The Noncommutative Ward Metric

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The $\mathbb{C}P^1$ model and its solitons

dynamics of maps $\quad u:\ (t,z,\bar z)\in\mathbb{R}^{1,2} \longrightarrow \mathbb{C}P^1 \simeq \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \simeq S^2$

other parametrization: $\quad u=\frac{p}{q}$

$$T=\begin{pmatrix}p\\q\end{pmatrix}\sim\begin{pmatrix}u\\1\end{pmatrix} \quad\Longrightarrow\quad P=P^\dagger=P^2=T\,\frac{1}{T^\dagger T}\,T^\dagger$$

hermitian rank-one projectors in $\mathbb{C}^2$

action
$$\begin{aligned}S &= -4\int \mathrm{d}^3x\ \mathrm{tr}\,\eta^{\mu\nu}\partial_\mu P\,\partial_\nu P\\ &= -4\int \mathrm{d}^3x\ (T^\dagger T)^{-1}\eta^{\mu\nu}\partial_\mu T^\dagger(\mathbb{1}-P)\,\partial_\nu T\\ &= -4\int \mathrm{d}^3x\ (1+\bar u u)^{-2}\eta^{\mu\nu}\partial_\mu\bar u\,\partial_\nu u\end{aligned}$$

static configurations have energy

$$\begin{aligned}E &= 8\int \mathrm{d}^2z\ (T^\dagger T)^{-1}\big\{\partial_{\bar z}T^\dagger(\mathbb{1}-P)\,\partial_z T + \partial_z T^\dagger(\mathbb{1}-P)\,\partial_{\bar z}T\big\}\\ &= 8\int \mathrm{d}^2z\ (1+\bar u u)^{-2}\big\{\partial_{\bar z}\bar u\,\partial_z u+\partial_z\bar u\,\partial_{\bar z}u\big\}\end{aligned}$$

classical static configurations: $\delta E = 0 \quad \Rightarrow \quad u = u(z)$ or $u = u(\bar{z})$

finite energy $\Rightarrow$ u rational of degree n $\Rightarrow$ $E = 8\pi|n|$ $\Leftrightarrow$ solitons

moduli space $\mathcal{M}_n$ has $\dim_{\mathbb{C}} = 2n{+}1$ (including domain and target isometries)

after removing isometries we remain with nontrivial moduli $\{\alpha\} = \{\beta, \gamma, \epsilon, \delta, \ldots\}$:

$n{=}1$: $\quad T(z) = \begin{pmatrix} \beta \\ z \end{pmatrix} \quad \Rightarrow \quad E = \int \mathrm{d}^2 z \, \frac{8\,|\beta|^2}{(|\beta|^2 + |z|^2)^2} = 8\pi$

with $\beta \in \mathbb{R}_+$ $\qquad$ $\dim_{\mathbb{R}} \mathcal{M}_1 = 1$

$n{=}2$: $\quad T(z) = \begin{pmatrix} \beta z + \gamma \\ z^2 + \epsilon \end{pmatrix} \quad \Rightarrow \quad E = \int \mathrm{d}^2 z \, \frac{8\,|\beta z^2 + 2\gamma z - \beta\epsilon|^2}{(|\beta z + \gamma|^2 + |z^2 + \epsilon|^2)^2} = 16\pi$

with $\beta, \gamma \in \mathbb{R}_+$ and $\epsilon \in \mathbb{C}$ $\qquad$ $\dim_{\mathbb{R}} \mathcal{M}_2 = 4$

we simplify to subclass $\beta{=}0$ $\quad \Rightarrow \quad T(z) = \begin{pmatrix} \gamma \\ z^2 + \epsilon \end{pmatrix}$ with $\gamma, \epsilon \in \mathbb{R}_+$

Moduli space metric

so far considered classical static finite-energy solutions (= solitons) $u = u(z\,|\,\alpha)$

bring back time dependence in adiabatic approximation (slow motion):

$u(t,z,\bar{z}) \;\approx\; u(z\,|\,\alpha(t))$ sequence of snapshots of static solitons

$$S\Big[u(\cdot\,|\,\alpha(t))\Big] \;+\; \text{const} \;=\; 4\int \mathrm{d}t\,\Big[\textstyle\int \mathrm{d}^2z\,(T^\dagger T)^{-1}\partial_{\bar\alpha}T^\dagger(\mathbb{1}-P)\,\partial_\alpha T\Big]\,\dot{\bar\alpha}\,\dot\alpha$$

$$= \;4\int \mathrm{d}t\,\Big[\textstyle\int \mathrm{d}^2z\,\frac{\partial_{\bar\alpha}\bar{u}\,\partial_\alpha u}{(1+\bar{u}u)^2}\Big]\,\dot{\bar\alpha}\,\dot\alpha \;=:\; \frac{1}{2}\int \mathrm{d}t\; g_{\bar\alpha\alpha}(\alpha)\,\dot{\bar\alpha}\,\dot\alpha$$

with Kähler metric $g_{\bar\alpha\alpha} \;=\; \partial_{\bar\alpha}\partial_\alpha\mathcal{K}$ deriving from a Kähler potential

$$\mathcal{K} \;=\; 8\int \mathrm{d}^2z\;\ln T^\dagger T \;=\; 8\int \mathrm{d}^2z\;\ln(1+\bar{u}u) \;\sim\; 8\int \mathrm{d}^2z\;\ln(\bar{p}p+\bar{q}q)$$

some moduli (like β) have infinite inertia ($g_{\bar\beta\beta}{=}\infty$) $\Rightarrow$ external parameters

$n{=}1:\quad T = \begin{pmatrix} \beta \\ z+\delta \end{pmatrix} \qquad \Rightarrow \qquad \mathcal{K} = 8\int \mathrm{d}^2z\ \ln\Big(1+\frac{|\beta|^2}{|z+\delta|^2}\Big) \stackrel{\text{reg}}{=} 8\pi\bar{\delta}\delta$ c.o.m.

$n{=}2:\quad T(z) = \begin{pmatrix} \gamma \\ z^2+\epsilon \end{pmatrix} \qquad \Rightarrow \qquad \mathcal{K} = 8\int \mathrm{d}^2z\ \ln\Big(1+\frac{|\gamma|^2}{|z^2+\epsilon|^2}\Big)$

evaluates to $\mathcal{K}_0 = 16\pi\,|\gamma| \int_0^{\pi/2} \mathrm{d}\theta\, \sqrt{1+|\tfrac{\epsilon}{\gamma}|^2 \sin^2\theta} = 16\pi\,|\gamma|\, E\big(-|\tfrac{\epsilon}{\gamma}|^2\big)$

expansions: 'ring' regime $|\epsilon| \ll |\gamma|$:

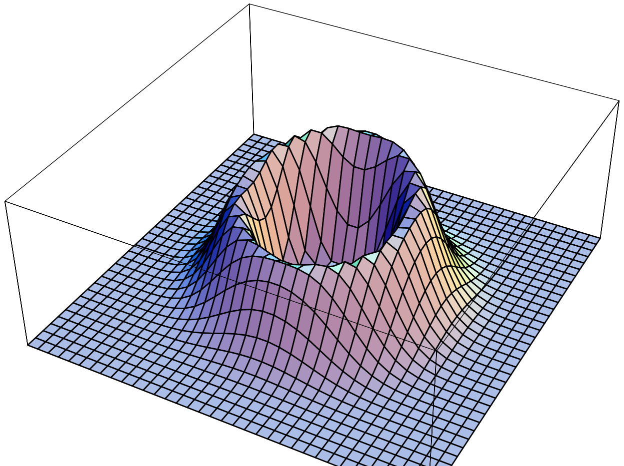
$$\mathcal{K}_0 = 8\pi^2|\gamma|\, \Big\{1+\frac14\Big|\frac{\epsilon}{\gamma}\Big|^2-\frac{3}{64}\Big|\frac{\epsilon}{\gamma}\Big|^4+\frac{5}{256}\Big|\frac{\epsilon}{\gamma}\Big|^6+\ldots\Big\}$$

'two-lump' regime $|\gamma| \ll |\epsilon|$:

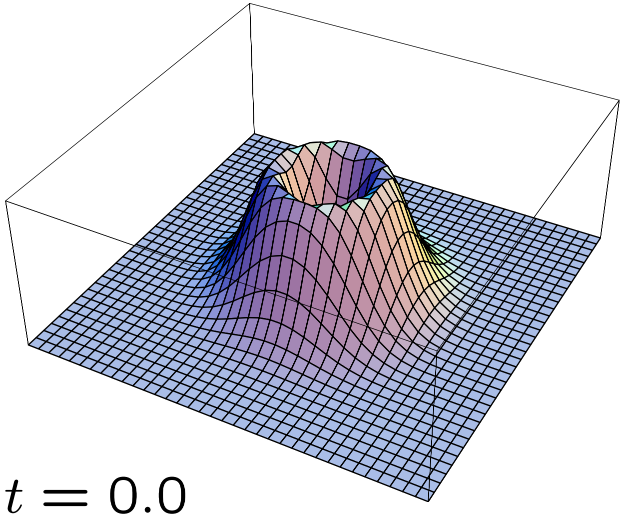
$$\mathcal{K}_0 = 16\pi\,|\epsilon|\,\Big\{1-\frac14\Big(-1+\ln\Big|\frac{\gamma}{4\epsilon}\Big|^2\Big)\Big|\frac{\gamma}{\epsilon}\Big|^2+\frac{1}{32}\Big(\frac32+\ln\Big|\frac{\gamma}{4\epsilon}\Big|^2\Big)\Big|\frac{\gamma}{\epsilon}\Big|^4-\frac{3}{256}\Big(2+\ln\Big|\frac{\gamma}{4\epsilon}\Big|^2\Big)\Big|\frac{\gamma}{\epsilon}\Big|^6+\ldots\Big\}$$

mild logarithmic singularities at $|\gamma| \to 0$

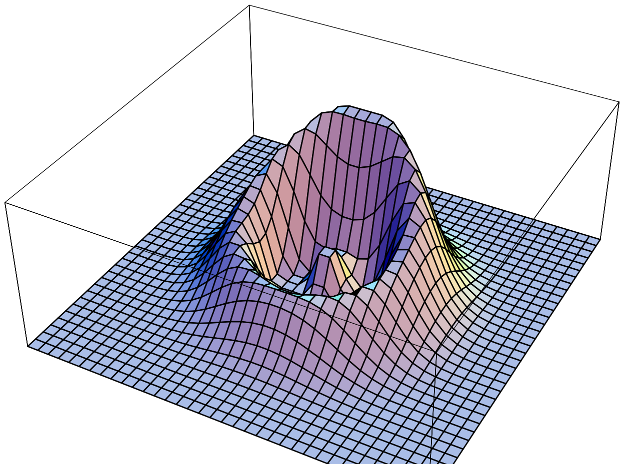
adiabatic motion in ring regime: energy density plots



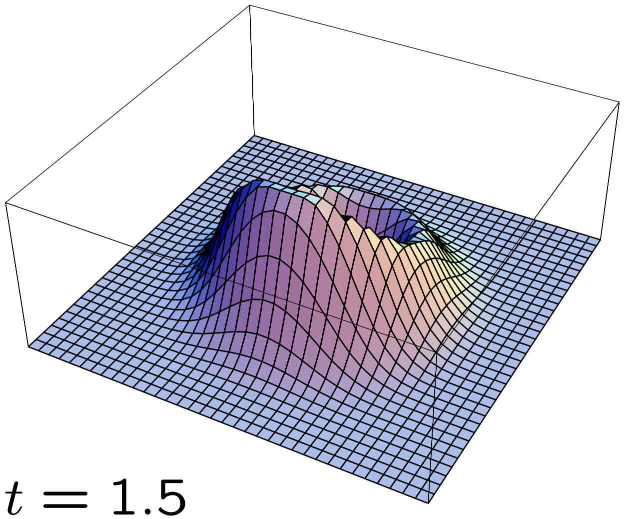
$t = -5.0$



$t = 0.0$



$t = -1.5$



$t = 1.5$

Moyal deformation

coordinates $(z,\bar z)$ $\longmapsto$ operators $(Z,\bar Z)$ with $[Z\,,\,\bar Z]=2\theta=\text{const}$

$$Z\;=\;\sqrt{2\theta}\,a\;=\;\sqrt{2\theta}\begin{pmatrix}0&0&&&\\ \sqrt{1}&0&0&&\\ &\sqrt{2}&0&0&\\ &&\sqrt{3}&0&\ddots\\ &&&\ddots&\ddots\end{pmatrix}\quad\&\quad \bar Z\;=\;\sqrt{2\theta}\,a^\dagger\;=\;\sqrt{2\theta}\begin{pmatrix}0&\sqrt{1}&&&\\ 0&0&\sqrt{2}&&\\ &0&0&\sqrt{3}&\\ &&0&0&\ddots\\ &&&\ddots&\ddots\end{pmatrix}$$

functions $f(z,\bar z)$ $\longmapsto$ operators $F\;=\;f(Z,\bar Z)\big|_{\text{sym}}$ $\qquad [a,a^\dagger]=\mathbb{1}$

$\partial_{\bar z}\;\mapsto\;\frac{1}{2\theta}[Z,\,\cdot\,]\;=\;\frac{1}{\sqrt{2\theta}}[a,\,\cdot\,]$ and $\int \mathrm{d}^2z\; f(z,\bar z)\;=\;2\pi\theta\,\mathrm{tr}\,F$

$a\,|0\rangle=0 \quad\Rightarrow\quad \mathcal{F}\;=\;\text{span}\Big\{|n\rangle=\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\;\Big|\;n=0,1,2,\ldots\Big\}$

$N\,|n\rangle\;=\;n\,|n\rangle$ and $\langle n|n\rangle\;=\;1$ for $n=0,1,2,\ldots$

dimensions: $Z=\sqrt{2\theta}\,a\,,\quad \beta=\sqrt{2\theta}\,b\,,\quad \delta=\sqrt{2\theta}\,d\,,\quad \gamma=2\theta\,g\,,\quad \epsilon=2\theta\,e$

$n{=}1$ warmup:

$$T = \binom{b}{a+d} \qquad \Rightarrow \qquad \mathcal{K} = 16\pi\theta\,\mathrm{tr}\,\ln T^\dagger T = 16\pi\theta\,\mathrm{tr}\,\ln\Big(\bar{b}b + (a^\dagger + \bar{d})(a+d)\Big)$$

formally independent of d via $a \mapsto a-d$ but divergent:

$$\frac{\mathcal{K}}{16\pi\theta} = \sum_{n=0}^{\infty} \langle n|\ln(\bar{b}b + N)|n\rangle = \sum_{n=0}^{\infty} \ln(\bar{b}b + n) = \infty \qquad \text{ambiguous}$$

fix the ambiguity by differentiating:

$$\frac{g_{\bar{d}d}}{16\pi\theta} = \frac{\partial_{\bar{d}}\partial_d\mathcal{K}}{16\pi\theta} = \mathrm{tr}\Big[(\bar{b}b+N)^{-1}\Big(1 - a\,(\bar{b}b+N)^{-1}a^\dagger\Big)\Big]$$

$$= \mathrm{tr}\Big[(\bar{b}b+N)^{-1}\Big(1 - (N+1)(\bar{b}b+N+1)^{-1}\Big)\Big]$$

$$= \bar{b}b\,\mathrm{tr}\Big[(\bar{b}b+N)^{-1}(\bar{b}b+N+1)^{-1}\Big]$$

$$= \sum_{n\geq 0} \frac{\bar{b}b}{(n+\bar{b}b)(n+1+\bar{b}b)} = 1 \qquad (!)$$

but $g_{\bar{b}b} = \infty \qquad \Rightarrow \qquad \mathcal{K} = 16\pi\theta\bar{d}d = 8\pi\bar{\delta}\delta$ c.o.m. as commutative case

Deformed rings

$$T = \begin{pmatrix} g \\ a^2 + e \end{pmatrix} \quad \Rightarrow \quad \mathcal{K} = 16\pi\theta \, \mathrm{tr} \ln T^\dagger T \qquad \text{with}$$

$$T^\dagger T = \bar{g}g + (a^{\dagger 2} + \bar{e})(a^2 + e) = \underbrace{\bar{g}g + N(N-1)}_{G} + \underbrace{e\, a^{\dagger 2} + \bar{e}\, a^2 + \bar{e}e}_{E}$$

abbreviating $G^{-1}a^{\dagger 2} =: \nwarrow$ and $G^{-1}a^2 =: \swarrow$ we get

$$\frac{\mathcal{K}}{16\pi\theta} = \mathrm{tr} \ln G + \mathrm{tr} \ln(\mathbb{1} + G^{-1}E) = \mathrm{tr} \ln G - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \mathrm{tr} \left(G^{-1}E\right)^k$$

$$= \exp\left(\bar{e}e\, \partial_{\bar{g}g}\right) \mathrm{tr} \Big\{ \ln G - \bar{e}e \nwarrow\swarrow - (\bar{e}e)^2 \left(\tfrac{1}{2} \nwarrow\swarrow\nwarrow\swarrow + \nwarrow\nwarrow\swarrow\swarrow\right) - (\bar{e}e)^3 \times$$

$$\times \left(\tfrac{1}{3} \nwarrow\swarrow\nwarrow\swarrow\nwarrow\swarrow + \nwarrow\nwarrow\nwarrow\swarrow\swarrow\swarrow + \nwarrow\nwarrow\swarrow\nwarrow\swarrow\swarrow + \nwarrow\swarrow\nwarrow\nwarrow\swarrow\swarrow\right) + \ldots \Big\}$$

$$\nwarrow |n\rangle = \frac{\sqrt{(n+1)(n+2)}}{\bar{g}g + n(n-1)} |n+2\rangle \qquad \text{and} \qquad \swarrow |n\rangle = \frac{\sqrt{n(n-1)}}{\bar{g}g + n(n-1)} |n-2\rangle$$

leading term $\mathrm{tr}\,\ln G = \sum_{n\geq 0} \ln\big(\bar{g}g + n(n-1)\big)$ determined via

$$\partial_{\bar{g}g}\mathrm{tr}\ln G = \mathrm{tr}\frac{1}{G} = \sum_{n\geq 0}\frac{1}{\bar{g}g+n(n-1)} = \frac{1}{\bar{g}g} + \frac{\pi^2}{2}\frac{\tan W}{W} \quad \text{with} \quad W = \frac{\pi}{2}\sqrt{1-4\bar{g}g}$$

expansion in $\bar{e}e$:

$$\frac{\mathcal{K}}{16\pi\theta} = \ln\bar{g}g + \ln\cos W + (\bar{e}e)^1\pi^2\frac{\bar{g}g}{4\bar{g}g+3}\frac{\tan W}{W}$$

$$+ (\bar{e}e)^2\pi^4\left\{\frac{48(\bar{g}g)^4+200(\bar{g}g)^3-33(\bar{g}g)^2+27\bar{g}g}{4\,(4\bar{g}g+3)^3(4\bar{g}g+15)}\frac{\tan W}{W^3} - \frac{(\bar{g}g)^2}{2\,(4\bar{g}g+3)^2}\frac{\sec^2 W}{W^2}\right\}$$

$$+ (\bar{e}e)^3\pi^6\left\{\frac{10240(\bar{g}g)^8+171520(\bar{g}g)^7+878336(\bar{g}g)^6+\ldots-13770(\bar{g}g)^2+6075\bar{g}g}{8\,(4\bar{g}g+3)^5(4\bar{g}g+15)^2(4\bar{g}g+35)}\frac{\tan W}{W^5}\right.$$

$$\left. - \frac{48(\bar{g}g)^5+200(\bar{g}g)^4-33(\bar{g}g)^3+27(\bar{g}g)^2}{4\,(4\bar{g}g+3)^4(4\bar{g}g+15)}\frac{\sec^2 W}{W^4} + \frac{(\bar{g}g)^3}{3\,(4\bar{g}g+3)^3}\frac{\tan W\sec^2 W)}{W^3}\right\}$$

$$+ O((\bar{e}e)^4)$$

exact in $\bar{g}g$

$\theta \to \infty \quad \Leftrightarrow \quad g, e \to 0$ but $|\frac{e}{g}| \ll 1$ fixed: $\bar{g}g\tan W \sim \bar{g}g\sec W$ finite!

$\theta \to 0 \quad \Leftrightarrow \quad g, e \to \infty$ but $|\frac{e}{g}| \ll 1$ fixed: $\mathcal{K} = \mathcal{K}_0 + O\Big(\frac{\theta^2}{|\gamma|}\Big)$ deformed rings

Deformed lumps

$$T = \begin{pmatrix} g \\ a^2 + e \end{pmatrix} \qquad \Rightarrow \qquad \mathcal{K} = 16\pi\theta\, \mathrm{tr}\, \ln T^\dagger T$$ with

$$T^\dagger T = (a^{\dagger 2} + \bar{e})(a^2 + e) + \bar{g}g = \underbrace{N(N-1) + e\, a^{\dagger 2} + \bar{e}\, a^2 + \bar{e}e}_{F} + \bar{g}g$$

find eigenvalues $\lambda_n(e)$ of F — represent F as differential operator on $L_2(\mathbb{R})$

$$a = \tfrac{1}{\sqrt{2}}(x+\partial_x) = \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{2}x^2}\partial_x\, \mathrm{e}^{\frac{1}{2}x^2} \quad \& \quad a^\dagger = \tfrac{1}{\sqrt{2}}(x-\partial_x) = \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{1}{2}x^2}\partial_x\, \mathrm{e}^{-\frac{1}{2}x^2}$$

$$\Big[\tfrac{1}{4}\partial_x^4 - x\,\partial_x^3 + (x^2 - \tfrac{1}{2} + \tfrac{e+\bar{e}}{2})\,\partial_x^2 - 2ex\,\partial_x + (2ex^2 - e + \bar{e}e)\Big]\, f(x) = \lambda(e)\, f(x)$$

via Fourier transformation and change of variables equivalent to

$$\Big[-\partial_z(1-z^2)\,\partial_z + \tfrac{1}{1-z^2} + \bar{e}e\,(1-z^2)\Big]\, f(z) = \lambda(e)\, f(z)$$ with $z \in [-1, 1]$

$$\left[-\partial_z(1-z^2)\,\partial_z + \tfrac{1}{1-z^2} + \bar{e}e\,(1-z^2)\right] f(z) \;=\; \lambda(e)\, f(z) \qquad \text{with} \quad z\in[-1,1]$$

recognize (oblate) spheroidal wave equation $\Rightarrow$ spheroidal eigenvalues:

$$\lambda_n = n(n{-}1)\Big\{1 + \tfrac{2}{(2n-3)(2n+1)}\bar{e}e + \tfrac{2(4n^4-8n^3-35n^2+39n+63)}{(2n-5)(2n-3)^3(2n+1)^3(2n+3)}(\bar{e}e)^2 + \ldots\Big\}$$

$$=: n(n{-}1)\,\tilde{\lambda}_n(e) \qquad \text{for} \quad n=0,1,2,\ldots \qquad \text{two zero modes} \quad \lambda_0=\lambda_1=0$$

deformed Kähler potential:

$$\frac{\mathcal{K}}{16\pi\theta} = \sum_{n=0}^{\infty} \ln\big[\lambda_n(e) + \bar{g}g\big]$$

$$= \sum_{n=0}^{\infty} \ln\big[n(n{-}1) + \bar{g}g\big] + \sum_{n=2}^{\infty} \ln\Big[1 + \tfrac{n(n-1)}{n(n-1)+\bar{g}g}\big(\tilde{\lambda}_n(e) - 1\big)\Big] \overset{\downarrow}{=} \ldots$$

precisely reproduces the expansion for the ring regime $|e| \ll |g|$

for two-lump regime $|g| \ll |e|$ want expansion of $\lambda_n(e)$ about $|e|{=}\infty$ unknown?

reorder and resum double series?

$$\frac{\mathcal{K}}{16\pi\theta} = 2\ln\bar{g}g + \Big\{\ln\pi + \tfrac{2}{3}\bar{e}e - \tfrac{4}{45}(\bar{e}e)^2 + \tfrac{64}{2835}(\bar{e}e)^3 - \tfrac{32}{4725}(\bar{e}e)^4 + \ldots\Big\}$$
$$+ \bar{g}g\Big\{1 - \tfrac{2}{9}\bar{e}e + \tfrac{56}{675}(\bar{e}e)^2 - \tfrac{656}{19845}(\bar{e}e)^3 + \tfrac{1216}{91125}(\bar{e}e)^4 + \ldots\Big\}$$
$$+ (\bar{g}g)^2\Big\{(\tfrac{3}{2} - \tfrac{\pi^2}{6}) + (\tfrac{62}{27} - \tfrac{2\pi^2}{9})\bar{e}e + (\tfrac{3742}{3375} - \tfrac{16\pi^2}{135})(\bar{e}e)^2 + \ldots\Big\}$$
$$+ (\bar{g}g)^3\Big\{(\tfrac{10}{3} - \tfrac{\pi^2}{3}) + (\tfrac{292}{81} - \tfrac{10\pi^2}{27})\bar{e}e + (\tfrac{254846}{151875} - \tfrac{112\pi^2}{675})(\bar{e}e)^2 + \ldots\Big\}$$
$$+ O\big((\bar{g}g)^4\big)$$

commutative small-$|g|$ divergencies gone ✓

managed to resum only the first line:

$$\lim_{g\to 0}\frac{\mathcal{K} - \mathcal{K}|_{e=0}}{16\pi\theta} = \sum_{n=2}^{\infty}\ln\tilde{\lambda}_n \overset{(?)}{=} -\ln\tilde{\lambda}_1 = \tfrac{2}{3}\,\bar{e}e - \tfrac{4}{45}(\bar{e}e)^2 + \tfrac{64}{2835}(\bar{e}e)^3 + \ldots$$
$$\overset{(*)}{=} \ln\Big\{1 + \sum_{\ell=1}^{\infty}\frac{(4\,\bar{e}e)^\ell}{(2\ell+1)!}\Big\} = \ln\frac{\sinh(2\sqrt{\bar{e}e})}{2\sqrt{\bar{e}e}}$$
$$= 2|e| - \ln|e| - 2\ln 2 + O(\mathrm{e}^{-4|e|}) \sim \frac{|\epsilon|}{\theta} + O(\mathrm{e}^{-2|\epsilon|/\theta})$$

proof of $(*)$ via Gel'fand-Yaglom theorem:

$$\Big[-\partial_z(1-z^2)\,\partial_z + \tfrac{1}{1-z^2} + \bar{e}e\,(1-z^2)\Big]\,f(z) \;=\; \lambda(e)\,f(z) \qquad \text{with} \quad z\in[-1,1]$$

substitute $z=\tanh y$

$$\Big[F(e)\,f\Big](y) \;\equiv\; \Big[-\cosh^2 y\,\partial_y^2 + \cosh^2 y + \tfrac{\bar{e}e}{\cosh^2 y}\Big]\,f(y) \;=\; \lambda(e)\,f(y) \qquad \text{with} \quad y\in\mathbb{R}$$

Gel'fand-Yaglom :

$$\lim_{g\to0}\exp\frac{\mathcal{K}-\mathcal{K}|_{e=0}}{16\pi\theta} \;\equiv\; \frac{\det F(e)}{\det F(0)} \;=\; \lim_{L\to\infty}\frac{\Phi(L)}{\Psi(L)} \qquad \text{with}$$

$$\Big[F(e)\,\Phi\Big](y) \;=\; 0\,, \qquad \Phi(-L) \;=\; 0\,, \qquad \Phi'(-L) \;=\; 1$$

$$\Big[F(0)\,\Psi\Big](y) \;=\; 0\,, \qquad \Psi(-L) \;=\; 0\,, \qquad \Psi'(-L) \;=\; 1$$

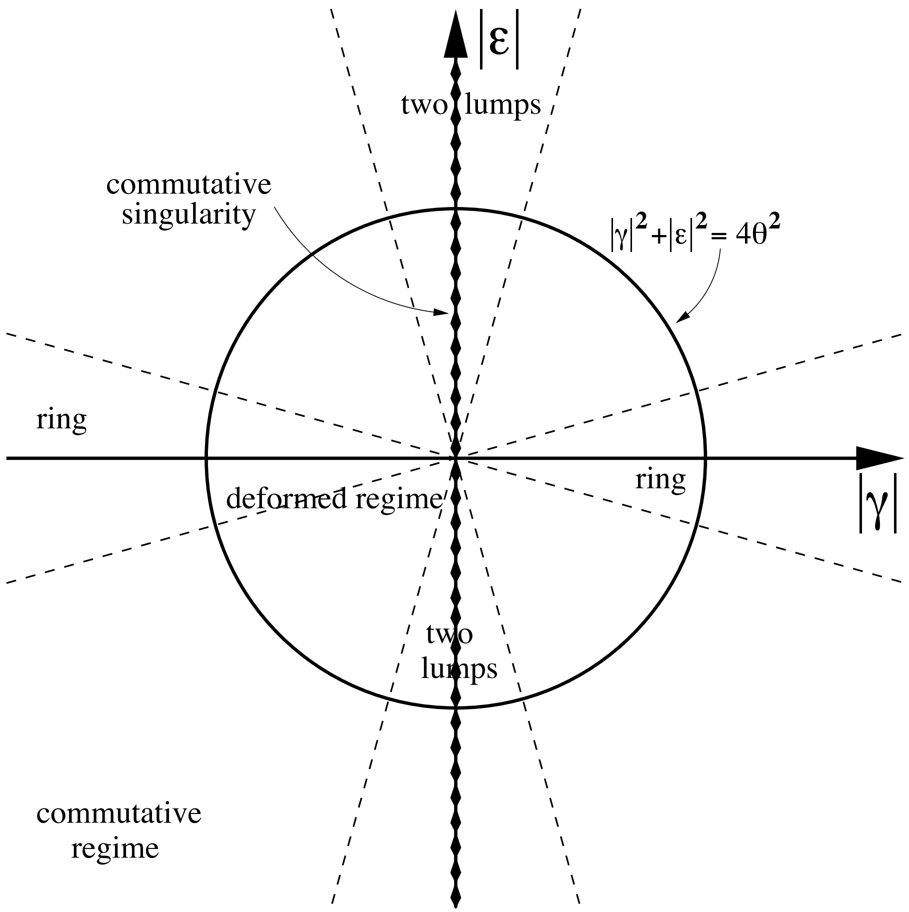
explicit solutions:

$$\Phi(y) \;=\; \tfrac{1}{\sqrt{\bar{e}e}}\sinh\Big[\sqrt{\bar{e}e}\,(\tanh y + \tanh L)\Big]\cosh y\cosh L$$

$$\Psi(y) \;=\; \sinh(y+L)$$

$$\frac{\Phi(L)}{\Psi(L)} \;=\; \frac{\sinh(2\sqrt{\bar{e}e}\tanh L)}{2\sqrt{\bar{e}e}\tanh L} \quad \xrightarrow{L\to\infty} \quad \frac{\sinh(2\sqrt{\bar{e}e})}{2\sqrt{\bar{e}e}} \;=\; \frac{\det F(e)}{\det F(0)} \qquad \square$$

Conclusions



modulus-of-moduli space

- ring regime: $x \equiv \left|\frac{\epsilon}{\gamma}\right|^2$

  $$\mathcal{K} = 8\pi^2\Big\{|\gamma|A_0(x) + \frac{\theta^2}{|\gamma|}A_1(x) + \dots\Big\}$$

- two-lump regime: $y \equiv \left|\frac{\gamma}{\epsilon}\right|^2$

  $$\mathcal{K} = 16\pi\Big\{|\epsilon|B_0(y) + \theta\, \mathrm{e}^{-2|\epsilon|/\theta}B_1(y) + \dots\Big\}$$

- logarithmic singularities pushed out
- nontrivial mathematics:
  - elliptic functions
  - spheroidal eigenvalues
  - Gel'fand-Yaglom