AND (IV) HAVE RIGHT STRUCTURES TO FORM AN $\mathbf{O(d,d)}$ INVARIANT PIECE.

FINALLY THE SIX TERMS IN (V) CAN BE REARRANGED TO OBTAIN AN $\mathbf{O(d,d)}$ INVARIANT EXPRESSION.

BASICALLY WE CONTRACT $\mathbf{O(d,d)}$ TENSORS WITH $\mathbf{O(d,d)}$ VECTORS LIKE $\mathcal{W}$ AND $\eta\mathcal{W}$. WHEN WE LOOK AT OTHER VERTEX FUNCTIONS WITH INTERNAL INDICES:

$$\mathbf{V_1^{(2)}} = \mathbf{A^{(2)}_{\alpha\beta,\alpha'}(X)\partial Y^\alpha \partial Y^\beta \bar\partial^2 Y^{\alpha'}}$$

$$\mathbf{V_1^{(3)}} = \mathbf{A^{(3)}_{\alpha,\alpha'\beta'}(X)\partial^2 Y^\alpha \partial Y^{\alpha'} \partial Y^{\beta'}}$$

$$\mathbf{V_1^{(4)}} = \mathbf{A^{(4)}_{\alpha,\alpha'}(X)\partial^2 Y^\alpha \bar\partial^2 Y^{\alpha'}}$$

WE NOTE THAT WHEN THE CONSTRAINTS SUCH AS TRANSVERSALITY CONDITIONS ARE ENFORCED, THESE INVOLVE PARTIAL DERIVATIVES WITH RESPECT TO INTERNAL COORDINATES, THESE VERTEX FUNCTIONS VANISH.

EVEN IF WE WERE TO EXPLORE THEIR T-DUALITY PROPERTIES, BEFORE IMPOSING THE REQUIREMENTS THAT THEY BE (1,1), WE ENCOUNTER ANOTHER DIFFICULTY.

NOTE THAT HIGHER ORDER DERIVATIVES OF

$\partial, \bar{\partial}$ ACT ON Y^α THUS WE DEAL WITH τ AND σ DERIVATIVES ON P AND Y' AND THEREFORE, NICE CORRESPONDENCE OF $\tau \leftrightarrow \sigma$ WITH $P \leftrightarrow Y'$ IS NO LONGER SO SIMPLE.

IN FACT BY CONSTRUCTING COMBINATIONS OF VARIOUS TERMS AND DEFINING THE $O(d,d)$ VECTOR $\mathcal{W}$ AND THEIR τ AND σ DERIVATIVES,

WE CAN SHOW VERTEX FUNCTION CAN BE CAST IN $O(d,d)$ INVARIANT FORM - A BIT TRICKY.

THIS IS NOT AN EFFICIENT METHOD WHEN WE CONSIDER HIGHER EXCITED MASSIVE LEVELS.

WE ENCOUNTER STRING OF TERMS OF TWO TYPES:

(a) PRODUCTS LIKE $\partial Y \partial Y ... \bar{\partial} Y \bar{\partial} Y ..$

(b) HIGHER DERIVATIVES AND THEIR PRODUCTS LIKE

$\partial^m Y \partial Y ... \bar{\partial}^n Y \bar{\partial} Y ..$

AN EXAMPLE: SECOND MASSIVE LEVEL

ONE OF THE VERTEX FUNCTIONS IS

$$\mathbf{V_2^{(1)} = C^{(1)}_{\alpha\beta\gamma,\alpha'\beta'\gamma'}(X)\partial Y^{\alpha}\partial Y^{\beta}\partial Y^{\gamma}\bar{\partial} Y^{\alpha'}\bar{\partial} Y^{\beta'}\bar{\partial} Y^{\gamma'}}$$

WE CAN EXPRESS $\mathbf{V_2^{(1)}}$ IN TERMS OF $\mathbf{P^{\alpha}, Y'^{\alpha}}$ AS IN THE EARLIER CASE AND EXPRESS IT IN A MANIFESTLY $\mathbf{O(d, d)}$ INVARIANT FORM; HOWEVER THE PROCEDURE IS QUITE TEDIOUS.

MOREOVER, THE VERTEX OPERATOR FOR SECOND MASSIVE LEVEL IS A SUM OF NINE VERTEX FUNCTIONS WITH HIGHER POWERS OF ∂ AND $\bar{\partial}$ ACTING ON $\mathbf{Y^{\alpha}}$.

WE PROPOSE AN EFFICIENT METHOD TO HANDLE THESE PROBLEMS BASED ON FOLLOWING OBSERVATIONS:

(a) THE BASIC BUILDING BLOCKS OF ANY VERTEX FUNCTION ARE
$\partial \mathbf{Y}^\alpha = \mathbf{P}^\alpha + \mathbf{Y}'^\alpha$ AND $\bar\partial \mathbf{Y}^\alpha = \mathbf{P}^\alpha - \mathbf{Y}'^\alpha$

(b) EACH VERTEX FUNCTION AT EACH MASS LEVEL IS EITHER STRING OF PRODUCTS OF THESE BASIC BLOCKS OR THESE BLOCKS ARE OPERATED BY ∂ AND $\bar\partial$ RESPECTIVELY SO THAT THE VERTEX FUNCTION HAS THE DESIRED DIMENSIONS.

(c) IN ORDER TO CAST VERTEX FUNCTIONS IN A T-DUALITY INVARIANT FORM, IT IS NOT CONVENIENT TO DEAL WITH $\mathbf{P} \pm \mathbf{Y}'$. WE ADOPT THE FOLLOWING STRATEGY:

INTRODUCE THE PROJECTION OPERATORS

$$\mathbf{P}_\pm = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} \pm \tilde{\sigma_3}), \qquad \tilde{\sigma_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

HERE 1 IS $\mathbf{2d} \times \mathbf{2d}$ MATRIX. THE DIAGONAL ENTRIES OF $\tilde{\sigma}_3$ ARE $\mathbf{d} \times \mathbf{d}$ UNIT MATRIX.

WE PROJECT OUT TWO $\mathbf{O(d,d)}$ VECTORS AS FOLLOWS

$$\mathbf{P} = \mathbf{P}_+ \mathcal{W}, \qquad \mathbf{Y}' = \mathbf{P}_- \mathcal{W}$$

THEREFORE,

$$\mathbf{P}+\mathbf{Y}'=\frac{1}{2}\Big(\mathbf{P}_{+}\mathcal{W}+\eta\mathbf{P}_{-}\mathcal{W}\Big),\quad \mathbf{P}-\mathbf{Y}'=\frac{1}{2}\Big(\mathbf{P}_{+}\mathcal{W}-\eta\mathbf{P}_{-}\mathcal{W}\Big)$$

NOTE THAT η FLIPS LOWER COMPONENT $\mathbf{Y}'$ OF $\mathcal{W}$ TO THE UPPER COMPONENT.

• **WHEN WE HAVE ONLY PRODUCTS OF $\mathbf{P}+\mathbf{Y}'$ AND $\mathbf{P}-\mathbf{Y}'$, WE FIRST EXPRESS THEM AS PRODUCT OF $\mathbf{O(d,d)}$ VECTORS AND THEN CONTRACT THESE VECTOR INDICES WITH INDICES OF SUITABLY CONSTRUCTED $\mathbf{O(d,d)}$ TENSORS.**

• **NOW WE DEAL WITH WORLDSHEET PARTIAL DERIVATIVES ∂ AND $\bar\partial$ WHICH KEEP ACTING ON $\mathbf{P}\pm\mathbf{Y}'$ IN SOME VERTEX FUNCTIONS. DEFINE**

$$\mathbf{\Delta}_{\tau}=\mathbf{P}_{+}\partial_{\tau},\ \ \mathbf{\Delta}_{\sigma}=\mathbf{P}_{+}\partial_{\sigma},\quad \mathbf{\Delta}_{\pm}(\tau,\sigma)=\frac{1}{2}(\mathbf{\Delta}_{\tau}\pm\mathbf{\Delta}_{\sigma})$$

WE HAVE TWO USEFUL RELATIONS

$$\partial(\mathbf{P}+\mathbf{Y}')=\mathbf{\Delta}_{+}(\tau,\sigma)\Big(\mathbf{P}_{+}\mathcal{W}+\eta\mathbf{P}_{-}\mathcal{W}\Big)$$

AND

$$\bar\partial(\mathbf{P}-\mathbf{Y}')=\mathbf{\Delta}_{-}(\tau,\sigma)\Big(\mathbf{P}_{+}\mathcal{W}-\eta\mathbf{P}_{-}\mathcal{W}\Big)$$

WE USE ABOVE TWO RELATIONS TO EXPRESS THE BASIC BUILDING BLOCKS AND THEIR WORLDSHEET DERIVATIVES IN TERMS OF $\mathbf{O(d,d)}$ VECTORS.

THESE PRODUCTS OF VECTORS WILL BE CONTRACTED WITH THE $\mathbf{O(d,d)}$ TENSORS. RECALL THAT THE M-MATRIX APPEARING IN THE $\mathbf{O(d,d)}$ FORM OF THE HAMILTONIAN IS DEFINED IN TERMS OF THE BACKGROUNDS $\mathbf{G}_{\alpha\beta}$ AND $\mathbf{B}_{\alpha\beta}$.

IT IS NOW STRAIGHT FORWARD TO EXPRESS $\mathbf{V}_1^{(2)}, \mathbf{V}_1^{(3)}$ AND $\mathbf{V}_1^{(4)}$ IN DUALITY INVARIANT FORM. WHEN WE CONSTRUCT VERTEX OPERATORS FOR HIGHER AND HIGHER EXCITED STATES A VARIETY OF VERTEX FUNCTIONS WILL APPEAR.

NOTE: FROM THE STRUCTURE OF THE VERTEX FUNCTIONS THAT EACH ONE OF THEM CAN BE EXPRESSED IN AN $O(d,d)$ INVARIANT FORM FOLLOWING OUR PRESCRIPTIONS.

CONSIDER THE n^{th} EXCITED LEVEL

- THE DIMENSION OF ALL RIGHT MOVERS CONSTRUCTED FROM ∂Y AND POWERS OF ∂ ACTING ON ∂Y SHOULD BE $n+1$
- SAME HOLDS FOR THE LEFT MOVING SECTOR.
- CONSIDER RIGHT MOVING SECTOR OF THE TYPE $\Pi_1^{n+1}\partial Y^{\alpha_i}$ LEFT MOVING SECTOR OF SAME TYPE: $\Pi_1^{n+1}\bar\partial Y^{\alpha_i}$ THE VERTEX FUNCTION IS WRITTEN AS

$$V_{\alpha_1,\alpha_2\ldots\alpha_{n+1},\alpha'_1\alpha'_2\ldots\alpha'_{n+1}}(X)\Pi_1^{n+1}\partial Y^{\alpha_i}\Pi_1^{n+1}\bar\partial Y^{\alpha'_i}$$

THESE PRODUCTS $\Pi_1^{n+1}\partial Y^{\alpha'_i}$ AND $\Pi_1^{n+1}\bar\partial Y^{\alpha'_i}$ CAN BE CONVERTED TO PRODUCTS OF PROJECTED $\mathcal{W}$ FOR RIGHT MOVERS AND ALSO FOR LEFT MOVERS.

A GENERIC VERTEX FUNCTION HAS THE STRUCTURE

$$\partial^p Y^{\alpha_i}\partial^q Y^{\alpha_j}\partial^r Y^{\alpha_k}\ldots\bar\partial^{p'} Y^{\alpha'_i}\bar\partial^{q'} Y^{\alpha'_j}\bar\partial^{r'} Y^{\alpha'_k}\ldots,$$

WITH THE CONSTRAINT

$p+q+r=n+1,\ p'+q'+r'=n+1$

REMARKS: THIS HAS TO BE CONVERTED INTO PRODUCTS OF $\mathcal{W}$ AND THEIR DERIVATIVES FORM AN $\mathbf{O(d,d)}$ TENSOR. THE RANK OF THE TENSOR IS DECIDED BY THE CONSTRAINTS SATISFIED BY p, q, r, etc. AS GIVEN ABOVE. THE RESULTING TENSOR WILL BE CONTRACTED WITH AN $\mathbf{O(d,d)}$ TENSOR TO GIVE THE CORRESPONDING VERTEX FUNCTION. THE CONSTRUCTION OF SUCH A VERTEX FUNCTION

STEP I: REWRITE

$\partial^{\mathbf{p}}\mathbf{Y} = \partial^{\mathbf{p-1}}(\mathbf{P}+\mathbf{Y}'), \quad \bar{\partial}^{\mathbf{p}'}(\mathbf{P}-\mathbf{Y}') = \bar{\partial}^{\mathbf{p}'-\mathbf{1}}(\mathbf{P}-\mathbf{Y}')$

STEP II: USING THE PROJECTION OPERATORS

$\partial^{\mathbf{p-1}}(\mathbf{P}+\mathbf{Y}') = \mathbf{\Delta_+}^{\mathbf{p-1}}(\mathbf{P}+\mathbf{Y}'), \quad \bar{\partial}^{\mathbf{p}'-\mathbf{1}}(\mathbf{P}-\mathbf{Y}') = \mathbf{\Delta_-}^{\mathbf{p}'-\mathbf{1}}(\mathbf{P}-\mathbf{Y}')$

STEP III:

$\mathbf{\Delta_+}^{\mathbf{p-1}}(\mathbf{P}+\mathbf{Y}') = \mathbf{\Delta_+}^{\mathbf{p-1}}\Big(\mathbf{P}_+\mathcal{W} + \eta\mathbf{P}_-\mathcal{W}\Big)$

$\mathbf{\Delta}^{\mathbf{p}'-\mathbf{1}}(\mathbf{P}-\mathbf{Y}') = \mathbf{\Delta_-}^{\mathbf{p}'-\mathbf{1}}\Big((\mathbf{P}_+\mathcal{W} - \eta\mathbf{P}_-\mathcal{W}\Big)$

$\mathbf{O(d,d)}$ INVARIANT FORM

A GENERIC VERTEX OPERATORS CAN BE EXPRESSED AS

$$\begin{aligned} V_{n+1} &= \mathcal{A}_{klm..,k'l'm'..}(X)\Delta_+{}^{p-1}\mathcal{W}_+^k\Delta_+^{q-1}\mathcal{W}_+^l\Delta_+{}^{r-1}\mathcal{W}_+^m.. \\ &\quad \Delta_-{}^{p'-1}\mathcal{W}_-^{k'}\Delta_-{}^{p'-1}\mathcal{W}_-^{l'}\Delta_-{}^{p'-1}\mathcal{W}_-^{m'} \end{aligned}$$

WHERE $\mathcal{W}_\pm = (\mathbf{P}_+\mathcal{W} \pm \eta\mathbf{P}_-\mathcal{W})$ **WITH**
$\mathbf{p+q+r=n+1}$ **AND** $\mathbf{p'+q'+r'=n+1}$

INDICES $\{k, l, m; k', l', m'\}$ APPEARING ON $\mathcal{W}_\pm$ REFER TO COMPONENTS OF THE $O(d,d)$ VECTORS.

$\mathcal{A}_{klm..,k'l'm'..}(X)$ IS $O(d,d)$ TENSOR.

THE VERTEX V_{n+1} WILL BE $O(d,d)$ INVARIANT IF THE TENSOR $\mathcal{A}_{klm..,k'l'm'..}(X)$ TRANSFORMS AS

$$\mathcal{A}_{klm..,k'l'm'..} \to \Omega_k^p \Omega_l^q \Omega_m^r ... \Omega_{k'}^{p'} \Omega_{l'}^{q'} \Omega_{m'}^{r'} \mathcal{A}_{pqr..,p'q'r'..}$$

SINCE THE EACH TERM IN THE PRODUCT $\Delta^{p-1}\mathcal{W}_+^k....\Delta_-^{p'-1}\mathcal{W}^{k'}$ TRANSFORMS LIKE AN $O(d,d)$ VECTOR.

SO FAR WE HAVE CONSTRUCTED VERTEX FUNCTIONS CONSTRUCTED OUT OF THE CONTRACTION OF $\partial Y^\alpha, \bar\partial Y^\alpha$ etc. WHOSE INDICES ARE CONTRACTED WITH X-DEPENDENT TENSORS WITH INTERNAL INDICES ONLY.

ALL THESE LEVELS TRANSFORM LIKE SCALARS UNDER $SO(D-1)$.

NOTE THAT ONCE WE ALLOW $\partial X^\mu, \bar\partial X^\mu$ AND THEIR DERIVATIVES TO APPEAR IN THE VERTEX FUNCTIONS WITH APPROPRIATE CONTRACTION OF LORENTZ INDICES, WE SHALL HAVE MANY MORE VERTEX FUNCTIONS.

LET US CONSIDER THE FIRST EXCITED MASSIVE LEVEL AND THE POSSIBLE VERTEX FUNCTIONS ASSOCIATED WITH IT.

RECALL TENSORS WITH SPACETIME INDICES TRANSFORM TRIVIALLY UNDER O(d,d) AND SO DOES X^μ.

$\tilde{V}_1^{(1)} = \tilde{A}^{(1)}_{\mu\nu,\mu'\nu'} \partial X^\mu \partial X^\nu \bar{\partial} X^{\mu'} \bar{\partial} X^{\nu'}$

$\tilde{V}_1^{(2)} = \tilde{A}^{(2)}_{\mu,\mu'\nu'} \partial^2 X^\mu \bar{\partial} X^{\mu'} \bar{\partial} X^{\nu'}, \tilde{A}^{(3)}_{\mu\nu,\mu'} \partial X^\mu \partial X^\nu \bar{\partial}^2 X^{\mu'}$

$\tilde{V}_1^{(3)} = \tilde{A}^{(3)}_{\mu,\mu'} \partial^2 X^\mu \bar{\partial}^2 X^{\mu'}$

ALL THESE VERTEX FUNCTIONS ARE $O(d,d)$ INVARIANT.

VERTEX FUNCTIONS OF THE TYPE LISTED BELOW:

(1). $\tilde{B}^{(1)}_{\mu\alpha,\alpha'\beta'} \partial X^\mu \partial Y^\alpha \bar{\partial} Y^{\alpha'} \bar{\partial} Y^{\beta'}$ AND OTHER SIMILAR TERMS.

(2). $\tilde{B}^{(2)}_{\mu\beta,\mu'\beta'} \partial X^\mu \partial Y^\beta \bar{\partial} X^{\mu'} \bar{\partial} Y^{\beta'}$ AND OTHER TERMS LIKE THESE.

(3). $\tilde{B}^{(3)}_{\mu\nu,\mu'\beta'} \partial X^\mu \partial X^\nu \bar{\partial} X^{\mu'} \bar{\partial} Y^{\beta'}$ AND OTHER TERMS LIKE THESE.

LET US EXAMINE THE TERMS (1), (2) AND (3) ABOVE. FIRST ONE IS A THREE INDEX TENSOR IN INTERNAL INDICES, SECOND WITH TWO INTERNAL INDICES AND THIRD IS RANK ONE. THESE INDICES ARE SATURATED WITH INDICES OF INTERNAL COORDINATES.

THUS

$\tilde{B}^{(1)}_{\mu\alpha,\alpha'\beta'}, \tilde{B}^{(2)}_{\mu\beta,\mu'\beta'}, \tilde{B}^{(3)}_{\mu\nu,\mu'\beta'}$

TRANSFORM AS RANKS THREE, TWO AND TENSORS AND THE LAST ONE AS A VECTOR.

SINCE LORENTZ INDICES ARE SATURATED BY CONTRACTION WITH SPACETIME COORDINATES, WHEN WE CONVERT $\partial \mathbf{Y}^{\alpha}, \bar{\partial} \mathbf{Y}^{\beta'}$ TO $\mathcal{W}$ WE NEED TO CONSTRUCT THIRD RANK, SECOND RANK AND RANK ONE $\mathbf{O(d,d)}$ TENSORS/VECTORS.

WE MAY ARGUE THAT ANY VERTEX FUNCTION AT A GIVEN LEVEL WITH ARBITRARY NUMBER OF LORENTZ INDICES AND INTERNAL INDICES WHICH ARE SATURATED WITH INDICES OF CORRESPONDING SPACETIME AND INTERNAL COORDINATES (ON WHICH ARBITRARY NUMBER OF DERIVATIVES ARE ACTING) i.e. $\partial, \bar{\partial}$, CAN BE EXPRESSED IN $\mathbf{O(d,d)}$ INVARIANT FORM.

FOR EXAMPLE A VERTEX FUNCTION OF THE FORM

$$\mathbf{T}_{\mu\alpha_{\mathbf{i}}\alpha_{\mathbf{j}}\mu'\alpha'_{\mathbf{i}}\alpha'_{\mathbf{j}}}\partial^{\mathbf{m}}\mathbf{X}^{\mu}\partial^{\mathbf{p}}\mathbf{Y}^{\alpha_{\mathbf{i}}}\partial^{\mathbf{q}}\mathbf{Y}^{\alpha_{\mathbf{j}}}...\bar{\partial}^{\mathbf{m}'}\mathbf{X}^{\mu'}\bar{\partial}^{\mathbf{p}'}\mathbf{Y}^{\alpha'_{\mathbf{i}}}\bar{\partial}^{\mathbf{q}'}\mathbf{Y}^{\alpha'_{\mathbf{j}}}..$$

CAN BE EXPRESSED IN A DUALITY INVARIANT FORM.

SUMMARY AND CONCLUSIONS

WE HAVE ATTEMPTED TO STUDY T-DUALITY SYMMETRY ASSOCIATED WITH EXCITED MASSIVE STATES OF A CLOSED BOSONIC STRING. THIS CONJECTURE IS VERIFIED IN A SIMPLE SCENARIO WHEN THE SPACETIME IS CONSIDERED TO BE FLAT AND INTERNAL DIRECTIONS ARE ALSO TAKEN TO BE FLAT.

WE INTRODUCED A METHOD TO CONSTRUCT $O(d, d)$ INVARIANT VERTEX FUNCTIONS FOR EACH EXCITED LEVEL.

HOWEVER, THIS TECHNIQUE IS NOT VERY GENERAL. IF ALLOW TO HAVE NONTRIVIAL INTERNAL METRIC $G_{\alpha\beta}, B\alpha\beta$ AND NONTRIVIAL GAUGE FIELDS ASSOCIATED WITH THE d-ISOMETRIES, THEN THINGS GET MORE COMPLICATED.

FURTHERMORE, IF THE SPACETIME IS CURVED AND ANTISYMMETRIC TENSOR IS NONZERO i.e. $g_{\mu\nu}, b_{\mu\nu}$ ARE NONTRIVIAL, WE FACE MORE COMPLICATIONS.

HASAN-SEN TYPE COMPACTIFICATION WITH FLAT INTERNAL DIRECTIONS $G_{\alpha\beta} = \delta_{\alpha\beta}$ AND $B\alpha\beta = 0$ COULD BE HANDLED USING THIS METHOD.